

# Neutrino mass, mixing and discrete symmetries

Alexei Y. Smirnov<sup>1</sup>

International Centre for Theoretical Physics, Strada Costiera 11, 34014 Trieste, Italy

E-mail: [smirnov@ictp.it](mailto:smirnov@ictp.it)

**Abstract.** Status of the discrete symmetry approach to explanation of the lepton masses and mixing is summarized in view of recent experimental results, in particular, establishing relatively large 1-3 mixing. The lepton mixing can originate from breaking of discrete flavor symmetry  $G_f$  to different residual symmetries  $G_\ell$  and  $G_\nu$  in the charged lepton and neutrino sectors. In this framework the *symmetry group condition* has been derived which allows to get relations between the lepton mixing elements immediately without explicit model building. The condition has been applied to different residual neutrino symmetries  $G_\nu$ . For generic (mass independent)  $G_\nu = \mathbf{Z}_2$  the condition leads to two relations between the mixing parameters and fixes one column of the mixing matrix. In the case of  $G_\nu = \mathbf{Z}_2 \times \mathbf{Z}_2$  the condition fixes the mixing matrix completely. The non-generic (mass spectrum dependent)  $G_\nu$  lead to relations which include mixing angles, neutrino masses and Majorana phases. The symmetries  $G_\ell$ ,  $G_\nu$ ,  $G_f$  are identified which lead to the experimentally observed values of the mixing angles and allow to predict the CP phase.

## 1. Introduction

A possibility to use the discrete flavor symmetries for understanding fermion masses and mixing had been proposed long time ago [1]. To a large extend the recent developments of this approach [2] was motivated by the tri-bimaximal (TBM) mixing [3]. The special values of elements of the TBM mixing matrix,

$$|U_{e3}|^2 = \sin^2 \theta_{13} = 0, \quad |U_{\mu 3}|^2 = |U_{\tau 3}|^2 = \sin^2 \theta_{23} = \frac{1}{2}, \quad |U_{e2}|^2 = \sin^2 \theta_{12} = \frac{1}{3}, \quad (1)$$

indicated possible geometric and group-theoretical origins of the mixing. The values in (1) also implied that there is no relation between mixing and masses (mass ratios). In this connection the most appealing framework is the one in which mixing originates from different ways of the flavor symmetry,  $G_f$ , breaking in the charged lepton and neutrino Yukawa sectors [4]:

$$G_f \rightarrow \text{breaking} \rightarrow \begin{cases} G_\nu & \text{neutrinos} \\ G_\ell & \text{charged leptons} \end{cases} . \quad (2)$$

The *residual* symmetries  $G_\nu$  and  $G_\ell$  of the neutrino and charged lepton mass matrices are different.  $G_\nu$  and  $G_\ell$  should be generic symmetries which exist for arbitrary values of masses. In this case appearance of TBM is, indeed, maximally controlled by symmetry.

<sup>1</sup> Talk given at the Symposium Discrete 2012, IST, Lisboa, Portugal, December 3 - 7, 2012.



Realizations of this framework in specific models are, however, rather complicated and not convincing. One should construct the Lagrangian invariant under symmetry  $G_f$  with certain field content and assignment of charges. Additional auxiliary symmetries are needed to forbid some interactions. Further model building is needed to achieve required vacuum alignment. Essentially, two different scalar sector should be introduced for spontaneous symmetry breaking in the neutrino and charged lepton sectors. Finally, after spontaneous symmetry breaking the mass matrices with certain residual symmetries emerge. Accidental symmetries and relations between mixing elements may show up in models which are not related to the original flavor symmetry. Complicated structure of models, many *ad hoc* parameters and new fields, additional symmetries, difficulty to include quarks, *etc.* cast doubt upon whole the approach [5].

Recent measurements of the neutrino oscillation parameters, and especially discovery of rather large 1-3 mixing [6], [7], [8], [9], [10], contrary to (1), have further disfavored TBM and the discrete symmetries behind. The TBM can be accidental and whole discrete flavor symmetry approach – phenomenologically irrelevant.

Further developments along this line were in two directions: (i) introduction of large corrections to TBM to reproduce results of measurements [11], (ii) modification of symmetries in such a way that they are consistent with nonzero 1-3 mixing [12]. In some situations these two lines coincide giving the same results. With this the symmetry effects become rather hidden.

In this connection in the framework of residual symmetries (2) a formalism has been developed [13], [14], [15] which allows to obtain consequences of flavor symmetries for mass and mixing without model building. The formalism allows to explore mixing patterns for wide class of different residual and covering symmetries. Inversely, it can be used to perform the “symmetry building” for a required mixing pattern. It helps to understand various features of mixing that are related to symmetries.

Symmetries and their consequences are the nicest part of the program, model building is the ugly one. In this review we will focus on the former: on the model independent part. The formalism [13, 14] will be explained in details and various related issues will be clarified. The applications to different residual neutrino symmetries will be described. Still one should remember that the problems with model building remain.

The paper is organized as follows. In Sec. 2 we summarize the relevant information on neutrino masses and mixing and its possible implications for discrete flavor symmetries. Next possible step in developments of the field will be outlined. In Sec. 3 the symmetry group conditions are derived which lead to relations between the lepton mixing elements without model building. Applications of these symmetry group conditions to different residual neutrino symmetries are described in Sec. 4. Discussion and conclusions are presented in Sec. 5.

## 2. Masses and mixing: status and implications

### 2.1. Data and Observations.

Results and observations from global fits [16], [17], [18] relevant for the discrete flavor symmetries can be summarized in the following way.

- 1). The 1-3 mixing deviates substantially from zero:

$$\sin^2 \theta_{13} = 0.022 - 0.024. \quad (3)$$

- 2). There are indications of significant deviation of the 2-3 mixing from maximal:  $\sin^2 \theta_{23} = 0.38 - 0.42$ , which can be characterized by

$$d_{23} \equiv 0.5 \cos 2\theta_{23} = 0.08 - 0.12. \quad (4)$$

The result follows from (i) the atmospheric neutrino data (included in the global fit); (ii) from MINOS direct measurements [19]:  $\sin^2 2\theta_{23} = 0.950^{+0.035}_{-0.036}$  ( $1\sigma$ ), and (iii) from comparison of

the T2K result [6] on  $\nu_\mu \rightarrow \nu_e$  oscillations ( $\propto \sin^2 \theta_{23} \sin^2 2\theta_{13}$ ) and the reactor disappearance results ( $\propto \sin^2 2\theta_{13}$ ). The atmospheric neutrino and T2K data being sensitive to  $\sin^2 \theta_{23}$  favor the first quadrant,  $\theta_{23} < \pi/4$ , at least for the normal mass hierarchy (NH). In the case inverted hierarchy (IH) different global fits give different results.

