

De Donder-Weyl Hamiltonian formulation and precanonical quantization of vielbein gravity

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Abstract. The De Donder-Weyl (DW) covariant Hamiltonian formulation of the Palatini first-order Lagrangian of vielbein (tetrad) gravity and its precanonical quantization are presented. No splitting into space and time is required in this formulation. Our recent generalization of Dirac brackets is used to treat the second class primary constraints appearing in the DW Hamiltonian formulation and to find the fundamental brackets. Quantization of the latter yields the representation of vielbeins as differential operators with respect to the spin connection coefficients and the Dirac-like precanonical Schrödinger equation on the space of spin connection coefficients and space time variables. The transition amplitudes on this space describe the quantum geometry of space-time. We also discuss the Hilbert space of the theory, the invariant measure on the spin connection coefficients, and point to the possible quantum singularity avoidance built in in the natural choice of the boundary conditions of the wave functions on the space of spin connection coefficients.

1. Introduction

There are several dominating approaches in the literature which aim at quantization of gravity or, more generally, a synthesis of general relativity and quantum theory. They can be conditionally classified according to their main strategies:

- 1) application of standard QFT techniques to the Lagrangians of general relativity theory or its alternatives (canonical QG, path integral, asymptotic safety),
- 2) adaptation of the classical GR to the technical requirements or limitations of QFT (LQG, shape dynamics),
- 3) postulating the fundamental microscopic dynamics so that classical GR would appear as an effective or emergent low energy theory (string theory, GFT, induced gravity, quantum/non-commutative space-times, causal networks).

However, considerably less efforts have been devoted to the fourth logical possibility:

- 4) a modification or improvement of quantum theoretic formalism and its adaptation to the geometric context of general relativity.

The distinguished role of time in the formalism and interpretation of quantum theory is one of the aspects to be overcome in the quantum formalism adapted to the goal of quantization of general relativity. As this feature of quantum description can be seen as inherited from the canonical Hamiltonian formalism whose structures underlie canonical quantization, one could try to find a generalization of canonical Hamiltonian formalism and its quantization in which



all space-time variables would be treated on an equal footing. Fortunately, such generalizations are known in the mathematical theory of the calculus of variations of multiple integrals and there is an infinite variety of covariant finite-dimensional Hamiltonian-like formulations given by different Lepage equivalents of the Poincaré-Cartan form [1–5]) which, from the point of view of physics, implement exactly this idea. The simplest of these formulations is known as the De Donder-Weyl (DW) theory (see e.g. [1, 6, 7]).

The DW Hamiltonian formulation of a field theory given by the first order Lagrangian $L = L(y^a, y_\mu^a, x^\nu)$ uses the covariant Legendre transformation to the new set of variables: *polymomenta*

$$p_a^\mu := \frac{\partial L}{\partial y_\mu^a}$$

and the *DW Hamiltonian function*

$$H(y^a, p_a^\mu, x^\mu) := y_\mu^a(y, p) p_a^\mu - L,$$

which, for regular theories with

$$\det ||\partial^2 L / \partial y_\mu^a \partial y_b^\nu|| \neq 0,$$

enable us to write the field equations in the DW covariant Hamiltonian form:

$$\partial_\mu y^a(x) = \frac{\partial H}{\partial p_a^\mu}, \quad \partial_\mu p_a^\mu(x) = -\frac{\partial H}{\partial y^a}. \quad (1)$$

The latter look like a multidimensional field theoretic analogue of the Hamilton equations with all space-time variables treated on an equal footing.

A generalization of Poisson brackets to the DW Hamiltonian formulation [8–10], which is suitable for quantization, is defined on semi-basic forms on the polymomentum phase space (with the space-time being the base manifold and the space of field and polymomentum variables being the fiber) and leads to the Poisson-Gerstenhaber algebra structure with respect to the graded Lie bracket and a special \bullet -product of forms:

$$A \bullet B := *^{-1}(*A \wedge *B), \quad (2)$$

called co-exterior.

Just as the canonical quantization proceeds from the mathematical structures of canonical Hamiltonian formalism, *precanonical quantization* starts from the mathematical structures underlying the DW Hamiltonian formulation: the polysymplectic form, Poisson-Gerstenhaber brackets, DW Hamilton-Jacobi theory, etc.

It was found in our earlier work [11–13] that the quantization of the subalgebra of precanonically conjugate variables (similar to the Heisenberg algebra) leads to the following representation of the operators of polymomenta:

$$\hat{p}_a^\nu = -i \hbar \varkappa \gamma^\nu \frac{\partial}{\partial y^a}, \quad (3)$$

which act on the Clifford-valued wave functions $\Psi(y, x)$ on the finite dimensional covariant configuration space of field and space-time variables y and x .

