Shell-model derivation of the shears mechanism

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Abstract. The geometry of the shears mechanism in nuclei is obtained by taking the limit of large angular momentum of shell-model matrix elements.

1. Introduction

In several lead isotopes, as well as in other regions of the nuclear chart near doubly-magic closures, regular sequences of γ rays are observed resembling those typical of a rotational band [1, 2, 3, 4]. These transitions are of magnetic character, in contrast to the usual quadrupole nature of nuclear rotors [5]. An explanation of this observation invokes the so-called 'shears' mechanism [6]. In a weakly deformed nucleus there exist low-energy configurations in which neutrons and protons combine into stretched structures ('blades') with angular momentum generated by the re-coupling of these blades, resembling the closing of a pair of shears. The breaking of the rotational symmetry in this case originates in an anisotropic distribution of nucleonic current loops (rather than electric charge), and thus the phenomenon is also referred to as magnetic rotation [7].

While an interpretation in terms of the shears mechanism provides an appealing, intuitive picture of these nuclear states, the question remains whether this geometry is borne out by microscopic calculations. A numerical shell-model calculation by Frauendorf *et al.* [8] confirmed the shears picture in the lead region. An analytic derivation of the geometry of the shears mechanism from the shell model is the purpose of the present contribution which presents results complementary to those reported elsewhere [9].

2. General shell-model expressions for the shears matrix element

Consider m nucleons of one type (say neutrons) in particle-like orbits $j_{1\nu}, j_{2\nu}, \ldots$, and m' nucleons of the other type (protons) in hole-like orbits $j_{1\pi}^{-1}, j_{2\pi}^{-1}, \ldots$, where $j_{k\rho}$ ($\rho = \nu, \pi$) is used as an abbreviation for the set of quantum numbers $n_{k\rho}$, $\ell_{k\rho}$ and $j_{k\rho}$ of singe-particle levels in a central potential. The m-particle (mp) and m'-hole (m'h) states are represented as $|N\rangle \equiv |j_{1\nu}j_{2\nu}(J_{12\nu})\ldots J_\nu\rangle$ and $|P^{-1}\rangle \equiv |j_{1\pi}^{-1}j_{2\pi}^{-1}(J_{12\pi})\ldots J_\pi\rangle$, where $J_{12\ldots\rho}$ are intermediate angular momenta occurring in some coupling scheme, and J_ν and J_π are the neutron and proton total angular momenta, respectively. The single-particle orbits $j_{k\rho}$ ($k = 1, 2, \ldots$) may or may not be identical and, in the former case, the states $|N\rangle$ and $|P^{-1}\rangle$ are assumed to be Pauli allowed. A shears band consists of the states $|NP^{-1}; J\rangle$ where J results from the coupling of J_ν and J_π.

A general shell-model hamiltonian has a neutron and a proton piece, and a neutron–proton interaction, and therefore can be written as

$$\hat{H} = \hat{H}_\nu + \hat{H}_\pi + \hat{V}_{\nu\pi}.$$

The quantity $\langle NP^{-1};J|\hat{H}|NP^{-1};J\rangle$ will be referred to as the (diagonal mp–m'h) shears matrix element. Due to the scalar character of the hamiltonian, the matrix elements of $\hat{H}_\nu$ and $\hat{H}_\pi$ depend only on the quantum numbers appearing in $|N\rangle$ and $|P\rangle$, respectively; $\hat{H}_\nu$ and $\hat{H}_\pi$ give constant energy contributions to all members of the shears band, irrespective of the total angular momentum J, and are therefore of no interest in the following. Any J dependence originates from the neutron–proton interaction $\hat{V}_{\nu\pi}$, specified by its matrix elements $\langle j_{k\nu}j_{l\pi};R|\hat{V}_{\nu\pi}|j_{k'\nu}j_{l'\pi};R\rangle$. A multipole expansion of $\hat{V}_{\nu\pi}$ leads to the following expression for the shears matrix element:

$$\begin{aligned}
&\langle NP^{-1};J|\hat{V}_{\nu\pi}|NP^{-1};J\rangle \\
&\quad= \sum_{klR\lambda}(-)^{j_{k\nu}+j_{l\pi}+J_\nu+J_\pi+J+R}(2R+1)V^R_{j_{k\nu}j_{l\pi}}\left\{\begin{array}{ccc} j_{k\nu} & j_{k\nu} & \lambda \\ j_{l\pi} & j_{l\pi} & R\end{array}\right\}\left\{\begin{array}{ccc} J_\nu & J_\pi & J \\ J_\pi & J_\nu & \lambda\end{array}\right\} \\
&\qquad\times\langle N||(a^\dagger_{j_{k\nu}}\tilde{a}_{j_{k\nu}})^{(\lambda)}||N\rangle\langle P^{-1}||(a^\dagger_{j_{l\pi}}\tilde{a}_{j_{l\pi}})^{(\lambda)}||P^{-1}\rangle, \qquad (1)
\end{aligned}$$

where $V^R_{j_{k\nu}j_{l\pi}} \equiv \langle j_{k\nu}j_{l\pi};R|\hat{V}_{\nu\pi}|j_{k\nu}j_{l\pi};R\rangle$. The operator $a^\dagger_{j_{k\rho}m_{k\rho}}$ creates a neutron ($\rho=\nu$) or a proton ($\rho=\pi$) in orbit $j_{k\rho}$ with projection $m_{k\rho}$ on the z axis; the corresponding annihilation operator $a_{j_{k\rho}m_{k\rho}}$ is modified to $\tilde{a}_{j_{k\rho}m_{k\rho}} \equiv (-)^{j_{k\rho}+m_{k\rho}}a_{j_{k\rho}-m_{k\rho}}$ to make it behave as a proper tensor under rotations in three dimensions.

The expression (1) for the shears matrix element between mp–m'h states can be further reduced with use of standard techniques of angular momentum. This reduction is carried out here for $m=m'$ with $m=1$ and $m=2$.

2.1. The 1p–1h shears matrix element

For $m=1$, the members of the shears band are written as $|NP^{-1};J\rangle = |j_\nu j_\pi^{-1};J\rangle$ and the expression (1) reduces to one for a particle–hole matrix element,

$$\langle NP^{-1};J|\hat{V}_{\nu\pi}|NP^{-1};J\rangle = -\sum_R(2R+1)V^R_{j_\nu j_\pi}\left\{\begin{array}{ccc} j_\nu & j_\pi & J \\ j_\nu & j_\pi & R\end{array}\right\}.$$

This is nothing but the Pandya relation which expresses a particle–hole matrix element as a sum of particle–particle matrix elements [10]. The object in curly brackets is a $6j$ symbol [11], a quantity which is scalar under rotations, depending on six angular momenta. It is in fact the simplest non-trivial *scalar* $3nj$ symbol, the case $n=1$ being the trivial triangular delta $\{j_1j_2j_3\}$ which is equal to one if j_1, j_2 and j_3 form a triad and zero otherwise [12]. For the purpose of comparing with the case $m=2$ given below, it is of some interest to convert the $6j$ symbol to the standard notation for $3nj$ symbols (see eq. (17.11a) of Yutsis *et al.* [12]), leading to

$$\langle NP^{-1};J|\hat{V}_{\nu\pi}|NP^{-1};J\rangle = -\sum_R(2R+1)V^R_{j_\nu j_\pi}\left\{\begin{array}{ccc} j_\nu & & j_\pi \\ & J & & R \\ j_\nu & & j_\pi\end{array}\right\}. \qquad (2)$$