3). With increase of accuracy the deviation of the 1-2 mixing from the TBM value  $1/3$  becomes stronger and more significant statistically:

$$\sin^2 \theta_{12} = 0.30 - 0.31. \quad (5)$$

4). First glimpses on the CP-violating phase show up in different global fits:  $\delta = 180^\circ$  from [16],  $\delta \sim 300^\circ$  for both hierarchies from [17], and  $\delta = 145^\circ$  (NH)  $\delta = 0$  (IH) from [18]. Being at  $1\sigma$  level or even lower all these are statistically insignificant. In some analyses one can trace the physical effects which favor certain value  $\delta$ , in other cases that can be artifact of the global fit without any clear reason behind or just fluctuation.

The deviation of mixing from TBM can be characterized by the values of elements of the third column of the PMNS matrix,  $U_{\alpha 3} \equiv \{U_{e3}, U_{\mu 3}, U_{\tau 3}\}$ :

$$U_{\alpha 3}^{TBM} = \{0, 0.71, 0.71\}, \quad \text{whereas} \quad U_{\alpha 3}^{exp} = \{0.15, 0.62, 0.77\}, \quad (6)$$

and for the “numerology bookkeeping”:  $U_{\tau 3} - U_{\mu 3} \approx U_{e3}$ .

## 2.2. The 1-3 mixing: relations and implications.

The same value of 1-3 mixing can be connected to other observables in various ways which have different theoretical implications.

(i) “Naturalness”:

$$\sin^2 \theta_{13} = A \frac{\Delta m_{21}^2}{\Delta m_{32}^2}, \quad (7)$$

where  $A = 0.78$  ( $\sim 1 - \sin \theta_C$ ) for the best fit value of  $\theta_{13}$ . This relation follows from “naturalness” of mass matrix [20]: the fact that there are two large mixings connecting neighboring generations, and from the following two assumptions: (i) normal mass hierarchy, (ii) absence of fine tuning between different elements of the mass matrix (e.g.,  $|m_{e\mu} - m_{e\tau}| \sim |m_{e\mu}|$ ).

(ii) “QLC”:

$$\sin \theta_{13} \approx \frac{1}{\sqrt{2}} \sin \theta_C (1 - V_{cb} \cos \delta) - V_{ub} \approx \frac{\theta_C}{\sqrt{2}}, \quad (8)$$

where  $\theta_C$  is the Cabibbo angle [21]. Varying  $\delta$  we have from (8)  $\sin^2 \theta_{13} = 0.025 \pm 0.003$  in agreement with observations (3, 4). Approximately,  $\sin^2 \theta_{13} \approx \frac{1}{2} \sin^2 \theta_C$ . This prediction is essentially result of permutation of the matrices of the maximal mixing 2-3 rotation and the 1-2 rotation on the Cabibbo angle:

$$U_{12}(\theta_C) U_{23} \left( \frac{\pi}{4} \right).$$

Here  $U_{12}(\theta_C)$  can follow from the charged leptons, whereas  $U_{23}(\frac{\pi}{4})$  – from neutrinos. The former implies a kind of quark-lepton symmetry, or unification, or common horizontal symmetry which lead to equality of the 1-2 mixings in the quark and lepton sectors. The second  $(\pi/4)$  rotation comes from the neutrino sector and can be part of the bi-maximal [21] or tri-bimaximal [22] mixings. The latter case is referred as the tribimaximal-Cabibbo mixing. The permutation is needed to reduce the mixing matrix to the standard form. The relation was first realized at the purely phenomenological level [23] and then in the Quark-Lepton Complementarity (QLC) framework [21]. Appearance of the bi-maximal or tri-bimaximal mixings can be a consequence of discrete flavor symmetries.

(iii) The  $\nu_\mu - \nu_\tau$  symmetry breaking:

$$\sin^2 \theta_{13} = \frac{1}{2} \cos^2 2\theta_{23} = 2d_{23}^2 \approx 0.022 \quad (9)$$

in perfect agreement with measurements (3,4). The relation (9) may follow from “universal” violation of the  $\nu_\mu - \nu_\tau$  symmetry. (In the case of exact symmetry  $\sin^2 \theta_{13} = d_{23} = 0$ .)

(iv) The self-complementarity [24]:

$$\theta_{12} + \theta_{13} = \theta_{23}. \quad (10)$$

It is purely leptonic relation which is also reproduced by QLC.

(v). “Anarchy”:

$$\sin^2 \theta_{13} > 0.025. \quad (11)$$

The inequality is realized at  $1\sigma$  level in the anarchy approach, in which values of mixing angles appear as random numbers [25].

(vi) “Quark-lepton universality”:

$$\theta_{13} \approx \frac{1}{2} \theta_{12} \theta_{23}.$$

This equality is similar to the one in the quark sector:  $V_{ub} = \frac{1}{2} V_{us} V_{cb}$ . It may indicate that the mixing patterns of quarks and leptons are organized in the same way but with certain rescaling. That would testify for (i) a kind of Fritzsch ansatz for mass matrices; (ii) normal mass hierarchy; (iii) relation between masses and mixing; (iv) flavor ordering or alignment in the mass matrix. The latter means that values of elements of the neutrino mass matrix in the flavor basis gradually decrease from  $m_{\tau\tau}$  to  $m_{ee}$ . In fact, due to deviation of the 2-3 mixing from maximal sharp difference of the elements of the dominant  $\mu\tau$ -block and the sub-dominant  $e$ -line is washed out. It seems that neutrino mass matrix is rescaling of the charged fermion mass matrices with weaker hierarchy of the elements. This can originate from power dependence of elements on large expansion parameter  $\lambda \sim 0.7 - 0.8$ . It looks like another complementarity:  $\lambda_\nu = 1 - \lambda_q \approx \theta_C$ . The power dependence could testify for a kind of Froggatt-Nielsen mechanism.

