

Quantization of the Schwarzschild geometry

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Abstract. The conditional symmetries of the reduced Einstein–Hilbert action emerging from a static, spherically symmetric geometry are used as supplementary conditions on the wave function. Based on their integrability conditions, only one of the three existing symmetries can be consistently imposed, while the unique Casimir invariant, being the product of the remaining two symmetries, is calculated as the only possible second condition on the wave function. This quadratic integral of motion is identified with the reparametrization generator, as an implication of the uniqueness of the dynamical evolution, by fixing a suitable parametrization of the r -lapse function. In this parametrization, the determinant of the supermetric plays the role of the measure. The combined Wheeler - DeWitt and linear conditional symmetry equations are analytically solved. The solutions obtained depend on the product of the two “scale factors”.

1. Introduction

A vast amount of work on the use of symmetries in the quantization of minisuperspace cosmological models fills the relevant literature of the last fifteen years. In fact, the application of the variational symmetries approach to classical Bianchi cosmologies has started in [1]. For the case of classical and quantum cosmology, in either Bianchi or higher derivative models, the first relevant works are [2], [3], [4], [5] and [6] (where conditional symmetries are used), while a revival of recent work on the subject can be found in [7], [8], [9] and [10]. Essentially, the method consists in the application of the standard theory of variational symmetries [11], [12] to the Lagrangian of some minisuperspace model. The steps of the usual procedure are (i) gauge fixing the lapse function to some convenient value, (ii) applying the first prolongation in the velocity phase space of a vector in the configuration space to this gauge fixed Lagrangian, and (iii) demanding its action to be zero. A slightly different approach exploiting the notions of the special projective and/or the homothetic group is adopted in [13], [14] and [15].

In the present work, we do not gauge fix the lapse function. This is justified by the fact that the full content of the corresponding reparametrization generator, i.e. the quadratic constraint, is kept intact by the presence of the lapse. This fact is important not only for the symmetries but also in the solution of the classical equations of motion, a task that is greatly facilitated by inserting the algebraic solution for N from the quadratic constraint into the other equations of motion [16], [17], [18] and [19]. In [20] a phase space point of view is adopted and a conditional symmetry is defined [21] as a conformal field of both the supermetric and the potential. Here, it is proven that this definition leads to the uncovering of all variational symmetries.

An application of the method to the case of static, spherically symmetric geometries is presented in this work. A minisuperspace Lagrangian for the aforesaid set of metrics has been



given in [22], [23] where a $3+1$ decomposition is introduced along the radial coordinate r which is taken to be the dynamical variable. In [24], [25] the symmetries of the reduced Schwarzschild action have been used in the quantization procedure. Here, the quantum analogues of the linear integrals of motion are also adopted as supplementary conditions imposed on the wave function. A careful examination of their role and integrability conditions leads to the unique Casimir invariant of their algebra. Through a particular redefinition of the lapse function, the Casimir invariant is identified to the corresponding r - reparametrization generator.

The paper is organized as follows: In section 2 the reduced valid Lagrangian reproducing Einstein's field equations for static, spherically symmetric configurations is given, and passing to its Hamiltonian formulation [26], the three conditional symmetries are revealed. In section 3, the dynamical system is quantized by use of Dirac's canonical quantization procedure for constrained systems [27]. Finally, some concluding remarks are presented in the discussion.

2. Hamiltonian formulation of static, spherically symmetric geometries

Our starting point is the static, spherically symmetric line element

$$ds^2 = -a(r)^2 dt^2 + n(r)^2 dr^2 + b(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

In the usual ADM $3+1$ decomposition one foliates space-time in t -hypersurfaces and the coefficient of dt^2 is the lapse function. Here we adopt a $3+1$ foliation in the r coordinate and therefore the role of the lapse is attributed to the coefficient of dr^2 in the line element. Thus we consider $n(r)$ in (1) to be the r -lapse function while $a(r)$ and $b(r)$ are the dependent "dynamical" variables on the r -hypersurface. Previous authors use also a "shift" term $2B(r) dr dt$. However, this is not relevant for the following discussion, as $B(r)$ does not enter the Einstein tensor while it can be absorbed by a time redefinition of the form $t = \tilde{t} + \int \frac{B(r)}{a(r)^2} dr$.