2.2. The 2p–2h shears matrix element

For $m=2$, the members of the shears band arise from the two-neutron-particle state $|N\rangle \equiv |j_{1\nu}j_{2\nu};J_\nu\rangle$ and the two-proton-hole state $|P^{-1}\rangle \equiv |j^{-1}_{1\pi}j^{-1}_{2\pi};J_\pi\rangle$ which are coupled to angular momentum J. The two-particle reduced matrix element in eq. (1) equals

$$\langle j_1j_2;J||(a^\dagger_{j_k}\tilde{a}_{j_l})^{(\lambda)}||j_1j_2;J\rangle = (-)^{j_1+j_2+J+\lambda}\sqrt{2\lambda+1}(2J+1)\hat{P}_{12}\left\{\begin{array}{ccc} j_1 & j_1 & \lambda \\ J & J & j_2\end{array}\right\}\delta_{k1}\delta_{l1},$$

where $\hat{P}_{12}$ is a symmetrizer defined as

$$\hat{P}_{12} f(j_1, j_2) = f(j_1, j_2) + f(j_2, j_1),$$

for any function f. The corresponding result for holes is obtained from eq. (14.39) of Talmi [11],

$$\langle j_1^{-1} j_2^{-1}; J || (a_{j_k}^\dagger \tilde{a}_{j_l})^{(\lambda)} || j_1^{-1} j_2^{-1}; J \rangle = (-)^{j_1+j_2+J+1} \sqrt{2\lambda+1} (2J+1) \hat{P}_{12} \left\{ \begin{array}{ccc} j_1 & j_1 & \lambda \\ J & J & j_2 \end{array} \right\} \delta_{k1} \delta_{l1}.$$

Insertion of these expressions for the two-particle and two-hole reduced matrix elements in eq. (1) leads to the shears matrix element

$$\frac{\langle NP^{-1}; J | \hat{V}_{\nu\pi} | NP^{-1}; J \rangle}{(2J_\nu + 1)(2J_\pi + 1)} = -\hat{P}_{12}^{\nu\pi} \sum_R (2R+1) V^R_{j_{1\nu} j_{1\pi}} \left\{ \begin{array}{ccccccc} j_{1\nu} & & J_\nu & & J_\pi & & j_{1\pi} \\ & j_{2\nu} & & J & & j_{2\pi} & & R \\ j_{1\nu} & & J_\nu & & J_\pi & & j_{1\pi} \end{array} \right\}, \qquad (3)$$

where $\hat{P}_{12}^{\nu\pi} \equiv \hat{P}_{12}^{\nu} \hat{P}_{12}^{\pi}$ now generates four terms according to

$$\begin{aligned} & \hat{P}_{12}^{\nu\pi} f(j_{1\nu}, j_{2\nu}, j_{1\pi}, j_{2\pi}) \\ = \ & f(j_{1\nu}, j_{2\nu}, j_{1\pi}, j_{2\pi}) + f(j_{1\nu}, j_{2\nu}, j_{2\pi}, j_{1\pi}) + f(j_{2\nu}, j_{1\nu}, j_{1\pi}, j_{2\pi}) + f(j_{2\nu}, j_{1\nu}, j_{2\pi}, j_{1\pi}). \end{aligned}$$

The object in curly brackets in eq. (3) is a $12j$ symbol of the *first* kind, a quantity which is scalar under rotations, depending on twelve angular momenta. The result (3) is obtained by expressing the shears matrix element as a sum over four $6j$ symbols, which can be related to a $12j$ symbol (see eq. (19.1) of Yutsis *et al.* [12]).

If the neutrons and protons are all particle-like (or all hole-like), a result similar to eq. (3) is obtained,

$$\frac{\langle NP; J | \hat{V}_{\nu\pi} | NP; J \rangle}{(2J_\nu + 1)(2J_\pi + 1)} = \hat{P}_{12}^{\nu\pi} \sum_R (2R+1) V^R_{j_{1\nu} j_{1\pi}} \left[\begin{array}{ccccccc} j_{1\nu} & & J_\nu & & J_\pi & & j_{1\pi} \\ & j_{2\nu} & & J & & j_{2\pi} & & R \\ j_{1\nu} & & J_\nu & & J_\pi & & j_{1\pi} \end{array} \right], \qquad (4)$$

where the object in square brackets is a $12j$ symbol of the *second* kind (see eq. (19.2) of Yutsis *et al.* [12]).