The constant \varkappa of the dimension ℓ^{1-n} in n space-time dimensions appears in precanonical quantization on the dimensional grounds. Its meaning as the inverse of a very small "elementary volume" is obvious e.g. in the representation of the basic $(n-1)$ -forms

$$\varpi_\nu := \partial_\nu \lrcorner (dx^0 \wedge \dots \wedge dx^{n-1})$$

in terms of the space-time Clifford algebra elements:

$$\widehat{\omega}_\nu = \frac{1}{\varkappa} \gamma_\nu.$$

Note that the approach of precanonical quantization does not modify the microscopic structure of space-time by any *ad hoc* assumptions.

The covariant analogue of the Schrödinger equation in precanonical quantization reads

$$i\hbar \varkappa \gamma^\mu \partial_\mu \Psi = \hat{H} \Psi, \quad (4)$$

where \hat{H} is the DW Hamiltonian operator composed from the partial differential operators with respect to the field variables, c.f. (3). For the free scalar field y

$$H = \frac{1}{2} p^\mu p_\mu + \frac{1}{2} \frac{m^2}{\hbar^2} y^2$$

and the operator \hat{H} corresponds to the harmonic oscillator along the field dimension y [11,12,14]:

$$\hat{H} = -\frac{1}{2} \hbar^2 \varkappa^2 \frac{\partial^2}{\partial y^2} + \frac{1}{2} \frac{m^2}{\hbar^2} y^2. \quad (5)$$

The self-adjointness of \hat{H} with respect to the inner product

$$\langle \Phi | \Psi \rangle := \int dy \, \overline{\Psi} \Psi, \quad (6)$$

where $\overline{\Psi} := \Psi^\dagger \gamma^0$, and Eq. (4) lead to the conservation law

$$\partial_\mu \int dy \, \overline{\Psi} \gamma^\mu \Psi = 0 \quad (7)$$

which makes the probabilistic interpretation of $\Psi(y, x)$ possible. Note however, that in pseudoeuclidean space-times the inner product in (6) is indefinite, while the conserved quantity

$$\int d\mathbf{x} \int dy \, \overline{\Psi} \gamma^0 \Psi \quad (8)$$

is positive definite (here the notation $x^\mu = (\mathbf{x}, t)$ is used). Hence, the approach of precanonical quantization implies a generalization of mathematical formalism of quantum theory with an indefinite metric Hilbert space, where γ^0 plays the role of J -operator (see e.g. [15]).

The particle interpretation of the free scalar field is suggested by the spectrum of DW Hamiltonian operator in (5): $\varkappa m(N + \frac{1}{2})$ with $N \in \mathbb{N}$, which implies that free particles of mass m correspond to the transitions between the neighbouring eigenstates of DW Hamiltonian operator.

The relation between the precanonical Schrödinger equation (4) and the functional differential Schrödinger equation following from canonical quantization:

$$i\hbar \partial_t \Psi = \hat{\mathbf{H}} \Psi,$$

where $\Psi = \Psi([y(\mathbf{x})], t)$ is the Schrödinger wave functional and $\hat{\mathbf{H}}$ is the functional derivative operator of the canonical Hamiltonian functional, is established by assuming that there is

a relation between Ψ and the precanonical wave function $\Psi(y, x)$ restricted to the subspace $\Sigma : (y = y(\mathbf{x}), t = \text{const})$:

$$\Psi([y(\mathbf{x})], t) = \Psi([\Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t)], [y^a(\mathbf{x})]),$$

and substituting the equation for $\partial_t \Psi_\Sigma$ following from the restriction of (4) to Σ into the chain rule differentiation

$$i\partial_t \Psi = \int d\mathbf{x} \text{Tr} \left\{ \frac{\delta \Psi}{\delta \Psi_\Sigma^T(y^a(\mathbf{x}), \mathbf{x}, t)} i\partial_t \Psi_\Sigma(y^a(\mathbf{x}), \mathbf{x}, t) \right\}. \quad (9)$$

