Some Aspects of Research on Mechanics of Thin Elastic Rod

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Abstract. Some aspects of research wok on the mechanics of thin elastic rod based on Kirchhoff-Cosserat's model were summarized. The analytical mechanics with arc-coordinate s and time t as double variables was established to formulate the motion of elastic rod. In stability analysis the difference and relationship between Lyapunov's stability and Euler's stability were discussed. The first approximate stability was determined by the characteristic equation with double eigenvalues in different domains, one of which can be determined by geometric conditions in static analysis. The Lyapunov's and Euler's stability conditions of the rod in space domain are the necessary conditions of Lyapunov's stability in time domain. As applications of the Kirchhoff's rod in molecular biology, the explanations of nucleosome structure and the chromosome coiling of DNA were given. Concerning the application in engineering the shape of a hanging rod under gravity and the coiling and stretching process of an extendable space mast were discussed. The motion of an axial moving beam with constant velocity and axial extensive force was discussed as an example of exact Cosserat's rod, the special case of small deformation is the Timoshenko's beam.

1. Introduction

The mechanics of a thin-long elastic rod has a practical background in molecular biology and engineering. The traditional model of a beam using Cartesian coordinates of centerline cannot be competent in analysis of a thin-long rod with large deformation. The Kirchhoff's kinetic analogy provides a different model of the rod by rotation of cross section along the centerline, and the attitude angles of cross section are used as unknown functions of arc-coordinates s. In analysis of dynamics the time variable is added and the model becomes a system with double independent variables s and t. The Cosserat's model is an exacter model, which abandons the hypothesis of inextensible and unshearable characters, and is closer to a real rod in practice. The Kirchhoff-Cosserat's model is suitable particularly to the large deformation of thin-long elastic rod, such as DNA, fiber, sub-ocean cable, oil-well drill string and other thin-long projects. In application of Kirchhoff-Cosserat's model all approaches of dynamics, such as the dynamics of rigid bodies, the analytical mechanics, the theory

of stability and others, can be used into the mechanics of elastic rod. In this paper some aspects of research on Kirchhoff-Cosserat's model are summarized.

2. Analytical mechanics of elastic rod [1~6]

As basic definitions in analytical mechanics of elastic rod, the virtual displacement of cross section of the rod is defined as: The imagined infinitesimal displacement of cross section, consistent with constraint, and irrelevant to the arc-coordinates and time. The principle of D'Alembert-Lagrange is defined as: For an elastic rod under ideal bilateral constraint, the true motion different to any possible motion consistent with constraint, that its virtual work for arbitrary virtual displacement is zero. The principle of minimal potential energy is used to take place of Hamilton principle, and the density of potential energy Γ of deformation of the rod plays a role of Lagrange's function in analytical statics.

$$\delta\int_0^L \Gamma \,\mathrm{d}s = 0\,,\quad \Gamma = \frac{1}{2}\left[A\omega_1^2 + B\omega_2^2 + C\left(\omega_3 - \omega_3^0\right)^2 \right] - F\gamma \tag{2.1}$$

When the rod under constraint condition $\boldsymbol{C}\left(\boldsymbol{q},\dot{\boldsymbol{q}}\right)=0$, the Lagrange's multiplier $\boldsymbol{\Lambda}$ can be used to compose a revised Lagrange's function $\tilde{\Gamma} = \Gamma + \boldsymbol{\Lambda}\cdot\boldsymbol{C}$. The Lagrange's equations are derived as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial\tilde{\Gamma}}{\partial q_i'}\right) - \frac{\partial\tilde{\Gamma}}{\partial q_i} = 0 \quad (j=1,2,\cdots,n) \tag{2.2}$$