No one of these results explicitly testifies for discrete flavor symmetries, although, as we will see later, consequences of symmetry can be hidden in rather complicated relations between mixing parameters. It seems that relations (i), (iv) and (vi) have no connection to discrete symmetries at all, the relation (ii) may require the symmetry to explain the maximal 2-3 rotation from neutrino sector, finally (iii) may or may not be a consequence of symmetry.

Existence of sterile neutrinos with the eV-scale mass can strongly affect our considerations. Their mixing with active neutrinos is not a small perturbation of the  $3\nu$ -pattern: all theoretical constructions should be reconsidered unless further complications and fine tunings are introduced [26].

### 2.3. Race for the mass hierarchy.

Future developments in the field will be related (apart from better measurements of known parameters) to

- clarification of the situation with sterile neutrinos;
- checks of the claim of the neutrinoless double beta decay observation in the Heidelberg-Moscow experiment; and
- establishing neutrino mass hierarchy.

All this will have serious impact on our understanding of mixings and masses and relevance of the discrete flavor symmetries.

As far as the mass hierarchy is concerned, in the  $2\nu$  approximation changing the hierarchy means flip of the sign:  $\Delta m_{31}^2 \rightarrow -\Delta m_{31}^2$ . The discrete symmetry transformation can be introduced which makes this flip. The case of inverted hierarchy is the special since it implies strong degeneracy of the two mass states  $\nu_1$  and  $\nu_2$ :  $\Delta m/m \sim \Delta m_{21}^2/2\Delta m_{31}^2 \sim 1.6 \cdot 10^{-2}$ . This strong degeneracy can be a consequence of certain discrete symmetry. However it is not accompanied by maximal mixing, which leads to certain tension.

A number of proposals are participating in the “race” for the mass hierarchy. Many are related to influence of matter effects on the 1-3 mixing: In uniform or nearly uniform medium the resonance enhancement of oscillations driven by the 1-3 mixing is in the neutrino channel for NH and in the antineutrino channel in the case of IH. To some extent this can be explored using the beam experiments, e.g., in NOvA [27]. The measured relatively large 1-3 mixing ensures that in supernova the 1-3 level crossing (H resonance) is highly adiabatic. Consequently, conversion effects in the cases of NH and IH are strongly different [28]. That determines also the oscillation effects of SN neutrinos in the matter of the Earth. Observation of the oscillation effect in the antineutrino channel would be the proof of NH [29].

Huge atmospheric neutrino detectors with low ( $\sim 1$  GeV) energy threshold such as PINGU [30] and ORCA have good chance to be the first and the cheapest ones [31]. These experiments have also good potential to measure precisely the 2-3 mixing and mass splitting.

### 3. Discrete symmetries and patterns of lepton mixing.

We assume that neutrinos are Majorana particles, and that the discrete flavor symmetry can be embedded into  $SU(3)$ . The latter requires that determinants of the transformation matrices should be 1.

#### 3.1. Residual symmetries and transformations.

In what follows we will consider the framework of the residual symmetries [4] in details. Let  $S_\nu$  and  $T$  be the transformations of the neutrino,  $M_\nu$ , and charged leptons,  $M_\ell$ , mass matrices which correspond to the residual symmetries  $G_\nu$  and  $G_\ell$ . The invariance means that

$$S_\nu^T M_\nu S_\nu = M_\nu, \quad (12)$$

$$T M_\ell M_\ell^\dagger T^\dagger = M_\ell M_\ell^\dagger. \quad (13)$$

The diagonalization of  $M_\nu$  and  $M_\ell$  gives mixing matrix which reflects symmetries  $G_\nu$  and  $G_\ell$ . In fact,  $S_\nu$  and  $T$  can be used as the generating elements of  $G_\nu$  and  $G_\ell$ .

The invariance conditions (12) and (13) should be satisfied in the basis of neutrino and charged lepton states in which the charged current (CC) interactions are diagonal. Only in this basis all the information about mixing is contained in the mass matrices. This basis is not unique: it is determined upto equal rotations of the left handed components of neutrinos and charged leptons. (The flavor basis is one possibility.) Only in this basis one can obtain information about mixing and masses exploring the mass matrices only <sup>2</sup>.

Most of the symmetry bounds on lepton mixing in the framework (2, 12, 13) have been obtained via *explicit model building*. That is, via construction of the Lagrangian invariant under  $G_f$ , spontaneous breaking of this symmetry, and diagonalization of the generated mass matrices. However, consequences of symmetry for mixing (as well as masses) can be obtained *without explicit model building* immediately from (2, 12, 13), *i.e.*, from the conditions that

<sup>2</sup> Obviously, in the neutrino and charged lepton mass bases all the information on mixing is in the charged current interactions, and the diagonal mass matrices do not contain information about mixing.

- the mass matrices of neutrinos and charged leptons (in the basis where the CC interactions are diagonal) have certain “residual” symmetries, and
- that these symmetries originate from breaking of the original flavor symmetry (or inversely, that they can be embedded into this flavor symmetry).

In other words, the relations between the mixing matrix elements can be obtained from the fact that the neutrino and charged lepton mass matrices are invariant under transformations which generate the flavor symmetry group. Relations obtained in specific models must coincide with relations obtained here, once  $G_\nu$  and  $G_l$  and the covering group  $G_f$  are the same.

### 3.2. Symmetry group condition.

The transformations  $S_\nu$  and  $T$  in the CC diagonal basis encode information about mixing (and masses) that originate from the flavor symmetry. The relations between mixing matrix elements follow from the conditions that  $S_\nu$  and  $T$  belong to the same *finite discrete* group  $G_f$  [13, 14]. This means that the product of  $S_\nu$  and  $T$

$$W \equiv S_\nu \cdot T \quad (14)$$

also belongs to  $G_f$ . Furthermore, since  $G_f$  is a *finite* group, there must exist integers  $n$ ,  $m$  and  $p$  such that  $S_\nu^n = \mathbf{I}$ ,  $T^m = \mathbf{I}$  and

$$W^p = (S_\nu \cdot T)^p = \mathbf{I}. \quad (15)$$

We will call this the *symmetry group condition*. The relations

$$S_\nu^n = T^m = W^p = \mathbf{I} \quad (16)$$

form a presentation of  $G_f$  which corresponds to the von Dyck group  $D(n, m, p)$ . The condition for the group  $D(n, m, p)$  to be finite is

$$\frac{1}{n} + \frac{1}{m} + \frac{1}{p} > 1, \quad (17)$$

and complete list of the finite von Dyck groups which have irreducible representation **3** includes  $\mathbf{D}_n = D(2, 2, n)$ ,  $\mathbf{A}_4 = D(2, 3, 3)$ ,  $\mathbf{S}_4 = D(2, 3, 4)$  and  $\mathbf{A}_5 = D(2, 3, 5)$ .