Then, in the limiting case $\gamma^0 \varkappa \rightarrow \delta(\mathbf{0})$, we are able to obtain the functional differential Schrödinger equation as the consequence of (4) and the expression of the Schrödinger wave functional in terms of the continuum product of precanonical wave functions [16,17] (c.f. [14,18]):

$$\Psi = \text{Tr} \left\{ \prod_{\mathbf{x}} e^{-iy(\mathbf{x})\gamma^i \partial_i y(\mathbf{x})/\varkappa} \Psi_\Sigma(y(\mathbf{x}), \mathbf{x}, t) \right\} \Big|_{\gamma^0 \varkappa \rightarrow \delta(\mathbf{0})}. \quad (10)$$

The existence of such a relation between precanonical quantization and functional Schrödinger representation suggests that the standard QFT based on canonical quantization is a singular limit of QFT based on precanonical quantization when the "elementary volume" $1/\varkappa$ is vanishing. Note also that the map $\gamma^0 \varkappa \rightarrow \delta(\mathbf{0})$ is actually the inverse of the "quantization map" from the exterior forms to Clifford numbers: $\widehat{\omega}_0 = \frac{1}{\varkappa} \gamma_0$, which underlie precanonical quantization, in the limit of infinite \varkappa .

2. DW Hamiltonian formulation of vielbein/tetrad gravity

Because the Dirac operator enters in the precanonical analogue of the covariant Schrödinger equation (4), the vielbein formulation of gravity is a more natural framework for precanonical quantization than the metric formulation used in our earlier work [19]. The latter essentially led to a hybrid quantum-classical theory (c.f. [20,21]) because a part of the spin connection term in the curved space-time Dirac operator in (4) can not be expressed and quantized in terms of the variables of the metric formulation.

Let us consider the first order Palatini type Lagrangian density of Einstein's gravity with the cosmological term:

$$\mathfrak{L} = \frac{1}{\kappa_E} \mathfrak{e} e_I^{[\alpha} e_J^{\beta]} (\partial_\alpha \omega_\beta^{IJ} + \omega_\alpha^{IK} \omega_{\beta K}^J) + \frac{1}{\kappa_E} \Lambda \mathfrak{e}, \quad (11)$$

where e_I^μ are the vielbein components, ω_α^{IJ} are torsion-free spin connection coefficients, $\kappa_E := 8\pi G$, and $\mathfrak{e} := \det ||e_\mu^a||$.

The polymomenta associated with the vielbein and spin connection field variables treated as independent dynamical variables:

$$\mathfrak{p}_{e_\beta^I}^\alpha = \frac{\partial \mathfrak{L}}{\partial_\alpha e_\beta^I} \quad \text{and} \quad \mathfrak{p}_{\omega_\beta^{IJ}}^\alpha = \frac{\partial \mathfrak{L}}{\partial_\alpha \omega_\beta^{IJ}},$$

yield the primary constraints of the DW Hamiltonian formalism, viz.

$$\mathfrak{p}_{e_\beta^I}^\alpha \approx 0, \quad \mathfrak{p}_{\omega_\beta^{IJ}}^\alpha \approx \frac{1}{\kappa_E} \mathfrak{e} e_I^{[\alpha} e_J^{\beta]}. \quad (12)$$

Consequently, not all space-time gradients of vielbein and spin connection fields can be expressed as functions of polymomenta and fields and we need to develop an analogue of the constraints analysis within the DW formalism.

Notwithstanding the fact that a mathematical literature related to the DW Hamiltonian theory with constraints exists [22–28], the analysis suitable for the purposes of quantization, though incomplete, seems to be found only in our paper [29]. The idea of that paper is to use the $(n-1)$ -forms constructed from the constraints and their Poisson-Gerstenhaber brackets found within the DW formalism in our earlier papers [8–10], and to try to find a generalization of the Dirac's treatment of constraints in the Hamiltonian formalism of mechanics to the DW Hamiltonian formalism in field theory.

Following this line and using the primary constraints (12), let us write down an extended DW Hamiltonian density

$$\mathfrak{H} = -\frac{1}{\kappa_E} \mathfrak{e}_I^{[\alpha} e_J^{\beta]} \omega_\alpha^{IK} \omega_{\beta K}^J - \frac{1}{\kappa_E} \Lambda \mathfrak{e} + \mu_{\alpha\beta}^I \mathfrak{p}_{e_\beta^I}^\alpha + \lambda_{\alpha\beta}^{IJ} \left(\mathfrak{p}_{\omega_\beta^{IJ}}^\alpha - \frac{1}{\kappa_E} \mathfrak{e}_I^{[\alpha} e_J^{\beta]} \right), \quad (13)$$

where μ and λ are the Lagrange multipliers. The DW Hamiltonian equations given by \mathfrak{H} yield:

$$\partial_\alpha e_\beta^I = \mu_{\alpha\beta}^I, \quad \partial_{[\alpha} \omega_{\beta]}^{IJ} = \lambda_{\alpha\beta}^{IJ}, \quad (14)$$

$$\partial_\alpha \mathfrak{p}_{e_\beta^I}^\alpha = -\frac{\partial \mathfrak{H}}{\partial e_\beta^I}, \quad \partial_\alpha \mathfrak{p}_{\omega_\beta^{IJ}}^\alpha = -\frac{\partial \mathfrak{H}}{\partial \omega_\beta^{IJ}}. \quad (15)$$