Note the striking similarity between the expressions (2) and (3) for the 1p–1h and 2p–2h shears matrix elements, raising the hope for a treatment of the general case with $m > 2$, a problem which is currently under study.

It is now a simple matter to introduce in the sums (2) or (3) values for the neutron–proton interaction matrix elements and to derive the J dependence of the shears matrix element. The geometric significance of these expressions can be understood by the taking the limit of large angular momenta. If these are large in comparison with $\hbar$, this can be considered as the classical limit of the quantum-mechanical expression for the shears matrix element.

3. Classical expressions for the shears matrix element

Classical limits of $3j$ and $6j$ symbols are known since the seminal study of Wigner [13], subsequently refined by Ponzano and Regge [14] whose work was put on a mathematically solid footing by Schulten and Gordon [15]. The classical limit of a $3j$ symbol is associated with the area of a triangle, while that of a $6j$ symbol involves the volume of a tetrahedron, with the lengths of the sides determined by the angular momenta. However, for $9j$ (let alone $12j$) symbols only partial results are known and therefore the classical limit of the expression (3) is difficult to obtain for an arbitrary neutron–proton interaction.

Instead of dealing with a general interaction, consider the modified surface delta interaction (MSDI),

$$\hat{V}^{\rm MSDI} = \sum_{k<l} \hat{V}^{\rm MSDI}(k,l), \quad \hat{V}^{\rm MSDI}(k,l) = -4\pi a'_T \delta(\bar{r}_k - \bar{r}_l)\delta(r_k - R_0) + b'\bar{\tau}_k \cdot \bar{\tau}_l + c',$$

which is known to be a reasonable approximation to the realistic nucleon–nucleon force in terms of the isoscalar and isovector strengths a'_0 and a'_1, and the strengths b' and c' of a charge-exchange and a constant interaction. The neutron–proton matrix element of the MSDI is [16]

$$V^R_{j_\nu j_\pi} = -\frac{(2j_\nu+1)(2j_\pi+1)}{2}\left[a_{01}\begin{pmatrix} j_\nu & j_\pi & R \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}^2 + a_0 \begin{pmatrix} j_\nu & j_\pi & R \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}^2\right] - b + c, \quad (5)$$

with

$$a_{01} = \frac{a_0 + a_1}{2} - (-)^{\ell_\nu + \ell_\pi + R}\frac{a_0 - a_1}{2},$$

and

$$a_T = a'_T C(R_0), \quad b = b'C(R_0), \quad c = c'C(R_0), \qquad C(R_0) = R^4_{n_\nu \ell_\nu}(R_0)R_0^2 = R^4_{n_\pi \ell_\pi}(R_0)R_0^2.$$

A geometric insight is obtained by introducing the neutron–proton matrix elements of the MSDI into the expressions for the shears matrix elements. The procedure is illustrated with the 1p–1h shears matrix element (2) and results are summarized for the 2p–2h shears matrix element (3).

3.1. *The 1p–1h shears matrix element*

When the neutron–proton matrix elements of the MSDI (5) are introduced in the shears matrix element (2), one encounters sums of the type

$$\sigma^{(\lambda)}_n \equiv \sum_R (-)^{\lambda R}(2R+1)\begin{pmatrix} j_\nu & j_\pi & R \\ \frac{1}{2} & n-\frac{1}{2} & -n \end{pmatrix}^2 \begin{Bmatrix} j_\nu & j_\pi & J \\ j_\nu & j_\pi & R \end{Bmatrix},$$

for $(\lambda, n) = (0,0)$, (0,1) and (1,0). These reduce to simple expressions in the classical limit. For example, for $\lambda = 0$, the sum can be exactly rewritten as

$$\sigma^{(0)}_n = \begin{pmatrix} j_\nu & j_\pi & J \\ \frac{1}{2} & -n+\frac{1}{2} & n-1 \end{pmatrix}^2.$$