For definiteness, let us consider the problem in the *flavor basis*, which means that the charged lepton mass matrix is diagonal and we will use the same notation for its symmetry transformations  $T$  as before. We denote by  $S_{\nu U}$  the transformation matrix that leaves invariant the neutrino mass matrix in the flavor basis:

$$S_{\nu U}^T M_{\nu U} S_{\nu U} = M_{\nu U}. \quad (18)$$

(The same result can be obtained in any basis where the CC interactions are diagonal).

It is the condition (15), that connects symmetry transformations of the neutrinos and charged leptons, that leads to relations between the mixing matrix elements. To see this we introduce  $S_m$  - the symmetry transformation of the neutrino mass matrix in the neutrino mass basis:

$$S_m^T m_\nu S_m = m_\nu, \quad m_\nu \equiv \text{diag}\{m_1, m_2, m_3\}. \quad (19)$$

Let us express  $S_{\nu U}$  in terms of  $S_m$ . In the flavor basis the neutrino mass matrix can be represented as

$$M_{\nu U} = U_{PMNS}^* m_\nu U_{PMNS}^\dagger, \quad (20)$$

where  $U_{PMNS}$  is the PMNS lepton mixing matrix. Using Eqs. (18), (19) and (20) it is easy to show that the transformation matrix  $S_{\nu U}$  equals

$$S_{\nu U} = U_{PMNS} S_m U_{PMNS}^\dagger. \quad (21)$$

Since  $S_m$  does not depend on mixing, whole information on the mixing is explicit in (21). Inserting  $S_\nu = S_{\nu U}$  into (15) we obtain

$$[U_{PMNS} S_m U_{PMNS}^\dagger T]^p = \mathbf{I}. \quad (22)$$

This is the main relation which connects the mixing matrix and generating elements of the group in the mass basis. It should give relations between the mixing matrix elements in terms of the group parameters. Let us stress that restrictions on mixing follow from the symmetry group condition which includes both the neutrino and charged lepton transformations.

One can obtain the relations for the matrix elements immediately from the Eq. (22), however simpler way to solve it and to analyze solutions is the following. It can be shown [13] that solution of (22) is equivalent to the solution of equation

$$\text{Tr} [U_{PMNS} S_m U_{PMNS}^\dagger T] = a, \quad (23)$$

where  $a$  is the sum of three  $p$ -th roots of unity,  $\lambda_\alpha$ :

$$a = \sum_{\alpha=1}^3 \lambda_\alpha, \quad (\lambda_\alpha)^p = 1. \quad (24)$$

The  $p$ -th roots of unity are a finite set of complex numbers, and in general (especially for large  $p$ ) several complex values for  $a$  can be found<sup>3</sup>. In fact,  $\lambda_\alpha$  are the eigenvalues of  $W$  they can be parameterized [32] as

$$\lambda_\alpha = e^{i\xi_\alpha}, \quad \xi_\alpha = 2\pi \frac{s_\alpha}{p}, \quad (\alpha = e, \mu, \tau), \quad (25)$$

where integers  $s_\alpha$  satisfy inequality  $s_\alpha < p$ . Since  $\text{Det}[W] = \lambda_e \lambda_\mu \lambda_\tau = 1$ , we have  $s_\tau = -s_e - s_\mu$ , and consequently,  $a = a(p, s_e, s_\mu)$ . For known  $a$  and given  $S_m$  and  $T$ , Eq. (23) provides a complex condition that the entries of  $U_{PMNS}$  must satisfy.

Notice that instead of (14) one can consider some other products of  $S_\nu$  and  $T$  or even two products simultaneously. The latter should not lead though to new relations between the mixing elements but will impose constraints on the group parameters (see below).

The symmetries  $G_\nu$  and  $G_l$  are broken in whole the theory, and therefore one expects corrections to the results obtained in the symmetry limit. Corrections can not be taken into account in this model independent formalism. If however, the residual symmetry of mass matrix exists (or imposed) after the corrections are taken into account, then the relations exist for mixing parameters with corrections.

### 3.3. Generic invariance of mass matrices.

There are certain symmetries of  $M_\nu$  and  $M_\ell$  which are always present. They play important role in our consideration. For arbitrary values of masses the neutrino mass matrix  $m_\nu \equiv \text{diag}\{m_1, m_2, m_3\}$  is invariant under the transformations

$$S_1 = \text{diag}\{1, -1, -1\}, \quad S_2 = \text{diag}\{-1, 1, -1\}, \quad (26)$$

<sup>3</sup> Here we use the same definition of  $a$  as in the paper [14] which has an opposite sign with respect to definition given in [13].

and  $S_3 = S_1 S_2$  with  $\text{Det}[S_i] = 1$ :

$$S_i^T m_\nu S_i = m_\nu. \quad (27)$$

The transformations  $S_1$  and  $S_2$  satisfy conditions  $S_i^2 = \mathbf{I}$  and generate the Klein group  $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ . Then in the flavor basis the mass matrix  $M_{\nu U}$  is invariant under the transformation  $S_{iU} = U_{PMNS} S_i U_{PMNS}^\dagger$ .

The charged lepton mass matrix in the flavor basis,  $m_\ell = \text{diag}(m_e, m_\mu, m_\tau)$ , has a full  $[U(1)]^3$  symmetry. We assume that the residual discrete symmetry is a  $\mathbf{Z}_m$  subgroup of  $[U(1)]^3$ . So, the corresponding transformation which satisfies the condition  $T^m = \mathbf{I}$ , can be written as

$$T \equiv \text{diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\}, \quad (28)$$

where

$$\phi_\alpha \equiv 2\pi \frac{\kappa_\alpha}{m}, \quad \alpha = e, \mu, \tau, \quad (29)$$

with  $\kappa_\alpha \leq m$ . Since the flavor group is a subgroup of  $SU(3)$ , we have  $\text{Det}[T] = 1$ , which gives

$$\phi_e + \phi_\mu + \phi_\tau = 0, \quad (30)$$

or equivalently,  $\kappa_\tau = -\kappa_e + \kappa_\mu$ .