The first equation in (15) and the second one in (14) reproduce, on the constraints subspace, the Einstein equations. The second equation in (15) and the first one in (14) lead to the covariant constancy condition:

$$\nabla_\beta (\mathfrak{e}_I^{[\alpha} e_J^{\beta]}) = 0, \quad (16)$$

which can be transformed into the expression of the spin connection in terms of vielbeins and their derivatives.

Equations (14), (15) are equivalent to the preservation of semi-basic $(n-1)$ -forms constructed from the constraints (12):

$$\mathfrak{e}_{e_\beta^I} := \mathfrak{p}_{e_\beta^I}^\alpha \varpi_\alpha, \quad \mathfrak{e}_{\omega_\beta^{IJ}} := \mathfrak{p}_{\omega_\beta^{IJ}}^\alpha \varpi_\alpha - \frac{1}{\kappa_E} \mathfrak{e}_I^{[\alpha} e_J^{\beta]} \varpi_\alpha. \quad (17)$$

By calculating the brackets of those forms using the local coordinate expression of the *polysymplectic form* introduced in our papers [8–10]:

$$\Omega = d\mathfrak{p}_{e_\beta^I}^\alpha \wedge de_\beta^I \wedge \varpi_\alpha + d\mathfrak{p}_{\omega_\beta^{IJ}}^\alpha \wedge d\omega_\beta^{IJ} \wedge \varpi_\alpha, \quad (18)$$

we obtain:

$$\begin{aligned} \{\mathfrak{e}_e, \mathfrak{e}_{e'}\} &= 0, \\ \{\mathfrak{e}_\omega, \mathfrak{e}_{\omega'}\} &= 0, \\ \{\mathfrak{e}_{e_\gamma^K}, \mathfrak{e}_{\omega_\beta^{IJ}}\} &= -\frac{1}{\kappa_E} \frac{\partial}{\partial e_\gamma^K} \left(\mathfrak{e}_I^{[\alpha} e_J^{\beta]} \right) \varpi_\alpha, \end{aligned} \quad (19)$$

where the bracket of two semi-basic Hamiltonian $(n-1)$ -forms F and G is defined as follows:

$$\{F, G\} := -X_F \lrcorner dG, \quad (20)$$

where

$$X_F \lrcorner \Omega := dF.$$

From (19) we conclude that the primary constraints in (12) are second class.

We use our generalization of Dirac bracket to the singular DW Hamiltonian formalism [29]:

$$\{F, G\}^D := \{F, G\} - \{F, \mathfrak{C}_U\} \bullet (\mathfrak{C}_{UV}^{-1} \wedge \{\mathfrak{C}_V, G\}), \quad (21)$$

where the indices U, V run over all components of e and ω and the inverse of the form-valued matrix $\mathfrak{C}_{UV} := \{\mathfrak{C}_U, \mathfrak{C}_V\}$ is defined by

$$\mathfrak{C}_{UV}^{-1} \wedge \mathfrak{C}_{VU'} := \sigma \varpi \delta_{UU'}, \quad (22)$$

where $\sigma = (-)1$ is the (pseudo)euclidean signature of the metric, $dx^\mu \wedge \varpi_\nu =: \delta_\nu^\mu \varpi$, and $\sigma \varpi \bullet$ is the unit operator when acting on any semi-basic form, Then we can obtain the following brackets of precanonically conjugate variables on the subalgebra of $(n-1)$ -forms:

$$\{\mathfrak{p}_e^\alpha \varpi_\alpha, e' \varpi_{\alpha'}\}^D = 0, \quad (23)$$

$$\{\mathfrak{p}_\omega^\alpha \varpi_\alpha, \omega' \varpi_{\alpha'}\}^D = \{\mathfrak{p}_\omega^\alpha \varpi_\alpha, \omega' \varpi_{\alpha'}\} = \delta_\omega^{\omega'} \varpi_{\alpha'}, \quad (24)$$