The Einstein - Hilbert action $A_{E-H} = \int \sqrt{-g} R d^4x$ for the geometries (1) leads to the reduced action $\mathcal{A} = \int L dr$ with the following Lagrange function $L(a, b, a', b', n)$:

$$L = 2 a n + \frac{4 b a' b'}{n} + \frac{2 a b'^2}{n}, \quad (2)$$

where $'$ denotes differentiation with respect to the spatial coordinate r . It is easy to verify that the Euler - Lagrange equations obtained from (2) are identical to Einstein's equations $\mathcal{G}_{\mu\nu} = 0$ for the line element (1).

In order to proceed with the Hamiltonian formalism we calculate the conjugate momenta,

$$\begin{aligned} \pi_n &= \frac{\partial \mathcal{L}}{\partial n'} = 0, \\ \pi_a &= \frac{\partial \mathcal{L}}{\partial a'} = \frac{4 b b'}{n}, \\ \pi_b &= \frac{\partial \mathcal{L}}{\partial b'} = \frac{4 b a'}{n} + \frac{4 a b'}{n}. \end{aligned} \quad (3)$$

Obviously, π_n is a primary constraint. The Legendre transformation leads to the Hamiltonian

$$H = n \mathcal{H}_c,$$

where

$$\mathcal{H}_c = -2 a - \frac{a \pi_a^2}{8 b^2} + \frac{\pi_a \pi_b}{4 b}. \quad (4)$$

The preservation of the primary constraint π_n in the r -evolution, i.e.

$$\pi'_n = \{\pi_n, H\} \approx 0,$$

leads to the secondary constraint

$$\mathcal{H}_c \approx 0. \quad (5)$$

The minisuperspace metric inferred from (4) is

$$G^{\alpha\beta} = \begin{pmatrix} -\frac{a}{4b^2} & \frac{1}{4b} \\ \frac{1}{4b} & 0 \end{pmatrix}. \quad (6)$$

According to our definition of a conditional symmetry [20], as a vector ξ on the configuration space the following relations must hold:

$$\mathcal{L}_\xi G^{\alpha\beta} = \phi(q) G^{\alpha\beta} \quad , \quad \mathcal{L}_\xi V(q) = \phi(q) V(q) \quad (7)$$

where q 's are the configuration space variables. Thus we are led to the following three conformal Killing fields of both $G_{\alpha\beta}$ and the potential $V = -2a$:

$$\xi_1 = (-a, b), \quad \xi_2 = \left(\frac{1}{ab}, 0 \right), \quad \xi_3 = \left(-\frac{a}{2b}, 1 \right) \quad (8)$$

which, contracted with (π_a, π_b) , provide us with the three integrals of motion:

$$Q_1 = -a \pi_a + b \pi_b, \quad Q_2 = \frac{\pi_a}{ab}, \quad Q_3 = -\frac{a \pi_a}{2b} + \pi_b. \quad (9)$$

We calculate the Poisson brackets of these conserved quantities with the canonical Hamiltonian H and the Poisson algebra that they satisfy:

$$\{Q_1, H\} = n \mathcal{H}_c, \quad \{Q_2, H\} = -\frac{n}{a^2 b} \mathcal{H}_c, \quad \{Q_3, H\} = \frac{n}{2b} \mathcal{H}_c, \quad (10a)$$

$$\{Q_1, Q_3\} = Q_3, \quad \{Q_2, Q_1\} = Q_2, \quad \{Q_3, Q_2\} = 0. \quad (10b)$$

As expected, the Poisson brackets (10a) are weakly vanishing on the constraint surface $\mathcal{H}_c \approx 0$ and therefore the three Q_I 's are constants of motion.

At this point it is interesting, and useful for what follows in the quantization, to adopt a new parametrization of the lapse $n(r) = \frac{\bar{n}(r)}{2a(r)}$ which makes the potential constant. The Lagrangian and the corresponding Hamiltonian are now given by

$$\bar{L} = \bar{n} + \frac{8ab a' b'}{\bar{n}} + \frac{4a^2 b'^2}{\bar{n}} \quad , \quad \bar{H} = \bar{n} \mathcal{H}_c = \bar{n} \frac{1}{2a} \mathcal{H}_c. \quad (11)$$

If the value of \bar{n} specified by the constrained equation is substituted into the Euler - Lagrange equations for $a(r)$ and $b(r)$ the system can be solved for only one acceleration, say $a''(r)$, and the general solution of the entire system is:

$$\bar{n}(r) = 2cb'(r) \quad , \quad a(r) = c \sqrt{1 - \frac{2M}{b(r)}}, \quad (12)$$

where the constants of integration have been rearranged so that the ensuing line element

$$ds^2 = -c^2 \left(1 - \frac{2M}{b(r)} \right) dt^2 + \left(1 - \frac{2M}{b(r)} \right)^{-1} b'(r)^2 dr^2 + b(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (13)$$

bears the closest possible resemblance to the standard form of the Schwarzschild metric, while the presence of the arbitrary function $b(r)$ reflects the r reparametrisation covariance of the system.