Notice that  $T$  and  $W$  enter the formalism in similar way and can be interchanged with substitution  $m \leftrightarrow p$ ,  $\kappa_\alpha \leftrightarrow s_\alpha$ .

Recall that symmetries generated by  $S_{iU}$  and  $T$  are always present and they can not be broken. The invariance does not depend on specific values of masses. Therefore the generic symmetries lead to relations between mixing parameters without connection to masses. In particular, it is for this reason the generic symmetry based on  $\mathbf{Z}_2 \otimes \mathbf{Z}_2$  was used in models of the TBM mixing. The symmetry transformations are present, but may or may not form the flavor group  $G_f$ , that is, they may or may not satisfy the symmetry group condition (15).

#### 4. Mixing patterns for different $G_\nu$

In what follows we will use  $T$  as in (28), and consider different  $G_\nu$ .

##### 4.1. $G_\nu = \mathbf{Z}_2$

Let us take one  $\mathbf{Z}_2$  as  $G_\nu$ , *i.e.* only one generic (“mass independent”) symmetry of the neutrino mass matrix. This means that only one transformation, *e.g.*,  $S_{1U}$  satisfies the group condition  $(S_{1U} \cdot T)^p = 1$ , whereas two others - don't:  $(S_{iU} \cdot T)^q \neq 1$ ,  $i = 2, 3$  for any  $q$ . They are outside of the group and do not lead to bounds on the mixing matrix. Obviously,  $S_i^2 = \mathbf{I}$  and therefore  $n = 2$  in the presentation (16).

With  $T$  from (28) the condition (23) can be written explicitly as

$$\sum_\alpha e^{i\phi_\alpha} (2|U_{\alpha i}|^2 - 1) = a, \quad (31)$$

or introducing  $a \equiv a_R + ia_I$ , as

$$\sum_\alpha \cos \phi_\alpha (2|U_{\alpha i}|^2 - 1) = a_R, \quad \sum_\alpha \sin \phi_\alpha (2|U_{\alpha i}|^2 - 1) = a_I. \quad (32)$$

These relations

- depend on the *moduli* of matrix elements, which is related to the diagonal form of the generating elements  $S_j$  and  $T$ ;



- relate elements of a *single column*  $j$ , and this index  $j$  is determined by the index  $j$  of the neutrino transformation matrix  $S_j$ ;
- determine the column  $j$  *completely*, since two relations (for real and imaginary parts of  $a$ ) plus unitarity are imposed.

Explicitly the relations read:

$$\begin{aligned}
 |U_{ej}|^2 &= \frac{a_R \cos \frac{\phi_e}{2} + \cos \frac{3\phi_e}{2} - a_I \sin \frac{\phi_e}{2}}{4 \sin \frac{\phi_{e\mu}}{2} \sin \frac{\phi_{\tau e}}{2}}, \\
 |U_{\mu j}|^2 &= \frac{a_R \cos \frac{\phi_\mu}{2} + \cos \frac{3\phi_\mu}{2} - a_I \sin \frac{\phi_\mu}{2}}{4 \sin \frac{\phi_{e\mu}}{2} \sin \frac{\phi_{\mu\tau}}{2}}, \\
 |U_{\tau j}|^2 &= \frac{a_R \cos \frac{\phi_\tau}{2} + \cos \frac{3\phi_\tau}{2} - a_I \sin \frac{\phi_\tau}{2}}{4 \sin \frac{\phi_{\tau e}}{2} \sin \frac{\phi_{\mu\tau}}{2}},
 \end{aligned} \tag{33}$$

where

$$\phi_{\alpha\beta} \equiv \phi_\alpha - \phi_\beta, \quad \alpha, \beta = e, \mu, \tau. \tag{34}$$

Due to the unitarity,  $\sum_\alpha |U_{\alpha j}|^2 = 1$ , there are two independent relation. For specific  $S_j$ , the equations give the absolute values of the  $j$ -th column of  $U_{PMNS}$ , if the values of  $m, p, \kappa_e, \kappa_\mu$  and  $a$  are given. In turn,  $a = a(p, s_e, s_\mu)$ , so that mixing elements are determined by 6 integer parameters (here  $n = 2$ ) which fix eigenvalues of  $T$  and  $W$ :  $(m, \kappa_e, \kappa_\mu, p, s_e, s_\mu)$ . A choice of these parameters is, however, restricted by the fact that  $T$  and  $W$  form a finite group, see Sec. 4.2 and [32]. Substituting the standard parametrization for  $U_{PMNS}$  in Eqs. (33) one obtains the two conditions that the mixing angles and the CP phase must satisfy.

If one of the charged leptons has the  $T$ -charge zero,  $\kappa_\alpha = 0$  (and so two others have opposite signs), the expressions (33) simplify:

$$|U_{\alpha j}|^2 = \eta, \tag{35}$$

$$|U_{\beta j}|^2 = |U_{\gamma j}|^2 = \frac{1-\eta}{2}, \quad \beta, \gamma \neq \alpha, \tag{36}$$

where

$$\eta \equiv \frac{1+a}{4 \sin^2 \left( \frac{\pi k}{m} \right)}. \tag{37}$$

Recall that here  $\alpha$  corresponds to the charged lepton which is invariant under  $T$  transformation,  $j$  corresponds to  $S_j$ . All the mixing parameters are determined by single quantity  $\eta$  which, in turn, is the function of the group parameters  $a(p)$ ,  $m$ ,  $\kappa_e$ , and  $\kappa_\mu$ . In other words  $\eta$  is determined by the group assignment:  $\{p, m, \kappa_e, \kappa_\mu, a\}$ . For small  $p$ , the parameter  $a$  is determined by  $p$  uniquely.