$$\{\mathfrak{p}_e^\alpha \varpi_\alpha, \mathfrak{p}_\omega\}^D = \{\mathfrak{p}_e^\alpha \varpi_\alpha, \omega' \varpi_{\alpha'}\}^D = \{\mathfrak{p}_\omega^\alpha \varpi_\alpha, e' \varpi_{\alpha'}\}^D = 0, \quad (25)$$

and similar precanonical brackets of $(n-1)$ - and 0-forms, which also constitute a subalgebra with respect to the Poisson-Gerstenhaber bracket operation:

$$\{\mathfrak{p}_e^\alpha \varpi_\alpha, e'\}^D = 0, \quad (26)$$

$$\{\mathfrak{p}_\omega^\alpha \varpi_\alpha, \omega'\}^D = \{\mathfrak{p}_\omega^\alpha \varpi_\alpha, \omega'\} = \delta_\omega^{\omega'}, \quad (27)$$

$$\{\mathfrak{p}_e^\alpha \varpi_\alpha, \mathfrak{p}_\omega\}^D = \{\mathfrak{p}_e^\alpha \varpi_\alpha, \omega\}^D = \{\mathfrak{p}_\omega^\alpha \varpi_\alpha, e\}^D = 0, \quad (28)$$

and

$$\{\mathfrak{p}_e^\alpha, e' \varpi_{\alpha'}\}^D = 0, \quad (29)$$

$$\{\mathfrak{p}_\omega^\alpha, \omega' \varpi_\beta\}^D = \{\mathfrak{p}_\omega^\alpha, \omega' \varpi_\beta\} = \delta_\beta^\alpha \delta_\omega^{\omega'}, \quad (30)$$

$$\{\mathfrak{p}_e^\alpha, \mathfrak{p}_\omega \varpi_{\alpha'}\}^D = \{\mathfrak{p}_e^\alpha, \omega \varpi_{\alpha'}\}^D = \{\mathfrak{p}_\omega^\alpha, e' \varpi_{\alpha'}\}^D = 0. \quad (31)$$

The following remarks regarding the above calculation are in order. Note that the formula in (21) assumes that \mathfrak{C}_{UV}^{-1} exists in the sense of (22) [29]. However, it is not the case for the matrix defined by (19):

$$\mathfrak{C}_{UV} := \begin{vmatrix} 0 & \mathfrak{C}_{e\omega} \\ \mathfrak{C}_{\omega e} & 0 \end{vmatrix}, \quad (32)$$

where $\mathfrak{C}_{e\omega} := \{\mathfrak{C}_e, \mathfrak{C}_\omega\}$ is a rectangular matrix (16×24 in $n = 4$ dimensions). In the usual Dirac's Hamiltonian formalism it would signal that not all of the second class constraints are found. However, it is not necessarily the case here, because the number of polymomenta is different from the number of field variables and the analogues of the symplectic matrix in the polysymplectic formalism are singular matrices similar to the higher dimensional Duffin-Kemmer-Petiau matrices (c.f. [30]) whose algebraic definition is actually tantamount to the statement that, up to a sign factor, they are generalized Moore-Penrose inverse to themselves. More generally than in (22) we can understand \mathfrak{C}_{UV}^{-1} as a generalized inverse such that

$$\mathfrak{C}_{UU'} \bullet (\mathfrak{C}_{U'V'}^{-1} \wedge \mathfrak{C}_{V'V}) = \mathfrak{C}_{UV}. \quad (33)$$

Then the specific structure of (32) ensures that the Moore-Penrose-type generalized inverse of \mathfrak{C}_{UV} has the same matrix block structure as (32) with $\mathfrak{C}_{e\omega}$ replaced by $\mathfrak{C}_{e\omega}^{-1}$, so that (22) is fulfilled on the e -subspace, viz.

$$\mathfrak{C}_{eU'}^{-1} \wedge \mathfrak{C}_{U'e'} = \delta_{ee'} \sigma \varpi. \quad (34)$$

This is the e -subspace inverse which is needed in order to calculate the brackets in (23), (26), and (29). For example, by denoting $\mathbf{p}_e := \mathbf{p}_e^\alpha \varpi_\alpha$, we obtain:

$$\llbracket e', \mathbf{p}_e \rrbracket^D = \llbracket e', \mathbf{p}_e \rrbracket - \llbracket e', \mathfrak{C}_{e''} \rrbracket \bullet (\mathfrak{C}_{e''\omega}^{-1} \wedge \llbracket \mathfrak{C}_\omega, \mathbf{p}_e \rrbracket) = -\delta_{e'}^e + \delta_{e'}^{e''} \bullet (\mathfrak{C}_{e''\omega}^{-1} \wedge \mathfrak{C}_{\omega e}) = 0, \quad (35)$$

which is the result in (26). All other brackets in (23)-(31) are vanishing as a consequence of the specific matrix block structure of \mathfrak{C}_{UV} and its generalized inverse.