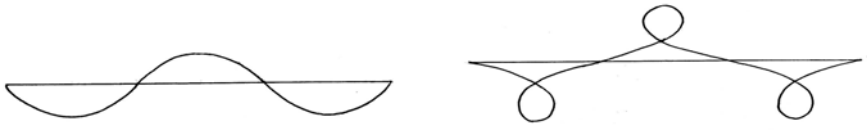
In dynamic analysis it is necessary to establish an analytical dynamics with double arguments s and t. Using the Lagrange's function $\Lambda = \Gamma - T$, composed of densities of potential energy Γ and kinetic energy T, the Lagrange's equations and the Nielson's equations can be derived as

$$\frac{\partial}{\partial s}\left(\frac{\partial\Lambda}{\partial q_j'}\right) + \frac{\partial}{\partial t}\left(\frac{\partial\Lambda}{\partial\dot{q}_j}\right) - \frac{\partial\Lambda}{\partial q_j} + f_j = 0 \quad (j=1,2,\cdots,n) \tag{2.3}$$

$$\frac{\partial\Lambda'}{\partial q_j'} + \frac{\partial\dot{\Lambda}}{\partial\dot{q}_j} - 3\frac{\partial\Lambda}{\partial q_j} + f_j = 0 \quad (j=1,2,\cdots,n) \tag{2.4}$$

3. Stability and vibration [7~17]

When the Lyapunov's stability theory is applied to analyze the equilibrium of elastic rod some unusual results can be obtained. As an example the equilibrium of an axial compressed elastic rod is stable by Lyapunov's stability definition (Fig.1a), and is unstable when the rod is extended (Fig.1b). The results contradict to the traditional knowledge of Euler's buckling concept of a compressed bar evidently. The different conclusions are caused by different definitions of stability. According to the Lyapunov's stability theory, the equilibrium is stable when the perturbed centerline is restricted in a small neighborhood of the unperturbed straight line. Nevertheless, from the viewpoint of Euler's stability, when the perturbation equations have nontrivial solution satisfying the boundary conditions on both ends, the equilibrium is unstable and the buckling occurs. The Lyapunov's stability is a concept of stability with respect to the initial perturbation, but the perturbed centerline is under constraints on both ends. When all boundary conditions on both ends are satisfied, the corresponding load is the Euler's critical load. Therefore the Euler's load can be calculated in analysis of Lyapunov's stability.



(a) Compressed rod (stable) (b) Extended rod (unstable)

Figure 1. Perturbed centerlines of compressed and extended rod

In stability analysis the linearized perturbed equations has double arguments s and t, it follows that the characteristic equation has double eigenvalues λ and w corresponding to different variables. When the Lyapunov's stability conditions in statics are satisfied, the eigenvalue in space domain $\lambda = \pm \mathrm{i}k$ can be determined by boundary conditions on both ends. Then the characteristic equation contains only one unknown eigenvalue w, which determines the stability of equilibrium in time domain.

As an example we consider an elastic rod with circular cross section and length L. The dynamical equations of a Kirchhoff's rod with the Euler's angles ψ, ϑ, φ and internal forces $F_i\ (i = 1,2,3)$ as unknown variables permit special solutions, corresponding to the helical equilibrium

$$\vartheta = \vartheta_0, \psi = \omega_0 s, \phi = 0, F_1 = 0, F_2 = F_0 \sin\vartheta_0, F_3 = F_0 \cos\vartheta_0 \tag{3.1}$$

where F_0 is the axial force required by the helical equilibrium

$$F_0 = -\frac{EI\omega_0^2}{\cos\vartheta_0} \tag{3.2}$$

where E, I are Young's modulus and inertial moment of the cross section. The characteristic equation of the linearized perturbed equations with perturbations $x_i\ (i = 1,2,\cdots,6)$ can be derived as

$$a(\lambda)w^4 + b(\lambda)w^2 + c(\lambda) = 0 \tag{3.3}$$

where $c(\lambda)$ is defined as

$$c(\lambda) = \lambda^4\left(1+\lambda^2\right)^2\left[1+\left(\lambda^2+3\right)\cos^2\vartheta_0\right] \tag{3.4}$$

The characteristic equation in space domain $c(\lambda) = 0$ has pure imaginary roots $\lambda = \pm \mathrm{i}k$, where

$$k = \left(\frac{1+3\cos^2\vartheta_0}{\cos^2\vartheta_0}\right)^{1/2} \tag{3.5}$$

Therefore the helical equilibrium is stable by Lyapunov's definition. The boundary conditions $x_i(L) = x_i(0)$ on both ends of the rod require $k = 2n\pi/\omega_0 L\ (n = 1,2,\cdots)$, then the Euler's critical load can be derived from Eqs.(3.5) and (3.2) as

$$\left|F_0\right|_{\mathrm{cr}} = \frac{4n^2\pi^2 EI\cos\vartheta_0}{L^2\left(1+3\cos^2\vartheta_0\right)} \tag{3.6}$$