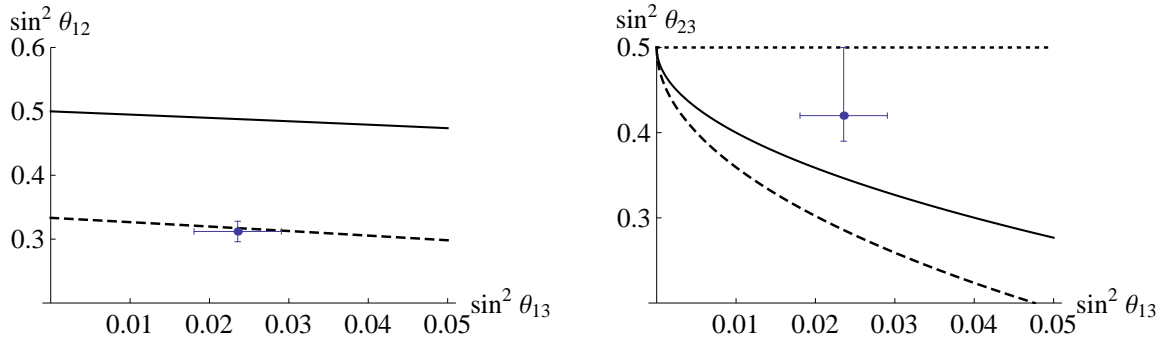
For finite groups we have found from (33) and (35 - 37) the following phenomenologically interesting possibilities

$$\frac{1}{6} \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{4} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \tag{38}$$

The first possibility corresponds to  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{4, 3, 0, 1, 1\}$  and is called the trimaximal mixing-1. It can be used as the first column ( $j = 1$ ) of the mixing matrix. The second one called the trimaximal mixing-2 can be obtained for  $\{p, m, \kappa_e, \kappa_\mu, a\} = (3, 3, 0, 1, 0)$ . It is viable solution for  $j = 2$ . Also the last possibility realized for  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{3, 4, 3, 0, 1\}$  with

certain corrections can be used for  $j = 2$ . Another class of possibilities can be obtained by  $(m \leftrightarrow p)$  permutation which corresponds to the exchange  $T \leftrightarrow W$  in a group presentation [14].

Notice that the first as well as the second columns in (38) coincide with columns of the TBM mixing. This means that TBM is special point in the parameter space of the solutions determined by one of the columns.



**Figure 1.** Symmetry relations between the mixing angles in the case of  $G_f = \mathbf{S}_4$  and  $G_\nu = \mathbf{Z}_2$ . Shown are dependences of  $\sin^2 \theta_{12}$  on  $\sin^2 \theta_{13}$  (left panel), and  $\sin^2 \theta_{23}$  on  $\sin^2 \theta_{13}$  for  $\delta = \pi$  (right panel) for two different symmetry assignment:  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{4, 3, 0, 1, 1\}$  and  $j = 1$  (dashed lines) and  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{3, 4, 0, 3, 0\}$  (solid lines). The dotted line in the right panel corresponds to  $\delta = \pi/2$  for both symmetry assignments. Crosses show the experimental results.

Using the standard parameterization for the PMNS matrix one can find from (35 - 37) relations between the mixing angles and the CP - phase. Eqs. (33) and (35) impose two relations on 4 parameters. Therefore using two known mixing angles one can predicts the CP-phase and the angle which is not yet well known (e.g.,  $\theta_{23}$ ).

As an example, we show in Fig. 1 the relations for two different symmetry assignments. The assignment  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{4, 3, 0, 1, 1\}$  with  $j = 1$  (it corresponds to  $G_f = \mathbf{S}_4$  and trimaximal-1) gives dependences which agree very well with the experimental data. Furthermore, according to the right panel of Fig. 1, one expects  $\delta = 90^\circ - 120^\circ$ .

#### 4.2. Finite subgroups of the infinite von Dyck groups.

Number of possibilities which lead to finite groups is very restricted. It can be extended if we consider big ( $> 5$ ) numbers  $m, n, p$ . This, however, makes group infinite, and therefore in general condition (16) can not be imposed. In order to render the group finite, an additional relation between the generating elements  $S_{iU}$  and  $T$  should be added to Eq. (16). One possibility is the relation of the form

$$X^q = 1, \quad X \equiv S_{jU} T^{-1} S_{jU} T \quad (39)$$

or explicitly

$$(S_{iU} T^{-1} S_{iU} T)^q = 1. \quad (40)$$

Together with (16), that would correspond to certain modular groups:  $PSL(2, \mathbf{Z}_7)$  if  $\{p, m\} = \{7, 3\}$  or  $\{7, 4\}$ ,  $\Delta(96)$  if  $\{p, m\} = \{8, 3\}$  and  $\Delta(384)$  for  $\{p, m\} = \{16, 3\}$  [33].

Using the same procedure as before for the matrix  $W$ , we obtain that Eq. (39) leads to the condition

$$\text{Tr}[X] = x, \quad (41)$$

where

$$x = \sum_{\beta} \lambda'_{\beta}, \quad \lambda'^q_{\beta} = 1. \quad (42)$$

The equation for  $X$  gives relations between the mixing elements of the same column  $j$  (since the same  $S_j$  is involved) which have already been completely fixed by the condition for  $W$ . The relations should be consistent. Therefore the condition Eq. (39) does not add new constraints on mixing angles, but imposes constraints on the group parameters to be consistent with conditions from  $W$ . Indeed,  $\text{Tr}[S_{iU}T^{-1}S_{iU}T]$  (and therefore  $x$ ) can be expressed in terms of  $\text{Tr}[W] = a$  and other group parameters as [14]

$$x = |a|^2 + a\text{Tr}[T^{\dagger}] + a^{\dagger}\text{Tr}[T] + A, \quad (43)$$

where

$$A = 2 + e^{i\phi_{e\mu}} + e^{-i\phi_{e\mu}} + e^{i\phi_{\mu\tau}} + e^{-i\phi_{\mu\tau}} + e^{i\phi_{\tau e}} + e^{-i\phi_{\tau e}}. \quad (44)$$

The equation (43) should be considered as condition for  $x$ ,  $a$  and phases  $\phi_{\alpha} = \phi_{\alpha}(m, k_{\alpha})$ . So, it is essentially a condition for the matrix  $T$  that fixes the values of  $k_e$  and  $k_{\mu}$  and makes the group finite. Instead of  $k_{\alpha}$  one can impose bounds on  $a$  or  $s_{\alpha}$ . (See discussion in [32] where the bounds on these parameters have been obtained by scanning of the groups.) Imposing the second condition gives systematic way to get bounds on the group parameters

Let us consider two examples [14].