The classical limit of the $3j$ symbol [13] leads to the approximations

$$\begin{aligned} \sigma^{(0)}_0 &\approx \frac{2[1+(-)^{j_\nu+j_\pi+J}]}{\pi(2j_\nu+1)(2j_\pi+1)}\frac{1}{\sin\theta_{\nu\pi}}, \\ \sigma^{(0)}_1 &\approx \frac{2}{\pi(2j_\nu+1)(2j_\pi+1)}\left(\frac{1}{\sin\theta_{\nu\pi}} + (-)^{j_\nu+j_\pi+J}\frac{1}{\tan\theta_{\nu\pi}}\right), \end{aligned}$$

where $\theta_{\nu\pi}$ is the shears angle, that is, the angle between the neutron and proton angular momenta,

$$\theta_{\nu\pi} = \arccos\frac{J(J+1) - j_\nu(j_\nu+1) - j_\pi(j_\pi+1)}{2\sqrt{j_\nu(j_\nu+1)j_\pi(j_\pi+1)}}.$$

For $(\lambda, n) = (1,0)$, a similar approximation is obtained from the relation $\sigma^{(1)}_0 = -(-)^J\sigma^{(0)}_1$.

With use of the preceding approximations, the expression for the 1p–1h shears matrix element (2) of a MSDI becomes

$$\langle NP^{-1};J|\hat{V}^{\mathrm{MSDI}}_{\nu\pi}|NP^{-1};J\rangle \approx (b-c)+\frac{s_1}{2\pi\sin\theta_{\nu\pi}}+\frac{t_1}{2\pi\tan\theta_{\nu\pi}}, \tag{6}$$

with

$$\begin{aligned} s_1 &= [1+(-)^{j_\nu+j_\pi+J}](a_0+a_1)+2a_0+(-)^{\ell_\nu+\ell_\pi+J}(a_0-a_1), \\ t_1 &= 2(-)^{j_\nu+j_\pi+J}a_0+(-)^{\ell_\nu+\ell_\pi+j_\nu+j_\pi}(a_0-a_1). \end{aligned}$$

3.2. The 2p–2h shears matrix element

A classical limit can also be obtained for the 2p–2h shears matrix element (3). The derivation is more complicated [9] and leads to the result

$$\langle NP^{-1};J|\hat{V}^{\mathrm{MSDI}}_{\nu\pi}|NP^{-1};J\rangle \approx 4(b-c)+\frac{s_2}{2\pi\sin\theta_{\nu\pi}}+\frac{t_2}{2\pi\tan\theta_{\nu\pi}}, \tag{7}$$

where $\theta_{\nu\pi}$ is now the angle between $\bar{J}_\nu$ and $\bar{J}_\pi$, and with

$$s_2 = 4(3a_0+a_1), \quad t_2 = 4\varphi(a_0-a_1), \qquad \varphi = \tfrac{1}{4}\hat{P}^{\nu\pi}_{12}(-)^{\ell_{1\nu}+j_{1\nu}+\ell_{1\pi}+j_{1\pi}}.$$

The classical limit of the 2p–2h shears matrix element (7) is remarkably similar to that of the 1p–1h shears matrix element (6). In both expressions the quantity $(b-c)$ appears and the constant as well as the charge-exchange interaction contribute equally to all members of the shears band. The coefficients s_m and t_m are expressed in terms of the isoscalar and isovector strengths a_0 and a_1 of the MSDI but the dependence for $m=1$ is different from that for $m=2$. It would be of interest to find the generalization to $m>2$.