After substitution of the determined eigenvalue $\lambda = \pm \mathrm{i}k$ into Eq.(3.3), one of the pure root conditions of eigenvalue w is $c(\lambda) > 0$. Comparing it to the characteristic equation $c(\lambda) = 0$, it follows that the axial force should be smaller than the Euler's critical load. Thus the Lyapunov's and Euler's stability conditions of the helical rod in space domain are the necessary conditions of Lyapunov's stability in time domain. When all stability conditions are satisfied letting $w = \pm \mathrm{i}\mu$, we obtain the angular frequency μ of lateral vibration of the helical rod as

$$\mu = \left(\frac{2n\pi}{L}\right)^2\sqrt{\frac{EI}{\rho S}}\Phi\left(k,\vartheta_0\right) \tag{3.7}$$

$$\Phi\left(k,\vartheta_0\right) = \sqrt{\frac{k^2+1+4\cos^2\vartheta_0}{k^2+\sin^2\vartheta_0}}\left\{1 \mp \sqrt{1+\frac{\left(k^2+\sin^2\vartheta_0\right)\left(k^2-1\right)^2\left[1-\left(k^2-3\right)\cos^2\vartheta_0\right]}{k^4\left(k^2+1+4\cos^2\vartheta_0\right)^2\cos^2\vartheta_0}}\right\}^{1/2} \tag{3.8}$$

4. Application in molecular biology [18~20]

4.1 Explanation of the nucleosome structure of DNA

As the structural unit of chromatin the nucleosome is composed of DNA and histone core HO with a small H1-histone (Fig.2, Cited from http://www.cvh.cc/onews.asp?id=29713). The DNA wraps around histones 1 and 3/4 circle and forms a nucleosome structure. Many nucleosomes are linked together like a beads. The nucleosome structure can be simplified as a thin elastic rod constrained by a cylinder with radius R and locked by a small bar at the entrance of DNA (Fig.3).

In order to explain the structure of nucleosome, at first we consider the equilibrium of the thin rod acted by the electric force f_e and contact force f_c with constraint conditions

$$\vartheta = \vartheta_0,\ \psi' = \frac{\sin\vartheta_0}{R},\ \varphi' = \omega_0 \tag{4.1}$$

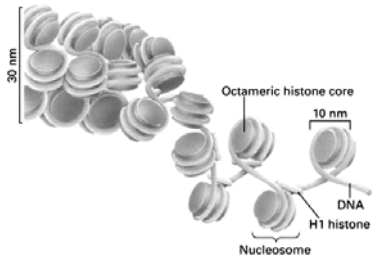


Figure 2. Nucleosome structure

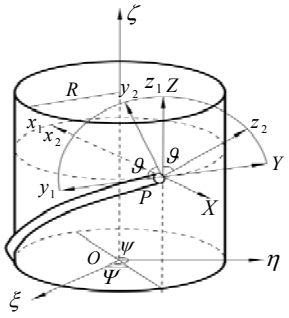


Figure 3. Thin rod constrained by cylinder

The internal forces and the contact forces can be solved as

$$\begin{gathered} F_1 = 0,\ F_2 = -\frac{A-C}{R^2}\sin^3\vartheta_0\cos\vartheta_0,\ F_3 = F_{30} \\ f_{c1} = \left(F_{30} + \frac{A-C}{4R^2}\sin^2 2\vartheta_s\right)\frac{\sin^2\vartheta_s}{R} + f_e,\ f_{c2} = f_{c3} = 0 \end{gathered} \tag{4.2}$$

where $A = EI$ and $C = GI_0$ are the bending and twisting stiffness of the rod. Considering the equilibrium of the cylinder HO wrapped by the rod we obtain the contact range of DNA

$$\left[\psi_1, 4\pi - \psi_1\right] = [102.5^\circ, 617.5^\circ] \tag{4.3}$$

After numerical calculation the range is about 1.43 circles (Fig.4). At last we consider the equilibrium of the H1-histone, which is simplified as a small bar in contact with the cylinder HO and DNA. The incline angle β of the bar can be obtained as

$$\tan\beta = -\frac{\mu R^3\cos\psi_1\tan\vartheta_0}{\mu R^3 + (C-A)\sin^4\vartheta_0} \tag{4.4}$$

where $\mu = f_{c1} - f_e$. When the electric force is balanced by the contact force, $\mu = 0$, and $\beta = 0$, the H1 bar is erect along ζ-axis (Fig.5).