The brackets in (23)-(31) are assumed to be the analogue of the fundamental Dirac brackets of canonical variables in constrained mechanics and they underlie the quantization procedure below.

3. Quantization

Usually, quantization of systems with second class constraints is performed by transforming the Dirac brackets into commutators according to the Dirac's quantization rule. However, in the present approach the latter has to be modified in order to make sure that densities are quantized as density valued operators, viz.

$$[\hat{A}, \hat{B}] = -i\hbar \widehat{\llbracket A, B \rrbracket^D}, \quad (36)$$

where (the operator of) \mathfrak{e} appears due to the fact that the polysymplectic form and polymomenta are densities.

Now, quantization of brackets in (23), (25), (26), (28), (29), (31) and the constraint $p_e \approx 0$ lead us to the conclusion that the operators of the conjugate polymomenta of vielbeins are zero: $\hat{p}_e = 0$. We can, therefore, set our precanonical wave function to depend only on the spin connection and space-time variables, i.e. $\Psi = \Psi(\omega_\alpha^{IJ}, x^\mu)$.

Further, quantization of the Dirac bracket in (27) yields

$$\widehat{p_{\omega_\beta^{IJ}}^\alpha \varpi_\alpha} = -i\hbar \hat{\mathfrak{e}} \frac{\partial}{\partial \omega_\beta^{IJ}}. \quad (37)$$

Moreover, by quantizing (30), which coincides with the familiar bracket of polymomenta and field variables that underlies precanonical quantization in flat space-time [11, 12], we obtain the formal representation of polymomenta:

$$\hat{\mathbf{p}}_{\omega_\beta^{IJ}}^\alpha = -i\hbar \mathfrak{x} : \hat{\mathfrak{e}} \hat{\gamma}^{[\alpha} \frac{\partial}{\partial \omega_{\beta]}^{IJ}} : , \quad (38)$$

where the density $\hat{\mathfrak{e}}$ and the curved space-time Dirac matrices $\hat{\gamma}^\alpha$ are yet unknown operators, and $:$ stands for a potential operator ordering ambiguity. Note that when obtaining (38) we still assumed that

$$\widehat{\varpi}_\nu = \frac{1}{\mathfrak{x}} \hat{\gamma}_\nu, \quad (39)$$

which is just a formal generalization of the relation known from precanonical quantization in flat space-time [11, 12], as far as the explicit operator representation of $\hat{\gamma}_\nu$ is not known explicitly.

Next, let us insert the precanonical operator representation of $\hat{\mathbf{p}}_\omega^\alpha$, Eq. (38), into the strong operator version of the second constraint in (12) and contract it with flat $\bar{\gamma}^{IJ}$ -s:

$$(\mathfrak{e} e_I^{[\alpha} e_J^{\beta]} \bar{\gamma}^{IJ})^{op} = \hat{\mathfrak{e}} \hat{\gamma}^{\alpha\beta} = \kappa_E (\mathbf{p}_{\omega_\beta^{IJ}}^\alpha \bar{\gamma}^{IJ})^{op}, \quad (40)$$

where $()^{op}$ replaces the hat over the longer expressions and

$$\hat{\gamma}^\nu := \hat{e}_I^\nu \bar{\gamma}^I,$$

where $\bar{\gamma}^I \bar{\gamma}^J + \bar{\gamma}^J \bar{\gamma}^I = 2\eta^{IJ}$, η^{IJ} is a fiducial flat Minkowski metric with the signature $+-\dots-$, and $\bar{\gamma}^{IJ} := \bar{\gamma}^{[I} \bar{\gamma}^{J]}$. A comparison with (38) yields the operator representation of the curved space-time Dirac matrices:

$$\hat{\gamma}^\beta = -i\hbar\kappa_E \bar{\gamma}^{IJ} \frac{\partial}{\partial \omega_\beta^{IJ}}, \quad (41)$$

vielbeins:

$$\hat{e}_I^\beta = -i\hbar\kappa_E \bar{\gamma}^J \frac{\partial}{\partial \omega_\beta^{IJ}}, \quad (42)$$

and the polymomenta conjugate to spin connection:

$$\hat{\mathbf{p}}_{\omega_\beta^{IJ}}^\alpha = -\hbar^2 \kappa_E^2 \hat{\mathbf{e}} \bar{\gamma}^{KL} \frac{\partial}{\partial \omega_{[\alpha}^{KL}} \frac{\partial}{\partial \omega_{\beta]}^{IJ}}, \quad (43)$$

where the operator of \mathbf{e} can now be constructed from (42):

$$\hat{\mathbf{e}} = \left(\frac{1}{n!} \epsilon^{I_1 \dots I_n} \epsilon_{\mu_1 \dots \mu_n} \hat{e}_{I_1}^{\mu_1} \dots \hat{e}_{I_n}^{\mu_n} \right)^{-1}. \quad (44)$$

We can also obtain the operators of $(n-1)$ -volume elements (39), that leads to a rather complicated non-local expression:

$$\hat{\omega}_\nu = \frac{1}{\kappa(n-1)!} \hat{\mathbf{e}} \epsilon_{\nu\mu_1 \dots \mu_{n-1}} \hat{\gamma}^{\mu_1} \dots \hat{\gamma}^{\mu_{n-1}}, \quad (45)$$

and the operator of the metric tensor $g^{\mu\nu}$:

$$\widehat{g^{\mu\nu}} = -\hbar^2 \kappa_E^2 \eta^{IJ} \eta^{KL} \frac{\partial^2}{\partial \omega_\mu^{IK} \partial \omega_\nu^{JL}}. \quad (46)$$

Finally, using (43) we construct the DW Hamiltonian operator \hat{H} which corresponds to the DW Hamiltonian density restricted to the subspace of constraints (12), $\mathbf{e}H := \mathfrak{H}|_C$:

$$\hat{H} = \hbar^2 \kappa_E^2 \bar{\gamma}^{IJ} \omega_{[\alpha}^{KM} \omega_{\beta]}^{ML} \frac{\partial}{\partial \omega_\alpha^{IJ}} \frac{\partial}{\partial \omega_\beta^{KL}} - \frac{1}{\kappa_E} \Lambda. \quad (47)$$

4. Covariant Schrödinger equation for quantum gravity

The precanonical covariant Schrödinger equation which generalizes Eq. (4) to the context of quantum gravity will have the form

$$i\hbar\kappa \hat{\nabla} \Psi = \hat{H} \Psi, \quad (48)$$

where $\hat{\nabla} := (\gamma^\mu (\partial_\mu + \omega_\mu))^{op}$ with the spin connection term $\omega_\mu := \frac{1}{4} \omega_{\mu IJ} \bar{\gamma}^{IJ}$ is what we called the "quantized Dirac operator", because the Dirac matrices and the spin connection term in it are now operators themselves. Using the operator representation of the curved space γ -matrices in (41) we obtain:

$$\hat{\nabla} = -i\hbar\kappa \bar{\gamma}^{IJ} \frac{\partial}{\partial \omega_\mu^{IJ}} \left(\partial_\mu + \frac{1}{4} \omega_{\mu KL} \bar{\gamma}^{KL} \right) \doteq. \quad (49)$$

Therefore, the precanonical counterpart of the Schrödinger equation for quantum gravity takes the form

$$\bar{\gamma}^{IJ} \doteq \left(\partial_\mu + \frac{1}{4} \omega_{\mu KL} \bar{\gamma}^{KL} - \omega_{\mu M}^K \omega_\beta^{ML} \frac{\partial}{\partial \omega_\beta^{KL}} \right) \frac{\partial}{\partial \omega_\mu^{IJ}} \Psi + \frac{\Lambda}{\hbar^2 \kappa_E^2} \Psi = 0 \quad (50)$$

and determines the wave function $\Psi(\omega, x)$ or, more generally, the transition amplitudes $\langle \omega, x | \omega', x' \rangle$. The latter provide an inherently quantum description of the geometry of space-time which generalizes classical differential geometry with its smooth connection fields $\omega(x)$.

Let us note that the combination of the constants including \varkappa in the last term of (50) is dimensionless. By fixing the operator ordering in (50) we would generate a dimensionless constant of the order $\sim n^6$ (the number of components of ω -s is $\sim n^3$) which can be interpreted as the cosmological constant Λ divided by $\hbar^2 \varkappa^2 \kappa_E^2$. In this case, however, the observable value of Λ is obtained (at $n = 4$) only if \varkappa is roughly at the nuclear energy scale, which is far away from our original expectation that \varkappa is at about the Planck scale and contradicts the experimental evidence that the usual relativistic space-time holds even at TeV scale. On the other hand, if we take \varkappa at the Planck scale then we arrive at the familiar 120 orders of magnitude error in the estimation of the cosmological constant, which is usually obtained by using the Planck scale cutoff in the momentum space integration of the zero point energies. This rather confirms that the constant \varkappa of precanonical quantization is related to the ultra-violet cutoff scale and that the cosmological constant problem is not related to the ground state of pure quantum gravity but rather to the particle composition of the universe.