1). The group  $PSL(2, \mathbf{Z}_7)$  is a subgroup of the infinite von Dyck group with the presentation (see Eq. (16))

$$S_{iU}^2 = T^7 = (S_{iU}T)^3 = \mathbf{I} \quad (45)$$

(i.e.,  $m = 7$ ) and with an additional condition

$$X^4 = \mathbf{I} \quad (46)$$

(Eq. (39) with  $q = 4$ ). For symmetry assignment  $\{p, m, \kappa_e, \kappa_{\mu}, a\} = \{3, 7, 5, 3, 0\}$ , we obtain  $x = 0$  and values of the mixing elements

$$|U_{\mu j}|^2 = \frac{1}{4[1 + \sin \frac{\pi}{14}]}, \quad |U_{\tau j}|^2 = \frac{1}{4[1 + \cos \frac{\pi}{7}]}, \quad (47)$$

and  $|U_{ej}|^2 = 1 - |U_{\mu j}|^2 - |U_{\tau j}|^2$ . For  $j = 2$  this gives good description of the experimental data. The corresponding relations for mixing angles are shown in Fig. 2. According to the right panel within  $1\sigma$  allowed region of mixing angles the CP phase equals  $\delta = (80^\circ - 90^\circ)$ .

2). The group  $\Delta(384)$  is a subgroup of the infinite von Dyck group with presentation

$$S_{iU}^2 = T^{16} = (S_{iU}T)^3 = \mathbf{I}, \quad (48)$$

and additional relation

$$X^3 = \mathbf{I}. \quad (49)$$

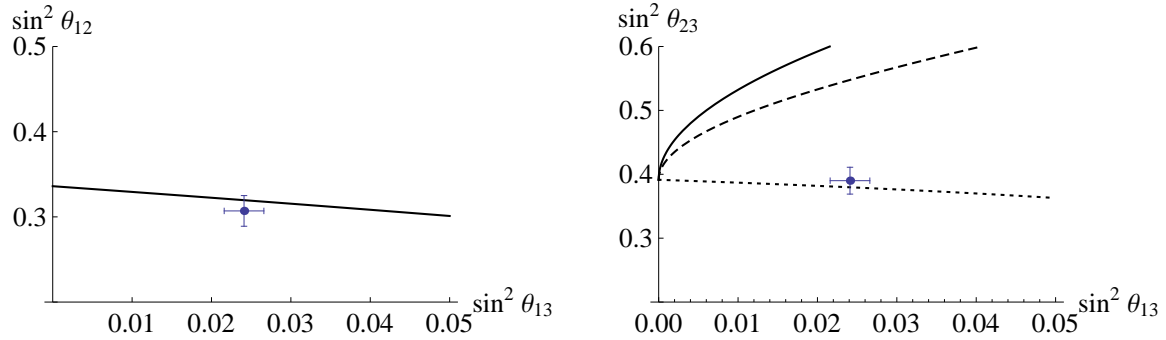
Again, we have  $x = 0$ . Viable mixing pattern can be obtained in the case of  $W \leftrightarrow T$  permuted version, when  $W$  is the symmetry of the charged leptons and  $T = S_{iU}W$  gives the symmetry group condition. Now  $p' = m$ ,  $m' = p$ ,  $a' = \text{Tr}[T]$  and  $\kappa_{\alpha} \rightarrow \xi_{\alpha}$ . For the set of the group parameters

$$\{p', m', s_e, s_{\mu}, a'\} = \left\{16, 3, 1, 1, \frac{-1+i}{\sqrt{2}}\right\} \quad (50)$$

we obtain

$$\{|U_{ej}|^2, |U_{\mu j}|^2, |U_{\tau j}|^2\} = \left\{\frac{4 + \sqrt{2} + \sqrt{6}}{12}, \frac{4 + \sqrt{2} - \sqrt{6}}{12}, \frac{2 - \sqrt{2}}{6}\right\}. \quad (51)$$

If  $j = 3$ , this model is able to fit all mixing angles for  $\delta \sim 50^\circ - 60^\circ$ .



**Figure 2.** Symmetry relations between mixing angles in the case of  $G_f = PSL(2, \mathbf{Z}_7)$  and  $G_\nu = \mathbf{Z}_2$ . *Left panel:* dependence of  $\sin^2 \theta_{12}$  on  $\sin^2 \theta_{13}$ , *right panel:* dependence of  $\sin^2 \theta_{23}$  on  $\sin^2 \theta_{13}$  for  $\delta = 0$  (solid line),  $\delta = \pi/4$  (dashed line) and  $\delta = \pi/2$  (dotted lines). The symmetry assignment is  $\{p, m, \kappa_e, \kappa_\mu, a\} = \{3, 7, 5, 3, 0\}$  and  $j = 2$ . Crosses show the experimental results.

#### 4.3. $G_\nu = \mathbf{Z}_2 \otimes \mathbf{Z}_2$ .

Let us consider the complete generic symmetry given by the Klein group  $G_\nu = \mathbf{Z}_2 \otimes \mathbf{Z}_2$ . A presentation of the group is given (for definiteness we take  $S_1, S_2$  as generating elements) by

$$S_{1U}^2 = S_{2U}^2 = T^m = W_{1U}^{p_1} = W_{2U}^{p_2} = \mathbf{I}, \quad [S_{1U}, S_{2U}] = 0. \quad (52)$$

They impose two sets of conditions on  $U_{PMNS}$  which can be written as

$$\text{Tr}[W_{jU}] = \sum_{\alpha} e^{i\phi_{\alpha}} (2|U_{\alpha j}|^2 - 1) = a_j, \quad j = 1, 2, \quad (53)$$

where  $a_j \equiv \sum_k \lambda_k^{(j)}$  are sums of three  $p_j$ -th roots of unity. Now there are four relations between the mixing matrix elements, corresponding to two columns of the mixing matrix, which determine  $U_{PMNS}$  completely [14]. An example of this case is given by two columns in Eq. (38) with  $\{p_1, p_2, m\} = \{4, 3, 3\}$ . The only mixing matrix compatible with first two columns in (38) is the TBM. In a sense the TBM is special being related to maximal generic neutrino symmetry group.