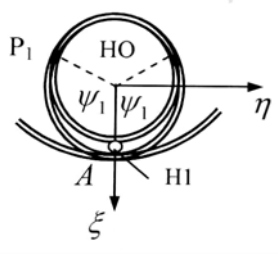


Figure 4. Contact range of DNA

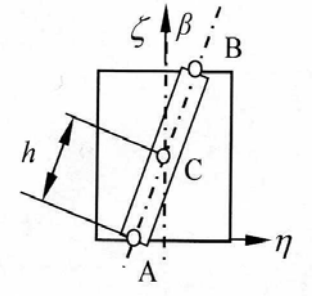


Figure 5. Equilibrium of H1-bar

4.2 Explanation of chromosome coiling of DNA

The formation of DNA in chromatin is coiled in helical state with different order. At first the nucleosome structure folds the DNA to a helix with diameter 11 nm, after that the helix twins itself in higher order coiling with diameter 30 nm into the chromatin (Fig.6, Cited from http://www. cchem. berkeley.edu/jehgrp/yms/). In order to explain the coiling process, a physical experiment was made by Thompson and Champney using a rubber rod under axial tension force F_0 and twist M_0. When the force F_0 increases the initial straight equilibrium becomes unstable and transforms to a helical equilibrium. When F_0 increases further the straight helical rod becomes unstable and twins to a helix with higher order. In explanation of the phenomenon we write the equilibrium equations of a Kirchhoff's rod under axial tension and twist, and uncouple it to a nonlinear equation of ϑ

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}s^2}+Q(\vartheta)=0,\quad Q(\vartheta)=\frac{l^2\sin\vartheta}{4}\left[\left(\cos\frac{\vartheta}{2}\right)^{-4}-\frac{2p}{l^2}\right] \tag{4.5}$$

Where $p=2F_0/A$ and $l=M_0/A$. The trivial solution $\vartheta_0=0$ or π of equation $Q(\vartheta_0)=0$ corresponds to a straight rod. In addition, when $2p/l^2>1$ a nontrivial solution exists as

$$\vartheta_0=2\arccos\left[\left(2p/l^2\right)^{-1/4}\right] \tag{4.6}$$

The solution (4.6) describes a helical equilibrium with pitch angle $\pi/2-\vartheta_0$, which decreases with increase of $2p/l^2$. The stability analysis confirms that ϑ_0 has a bifurcation point of $2p/l^2=1$, after which the stable straight state becomes unstable and a stable helical state exists (Fig.7). When F_0 increases further the straight thin helical rod becomes unstable and twins to a coiling state of 2-nd order. The bifurcation point occurs on $2\tilde{p}/\tilde{l}^2=1$, where $\tilde{p}=2F_0/\tilde{A}$, $\tilde{l}=M_0/\tilde{A}$, and $\tilde{A}$ is an equivalent bending stiffness of a thin helical rod

$$\tilde{A}=A\cos\vartheta_0\left[1+\frac{(1-\lambda)}{2\lambda}\sin^2\vartheta_0\right]^{-1},\quad \lambda=\frac{C}{A} \tag{4.7}$$

Continuing the similar process the forming of different order helix in chromosome coiling can be explained qualitatively.

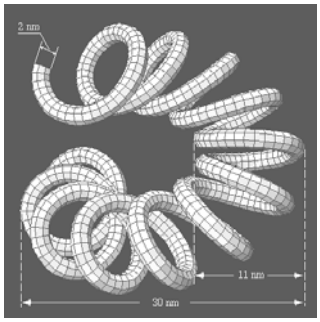


Figure 6. Chromosome coiling of DNA

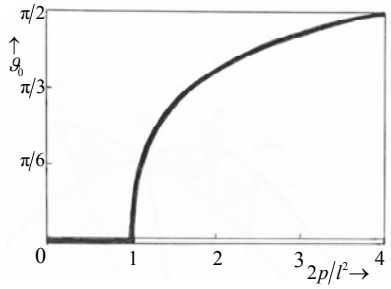


Figure 7. Bifurcation of ϑ_0 vs $2p/l^2$

5. Application in engineering [21~24]

5.1 Shape of hanging rod under gravity

The shape of a flexible cord under gravity was expressed by hyperbolic function in seventeen century as a successful application of mathematical calculus. The well-known curve of catenary can be utilized in engineering as an approximate model of hanging cable or pipe. When the hanging cable has large