5. Hilbert space

It is natural to assume that the wave functions $\Psi(\omega, x)$ vanish at large values of ω -s. Then the probability amplitude of observing the regions of space-time with very large curvature is very small, so that the quantum gravitational singularity avoidance is essentially built in in the choice of the boundary condition in ω -space.

The scalar product is expected to have the form:

$$\langle \Phi | \Psi \rangle := \int [d\omega] \bar{\Phi} \Psi,$$

where $[d\omega]$ is an invariant measure on the space of spin connection coefficients. Using the arguments similar to those used by Misner to obtain the invariant measure on the space of metrics [31], we found:

$$[d\omega] = \epsilon^{-n(n-1)} \prod_{\mu, I < J} d\omega_{\mu}^{IJ}. \quad (51)$$

Because in the present picture ϵ is an operator given by (44), the measure $[d\omega]$ is operator valued and the scalar product of the theory has the form

$$\langle \Phi | \Psi \rangle := \int \bar{\Phi} [\widehat{d\omega}] \Psi. \quad (52)$$

Then the most natural definition of the expectation values of operators using the scalar product with the operator valued measure implies the Weyl ordering, viz.

$$\langle \widehat{O} \rangle := \int \bar{\Psi} \left([\widehat{d\omega}] \widehat{O} \right)_W \Psi. \quad (53)$$

When discussing the specific physical problems using the formalism of this paper we will have to distinguish between the physical aspects and those attributed to the choice of coordinates. The latter are a macroscopic notion due to the observer's choice and, therefore, can be implemented on the average. For example, the choice of the harmonic coordinates on the average leads to the following condition on the wave function $\Psi(\omega, x)$:

$$\partial_{\mu} \left\langle \Psi(\omega, x) \left| \widehat{\epsilon g^{\mu\nu}} \right| \Psi(\omega, x) \right\rangle = 0, \quad (54)$$

which should be solved together with the covariant Schrödinger equation, Eq. (50), and that makes the problem more complicated.

6. Conclusion

Quantization of vielbein gravity using the approach of precanonical quantization, which is based on the De Donder-Weyl covariant Hamiltonian formulation, is discussed. All space-time variables are treated on an equal footing as generalizations of the time parameter in non-relativistic mechanics. No global splitting to space and time is required.

The DW Hamiltonian formulation of the first order Palatini action of vielbein gravity leads to the second class primary constraints which are treated using our recent generalization of Dirac bracket to the DW formalism [29]. The consideration of the fundamental generalized Dirac brackets of precanonically conjugate variables, Eqs. (23)-(31), leads to the conclusion that the quantum dynamics of gravity can be formulated using the wave functions on the space of spin connection coefficients and space-time variables. The operators of vielbeins, metric tensor, DW Hamiltonian operator and the quantized Dirac operator which enters the covariant precanonical Schrödinger equation of quantum gravity are explicitly constructed. We also discuss the Hilbert space of the theory and the invariant operator-valued integration measure on the space of spin connection coefficients. Let us note that the resulting (still tentative) formulation of quantum gravity is non-perturbative, covariant and background independent.

However, it is not clear at this stage if the consideration of the fundamental Dirac brackets in (23)-(31) is sufficient, because on the subalgebra of 0- and $(n-1)$ -forms we can also calculate brackets between the forms composed from vielbeins and spin connection coefficients, such as $\llbracket e, \omega \varpi_\mu \rrbracket^D \sim \partial_\mu \lrcorner \mathfrak{C}_{ew}^{-1}$, which explicitly depend on the complicated nonlinear expression of the generalized inverse of the rectangular matrix \mathfrak{C}_{ew} in (19) and cannot be quantized directly using the Dirac's quantization rule (36).

Among the issues left beyond the scope of the paper there are the details of the choice of the coordinate (gauge) conditions on the average (c.f. [32]), which are not sufficiently clear to us, and the issues related to the indefinite inner product Hilbert space appearing in the formalism of the theory. We were also unable so far to demonstrate that the present formulation reproduces the Einstein equations on the average or in the classical limit. We hope to elaborate on those issues in the forthcoming publications.

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