Complete scan of the discrete finite groups with order less than 1536 and  $G_\nu = \mathbf{Z}_2 \otimes \mathbf{Z}_2$  has been performed in [34] and for these groups the mixing patterns has been computed<sup>4</sup>.

The formalism allows to explain an observation [33] that in some cases, the mixing matrix derived from Eqs. (53) has the property that the absolute values of entries of two columns were equal up to a permutation:

$$|U_{\alpha i}|^2 = |U_{f(\alpha)j}|^2, \quad i \neq j. \quad (54)$$

Here  $f(\alpha)$  is the permutation operation of the flavor indices. An example, which is realized in the case of  $PSL(2, \mathbf{Z}_7)$  group is

$$|U_{\alpha i}|^2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ c_3 & c_1 & c_2 \\ c_2 & c_3 & c_1 \end{pmatrix} \quad (55)$$

<sup>4</sup> The approach in [34] is, however, different from the one presented here: the starting point in [34] is the explicit form of the generators of the residual symmetries for leptons and neutrinos in the basis of diagonal CC interaction. The generators satisfy presentation of the full selected flavor group. The generators are diagonalized by rotations  $\Omega_\ell$  and  $\Omega_\nu$ . Then, as can be shown, the mixing matrix is given by  $\Omega_\ell^\dagger \Omega_\nu$ . See also discussion in [32].

with

$$c_2 = \frac{1}{4(1 + \sin \frac{\pi}{14})}, \quad c_3 = \frac{1}{4(1 + \cos \frac{\pi}{7})}, \quad c_1 = 1 - c_2 - c_3. \quad (56)$$

#### 4.4. Non-generic neutrino symmetry.

We can introduce a non-generic neutrino symmetry  $G_\nu$  which includes transformation,  $Q_m$ , under which neutrino mass matrix in the mass basis is invariant only for some specific values of masses. Inversely, introduction of such a symmetry in the flavor symmetry imposes conditions on neutrino masses. The symmetry group condition for  $Q_m$  reads  $(Q_{mU} \cdot T)^r = 1$ , where  $r$  is integer and  $Q_{mU} = U_{PMNS} Q_m U_{PMNS}^\dagger$  is the symmetry transformation in the flavor basis. Also some of generic symmetries  $\mathbf{Z}_2$  could be a part of  $G_f$ . In the latter case one should impose also the relation  $(Q_{mU} \cdot S_U)^q = (Q_m \cdot S)^q = 1$ . The mixing matrix disappears in the last equality and therefore it should be considered as a consistency condition.

### 5. Conclusion

1. The appealing discrete flavor symmetry framework (inspired by TBM) is based on idea that the lepton mixing originates from different ways of the discrete flavor symmetry breaking in the neutrino and charged lepton Yukawa sectors. These different ways lead to different residual symmetries of the neutrino and charged lepton mass matrices which ensure special forms of these matrices, and consequently, special form of the mixing matrix. Here the symmetry transformation are generic transformations valid for arbitrary masses.

2. Recent measurements of the neutrino oscillation parameters show substantial deviations from the TBM mixing. This may further indicate that TBM is accidental and whole discrete flavor symmetry approach is phenomenologically irrelevant. Experimental value of the 1-3 mixing can be connected with other observables in various ways which have different implications to theory. It is not clear however if these relations imply discrete symmetries: some – certainly not, although symmetry effects can be rather hidden.

Further developments in the field can be related to establishing the neutrino mass hierarchy. “Race” for the mass hierarchy and CP has started: studies of the atmospheric neutrinos with multi-megaton mass detectors having low energy thresholds can provide fast, inexpensive and reliable answer.

3. Discrete symmetries still may play important role in formation of the lepton flavor structures. TBM can be treated as the lowest order structure which requires corrections. Discrete symmetries can be consistent with the non-zero 1-3 mixing and deviation of the 2-3 mixing from maximal.

4. Consequences of symmetries for mixing, and in general, for masses in this framework can be obtained immediately from the *symmetry group condition(s)*, without explicit model building. For this the knowledge of (assumption about) the residual symmetries and the covering group is enough. The symmetry group condition includes the PMNS matrix and the generating elements of the residual symmetries in the mass basis.

From the point of view of specific models, here it is assumed that the relevant model-building has already been done and the mass matrices of neutrinos and charged leptons with certain symmetries obtained.

5. The symmetry group conditions have been applied to different residual neutrino symmetries and full flavor groups. (i) The generic  $G_\nu = \mathbf{Z}_2$  with finite von Dyck group as well as finite subgroup of the infinite von Dyck group as  $G_f$  lead to two relations between the mixing parameters. This fixes one of the columns of mixing matrix. (ii) The Klein group  $\mathbf{Z}_2 \times \mathbf{Z}_2$  as  $G_\nu$  fixes the mixing matrix completely. (iii) The formalism can be generalized to include non-generic neutrino symmetries  $G_\nu$  which lead to relations between mixings and neutrino masses.

6. The relations between mixing parameters obtained here include the CP-phase and therefore the phase can be predicted using measured values of mixing angles. In the examples we discussed  $\delta$  can be around  $60^\circ$  or  $90^\circ$ . So, future measurements of the phase can provide important test of the framework.

7. The formalism presented here can be used in various ways. In specific model, once the residual symmetries are identified the consequences on mixing and masses can be obtained immediately. The formalism allows to analyze systematically possible mixing patterns that can be extracted from the finite groups or subgroups of the von Dyck type. Inversely, for a given pattern of mixing the formalism allows to perform the “symmetry building” - identify the residual and covering symmetry groups.

The formalism allows also to explain various features of mixing matrix which follow from symmetries. In particular, in the case of Klein group in the neutrino sector it can explain appearance of mixing patterns in which two or three columns of  $U_{PMNS}$  have equal but permuted elements.

8. Further generalizations of the formalism are possible. On the other hand, discrete symmetries can be realized in some other way in frameworks which differs from (2). Mixing (or its zero order structure) can originate from different nature of the mass terms of the charged leptons (Dirac) and neutrinos (Majorana), or from neutrino mixing with new degrees of freedom (e.g., singlets of SM).

### 5.1. Acknowledgments

Large part of this talk is based on papers written with in collaboration with D. Hernandez. I am grateful to C. Hagedorn, A. Pilaftsis and R. Mohapatra for useful discussions during the Symposium.

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