cross section the influence or bending rigidity cannot be neglected. The equilibrium equation of Kirchhoff's rod under gravity can be written with consideration of the bending rigidity

$$EI\vartheta'' + (\rho Sgs)\cos\vartheta - F_0 \sin\vartheta = 0 \tag{5.1}$$

Where ϑ is the incline angle of the tangent of centerline to the horizon, ρ and S are the density and area of cross section (Fig.8). Using the perturbation method with a small parameter $\varepsilon = EI/mgl^2$ and taking the traditional solution of catenary as zero-th approximate solution, where l is the length of the rod, we obtain the solution corrected by the bending stiffness

$$\vartheta(s) = \arctan(\beta s) - \frac{2\varepsilon\beta^4 l^3 s}{\left(1+\beta^2 s^2\right)^{3/2}} \tag{5.2}$$

where $\beta = \rho Sg/F_0$. Introduce a function $\phi(\beta x)$

$$\phi(\beta x) = 2\beta^3 l^3 \int_0^{\beta x} \frac{\mathrm{d}u}{\mathrm{ch}^3 u} \tag{5.3}$$

The shape of the hanging rod can be derived in analytical form as

$$y(x) = \frac{1}{\beta}\left[\mathrm{ch}(\beta x) - 1 - \varepsilon\int_0^{\beta x} \mathrm{ch}u\,\phi(u)\,\mathrm{d}u\right] \tag{5.4}$$

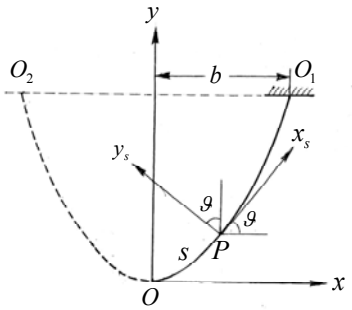


Figure 8. The hanging rod under gravity

5.2 Coiling and stretching of extendable space mast

The coiling and stretching process of the main element of extendable space mast in astronautic technique can be simplified by the motion of an elastic rod under unilateral constraint of a cylinder (Fig.9). The large deformation from a straight rod to a flat helical rod can be analyzed by the Kirchhoff's model with consideration of distributed contact force. In order to simplify the problem we assume that the rod maintains helical shape with unchanged radius R as the result of cylindrical constraint, and only the variation of angle ϑ and twisting of the rod are considered. In discussion of coiling process the inertial effect can be neglected approximately when the deformation is slow enough. The necessary external forces and torques in coiling process can be determined by some first integrals as

$$F(\vartheta) = \frac{A(1-\lambda)}{R^2}\cos\vartheta\sin^2\vartheta\,,\quad M(\vartheta) = \frac{A\sin\vartheta}{R}\left(\lambda\cos^2\vartheta + \sin^2\vartheta\right) \tag{5.5}$$

The stretching of the rod in space is a fast process, and the inertial effect should be considered. Based on the hypothesis of maintaining helical shape, the dynamical equations can be simplified to a nonlinear equation of $\vartheta(t)$

$$\ddot{\vartheta} - g(\vartheta)\dot{\vartheta}^2 = 0\,,\quad g(\vartheta) = \varepsilon\cos\vartheta\sin\vartheta\left(1+\cos^2\vartheta\right) \tag{5.6}$$

where $\varepsilon = (a/2R)^2$, a is the diameter of the rod. Eq.(5.6) permits a first integral to determine the

phase trajectories in phase plane $\left(\vartheta,\dot{\vartheta}\right)$ (Fig.9)

$$\dot{\vartheta}\left(\vartheta\right)=\dot{\vartheta}_0\left\{1+\frac{\varepsilon}{4}\left[2\left(\cos^2\vartheta_0-\cos^2\vartheta\right)+\cos^4\vartheta_0-\cos^4\vartheta\right]\right\} \tag{5.7}$$