4. Comparison of quantum-mechanical and classical expressions

In Fig. 1 the exact 1p–1h and 2p–2h shears matrix elements of the SDI (*i.e.*, MSDI with $b=c=0$) are compared with their classical approximations. To illustrate the quality of the classical approximation, rather large values for the neutron and proton angular momenta are chosen, $j_\rho=\frac{41}{2}$ for the 1p–1h and $J_\rho=20$ for the 2p–2h case. Even for such unrealistically high single-particle angular momenta, the 1p–1h matrix element does not display a shears behaviour since the minimum energy is *not* attained for a configuration with orthogonal neutron and proton angular momenta. The 2p–2h matrix element, on the other hand, does behave (qualitatively at least) as expected for the energies of the members of a shears band. In fact, for $t_2=0$ (or $a_0=a_1$) a minimum energy is reached for a shears angle $\theta_{\nu\pi}=90^{\mathrm{o}}$ between the neutron and proton angular momentum vectors. Furthermore, it can be shown [9] that the shears picture remains valid for smaller values of the angular momenta, as long as these are made from nearly stretched configurations with $J_\rho \approx j_{1\rho}+j_{2\rho}$.

5. Conclusion

In this contribution 1p–1h and 2p–2h shears matrix elements of a modified surface delta interaction (MSDI) are studied in the limit of large angular momentum. The geometric picture underlying the shears mechanism is confirmed in the latter case and the shears angle at which the 2p–2h matrix element reaches a minimum energy can be related to the isoscalar and isovector strengths of the MSDI.

Future studies are called for along the same lines. They include the extension to more complicated configurations (*i.e.*, mp–m'h and mixed configurations) and to other components of the interaction (*e.g.*, the tensor force), and the study of electromagnetic properties.

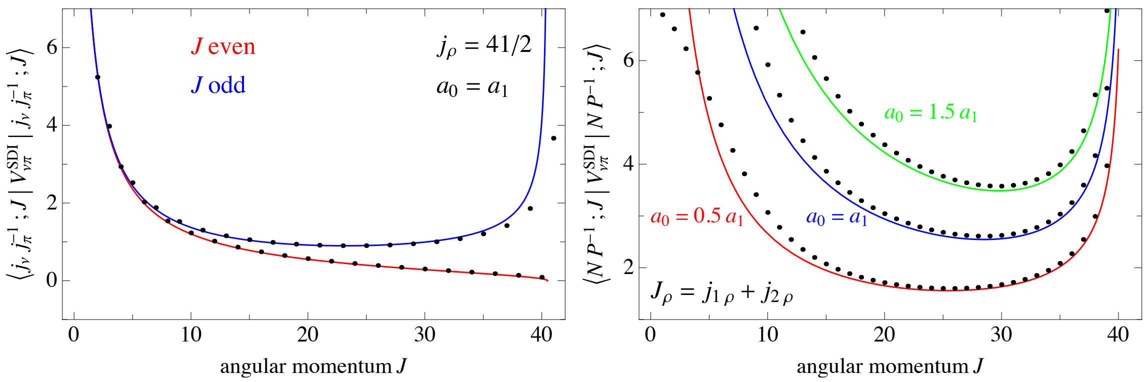

Figure 1. Left: The exact expression (2) for the 1p–1h shears matrix element of the SDI (dots) compared with its classical approximation (6) (lines). The single-particle angular momenta are $j_\rho = \frac{41}{2}$. The matrix element is in units $a_0 = a_1$. Right: The exact expression (3) for the 2p–2h shears matrix element of the SDI (dots) compared with its classical approximation (7) (lines). The single-particle angular momenta are $j_{1\rho} = \frac{19}{2}$ and $j_{2\rho} = \frac{21}{2}$, and the total neutron and proton angular momenta $J_\rho = 20$. Results are shown for three choices of the ratio a_0/a_1 and the matrix element is in units a_1.

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