The new supermetric is now given by

$$\bar{G}^{\alpha\beta} = \begin{pmatrix} -\frac{1}{8b^2} & \frac{1}{8ab} \\ \frac{1}{8ab} & 0 \end{pmatrix}, \quad (14)$$

and the fields (8) are now turned into Killing fields of $\bar{G}^{\alpha\beta}$; they are also, trivially, symmetries of the constant potential $\bar{V} = 1$ as well. The algebra satisfied by the new quantities $\bar{Q}_I = \xi_I^\alpha \bar{\pi}_\alpha$ and $\bar{\mathcal{H}}_c$ can be easily seen to be

$$\{\bar{Q}_1, \bar{\mathcal{H}}_c\} = 0, \quad \{\bar{Q}_2, \bar{\mathcal{H}}_c\} = 0, \quad \{\bar{Q}_3, \bar{\mathcal{H}}_c\} = 0. \quad (15)$$

Therefore the quantities \bar{Q}_I still remain constants of motion. It is in this conformal gauge that one can explicitly verify that the prolonged ξ_I 's, X_I , are satisfying a form of the standard condition of variational symmetries $pr^{(1)}X_I(\bar{L}) = 0$, thus ensuring that they are generators of Noether symmetries. It is important to note that if one had inserted the X_I 's into the full relation $pr^{(1)}X(L) + L\text{Div}\Xi = \text{Div}F$ with L given by (2) the result would be negative.

If we invert the definitions for $\bar{\pi}_\alpha$ and use (12) in \bar{Q}_I we calculate the integrals' values on the solution space:

$$Q_1 = \bar{Q}_1 = 4cM, \quad Q_2 = \bar{Q}_2 = \frac{4}{c}, \quad Q_3 = \bar{Q}_3 = 2c \quad (16)$$

As it is known the Schwarzschild solution involves only one essential constant, the mass M . The second constant appearing in the solution (12), (13) can be seen to be absorbable by a scaling of the time coordinate $t \rightarrow \frac{t}{c}$, allowing to set $c = 1$ but not $c = 0$. So, on the solution space we can set $\bar{Q}_1 = 4M$, $\bar{Q}_2 = 4$, $\bar{Q}_3 = 2$. It is noteworthy that the values of the last two integrals of motion can be changed at will and that they do not involve the essential parameter M which characterizes the geometry. The above argument relies on the underlying geometry.

But what if we were deprived of the line element and we were just given the dynamical system (11)? How could we differentiate between the constants κ_1 and κ_2, κ_3 ? Interestingly enough, there is an argument that leads to a distinction between them: The crucial observation is that \bar{Q}_1 has a vanishing Poisson bracket with each kinetic term of \bar{H} , while \bar{Q}_2, \bar{Q}_3 need the entire kinetic term in order to produce vanishing Poisson brackets. This fact is reflected in the following property concerning the one-parameter family of canonical transformations generated by the charge Q_1 (see [29], [30] for the generalization of Noether symmetries for constrained systems and for Noether's theorem in phase space),

$$a \rightarrow e^\lambda a, \bar{\pi}_a \rightarrow e^{-\lambda} \bar{\pi}_a, b \rightarrow e^{-\lambda} b, \bar{\pi}_b \rightarrow e^\lambda \bar{\pi}_b \quad (17)$$

Under such transformations \bar{Q}_1 remains, of course, unchanged, while \bar{Q}_2, \bar{Q}_3 are scaled by $e^{-\lambda}$ and e^λ respectively. One can thus use the freedom of λ to arbitrarily change the values κ_2, κ_3 , but not κ_1 . Furthermore, the Hamiltonian $\bar{\mathcal{H}}_c$ remains, due to the particular scaling of \bar{Q}_2, \bar{Q}_3 , unchanged.