Thus the stretching velocity $v_\zeta\left(s\right)$ and the time t_{str} of the duration from a flat helix to a straight rod are obtained in analytical form

$$v_\zeta=-\left(\sqrt{\frac{E}{\rho}}\dot{\vartheta}\sin\vartheta\right)s,\quad t_{\mathrm{str}}=\frac{\pi Sl^2}{4I}\sqrt{\rho E}\left(1+\frac{11\varepsilon}{32}\right) \tag{5.8}$$

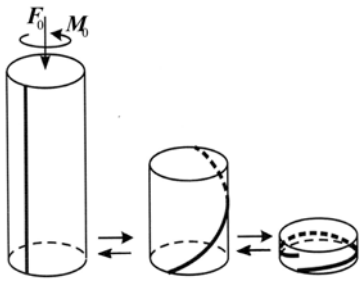


Figure 9. Coiling and stretching of a rod

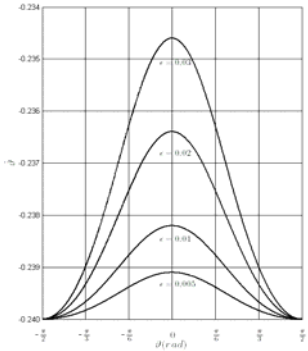


Figure 9. Phase trajectories in $\left(\vartheta,\dot{\vartheta}\right)$ plane

6. Exact Cosserat's model [25~28]

The inextensible and unshearable hypotheses of Kirchhoff's model are not adaptable to the real soft materials. The exact Cosserat's model takes the extension of centerline and the shear deformation of cross section in consideration, and is closer to a real elastic rod in practice. As the result of axial extension the arc-coordinate s of an arbitrary point P in centerline is changed by $s^*=s\left(1+\varepsilon_3\right)$, where ε_3 is the normal strain of centerline. The shear strains $\varepsilon_1,\varepsilon_2$ of cross section causes an additional rotation of the tangent vector of centerline, which deviates from the normal axis of cross section. Therefore the curvature of centerline is determined not only by the rigid rotation of cross section, as well as the shear deformation. The internal force and torque are proportional to the strains and curvature-twist respectively. The dynamical equations of the Cosserat's rod can be established with Cardan's angles ψ,ϑ,φ and deflections $w_i\ \left(i=1,2,3\right)$ as unknown variables. The configuration of the rod after deformation is determined by adding the deflections to the initial shape.

As an example we consider the motion of an axial moving beam with constant velocity v_0 and axial extensive force F_0 (Fig.10). Since the mass of beam flows along the centerline, the Euler's concept of velocity field is applied in mathematic formulation. As a special case of Cosserat's model with small deformation, the dynamical equation of an axial moving Timoshenko's beam can be written directly

$$a_1\frac{\partial^4 w}{\partial t^4}+a_2\frac{\partial^4 w}{\partial t^3\partial s}+a_3\frac{\partial^4 w}{\partial t^2\partial s^2}+a_4\frac{\partial^2 w}{\partial t^2}+a_5\frac{\partial^4 w}{\partial t\partial s^3}+a_6\frac{\partial^2 w}{\partial t\partial s}+a_7\frac{\partial^4 w}{\partial s^4}+a_8\frac{\partial^2 w}{\partial s^2}=0 \tag{6.1}$$

where the complex variable $w=w_1+\mathrm{i}w_2$ is applied to describe the three-dimensional deformation of the beam. In the stability analysis of quasi-stationary state of the beam the critical axial velocity before buckling can be derived as

$$v_{0,\mathrm{cr}}=\left[\frac{F_0}{\rho S}\left(\frac{1-\alpha\sigma_n}{1-\sigma_n}\right)\right]^{1/2},\quad \alpha=\frac{\kappa GS}{F_0},\sigma_n=\frac{EIn^2\pi^2}{F_0l^2\left(1-\alpha\right)}\quad \left(n=1,2,\cdots\right) \tag{6.2}$$

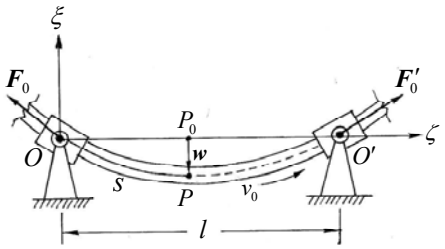


Figure 10. The axial moving beam

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