If we return to the phase space, we can write

$$\bar{Q}_{23} = \bar{Q}_2 \bar{Q}_3 = -\frac{1}{2b^2} \bar{\pi}_\alpha^2 + \frac{1}{ab} \bar{\pi}_\alpha \bar{\pi}_\beta \quad (18)$$

with the relevant Poisson bracket algebra now becoming

$$\{\bar{Q}_{23}, \bar{\mathcal{H}}_c\} = 0 \quad (19a)$$

$$\{\bar{Q}_I, \bar{Q}_{23}\} = 0, \quad I = 1, 2, 3. \quad (19b)$$

As expected \bar{Q}_{23} is a quadratic integral of motion. From a group theoretical view, \bar{Q}_{23} is an element of the centre of the universal enveloping algebra (uea) generated by \bar{Q}_I 's, i.e. it is the Casimir invariant. The Hamiltonian $\bar{\mathcal{H}}_c$ belongs also to the centre of uea (it commutes with all \bar{Q}_I 's), thus it can only differ from \bar{Q}_{23} by an additive and/or a multiplicative constant. It is an easy matter for one to check that indeed

$$\bar{Q}_{23} = 8(\bar{\mathcal{H}}_c + 1). \quad (20)$$

To sum up, we have constructed a gauge independent, quadratic in the momenta, integral of motion which commutes with the only integral of motion that entangles the sole essential constant of the Schwarzschild solution. In the next section, and in order to proceed with the quantization, we will rely on these two quantities.

3. Quantization

In order to quantize our system, we must turn into operators $\bar{\mathcal{H}}_c = \frac{1}{8}(\bar{Q}_{23} - 8)$, \bar{Q}_1 , \bar{Q}_2 and \bar{Q}_3 (hereafter, for the shake of simplicity, we will omit the *bars* from the symbols of the corresponding operators). The corresponding quantum operators can be inferred from the general definitions

$$\hat{\mathcal{H}}_c := -\frac{1}{2\mu} \partial_\alpha (\mu G^{\alpha\beta} \partial_\beta) + V(q) \quad (21)$$

$$\hat{Q}_I := -\frac{\hbar}{2\mu} (\mu \xi_I^\alpha \partial_\alpha + \partial_\alpha \mu \xi_I^\alpha), \quad (22)$$

where μ is a suitable measure which transforms as a scalar density. Definition (22) supplies us with a Hermitian operator [28] for the previously mentioned conditional symmetries (8). It can be seen [20] that, the measure entering the quantum operators ought to be taken as $\mu(a, b) = \lambda \sqrt{\det|\bar{G}_{\alpha\beta}|} \propto ab$. The constancy of λ is forced by the combined requirement that the Q_I 's must be realized as Hermitian operators and at the same time retain their classical geometrical character by acting as derivatives. Thus the extra term $\xi_I^\alpha \partial_\alpha \lambda$ must vanish for all $I = 1, 2, 3$, which leads to a constant λ . Two further arguments in favor of this choice of measure are:

- The fact that the quantum analogue of the algebra (19b) is made isomorphic to the classical, i.e.

$$[\hat{Q}_I, \hat{Q}_{23}]F(a, b) = 0, \quad I = 1, 2, 3 \quad \text{for any } F(a, b) \quad (23)$$

a fact that is highly non trivial, since it depends on the choice of both the factor ordering and the measure.

- At the classical level, the only linear integral of motion involving the essential constant is Q_1 . If we seek the functions on the configuration space which are invariant under the point transformations generated by Q_1 we find $\{Q_1, f(a, b)\} = 0 \Rightarrow f(a, b) = f(ab)$.

The above arguments lead to the following linear operators corresponding to the elements of the classical algebra, the Casimir invariant and the Hamiltonian:

$$\hat{Q}_1 = -\hbar(b \partial_b - a \partial_a) \quad (24)$$

$$\hat{Q}_2 = -\frac{\hbar}{ab} \partial_a \quad (25)$$

$$\hat{Q}_3 = -\hbar \left(\partial_b - \frac{a}{2b} \partial_a \right) \quad (26)$$

$$\hat{Q}_{23} = \frac{2}{b^2} \partial_a \partial_a - \frac{1}{ab} \partial_a \partial_b + \frac{1}{2ab^2} \partial_b \quad (27)$$

$$\hat{\mathcal{H}}_c = \frac{1}{8} (\hat{Q}_{23} - 8) \quad (28)$$

It is an easy task to check that these operators satisfy not only the relations

$$[\widehat{Q}_I, \widehat{Q}_J]F(a, b) = \pm C^K_{IJ} \widehat{Q}_K F(a, b) \quad (29)$$

for any test function $F(a, b)$, but also (23) as well.

The integrability condition for the equations emerging from the enforcement of the \widehat{Q}_I 's upon the wave function

$$\widehat{Q}_I \Psi = \kappa_I \Psi \quad (30)$$

leads to the selection rule

$$C^M_{IJ} \kappa_M = 0. \quad (31)$$

Due to the constraint condition (31), applied to the specific structure constants inferred from (10b), we conclude that only the eigenvalue κ_1 is free and κ_2, κ_3 must necessarily be zero. However, the latter is impossible since on the classical solution space (16) hold. This means that not all three operators can be applied simultaneously. The latter results in the need to consider the two- and/or one- dimensional subalgebras together with the Wheeler - DeWitt equation $\widehat{\mathcal{H}}_c \Psi = 0$. The investigation of these cases can be easily carried out.

As far as the $2d$ subalgebras are concerned the results obtained are briefly the following:

- (a) For the two non Abelian subgroups either κ_2 or κ_3 are forced to be zero, something that is inconsistent with their classical values.
- (b) For the Abelian subgroup, the two linear equations lead to the solutions $\Psi(a, b) = A \exp\left(\frac{i}{2}(\kappa_2 a^2 b + 2\kappa_3 b)\right)$, where A is constant and the quadratic constraint enforces the restriction $\kappa_2 \kappa_3 = 8$. It is of course doubtful if one can accept such a wave function to represent the geometry, knowing that it does not contain the essential constant M . However, one could interpret it as plane waves representing the limiting flat space-time $M = 0$.

The one-dimensional subalgebras spanned by $\widehat{Q}_2, \widehat{Q}_3$ give solutions which are special cases of the solution described in b), as expected since they commute.

Consequently, the only possibility is to adopt $\widehat{\mathcal{H}}_c$ and \widehat{Q}_1 as conditions on the wave function. This is indeed possible, since they commute with each other and therefore can be considered as physical quantities on the phase space that can be measured "simultaneously" (our dynamical parameter is the distance r). The ensuing eigenvalue equations are

$$\widehat{\mathcal{H}}_c \Psi = 0 \Rightarrow a \partial_a \partial_a \Psi - 2b \partial_a \partial_b \Psi + \partial_a \Psi - 16ab^2 \Psi = 0 \quad (32a)$$

$$\widehat{Q}_1 \Psi = \kappa_1 \Psi \Rightarrow \pm(-a \partial_a \Psi + b \partial_b \Psi) = \kappa_1 \Psi \quad (32b)$$

The solution of the linear partial differential equation (32b) is

$$\Psi(a, b) = a^{\pm \kappa_1} S(ab). \quad (33)$$

If we insert the above solution into the Hamiltonian constraint (32a) we arrive at the following ordinary differential equation for $S(u)$ ($u = ab$):

$$u^2 S''(u) + u S'(u) + (\kappa_1^2 + 16u^2) S(u) = 0 \quad (34)$$

which has the general solution

$$S(u) = c_1 J_{\pm \kappa_1}(4u) + c_2 Y_{\pm \kappa_1}(4u), \quad (35)$$

in terms of the Bessel functions of imaginary order.

In order to gain some insight on the normalizability of the formal probability, instead of these Bessel functions and because of their imaginary order, we can use the functions $F_{\mathfrak{z}\kappa_1}(4u)$ and $G_{\mathfrak{z}\kappa_1}(4u)$ defined in [31] through the Hankel functions $H_\mu^{(1)}(u) = J_\mu(u) + \mathfrak{z}Y_\mu(u)$ and $H_\mu^{(2)}(u) = J_\mu(u) - \mathfrak{z}Y_\mu(u)$, $\mu \in \mathbb{C}$. Thus, the solution can be written as

$$S(u) = c_1 F_{\mathfrak{z}\kappa_1}(4u) + c_2 G_{\mathfrak{z}\kappa_1}(4u) \quad (36)$$

with

$$F_{\mathfrak{z}\kappa_1}(4u) = \frac{1}{2} \left(e^{-\kappa_1\pi/2} H_{\mathfrak{z}\kappa_1}^{(1)}(4u) + e^{\kappa_1\pi/2} H_{\mathfrak{z}\kappa_1}^{(2)}(4u) \right) \quad (37)$$

$$G_{\mathfrak{z}\kappa_1}(4u) = \frac{1}{2\mathfrak{z}} \left(e^{-\kappa_1\pi/2} H_{\mathfrak{z}\kappa_1}^{(1)}(4u) - e^{\kappa_1\pi/2} H_{\mathfrak{z}\kappa_1}^{(2)}(4u) \right). \quad (38)$$

These functions are linearly independent solutions of (34) and have the following properties: a) when $u \in (0, +\infty)$ they are real, b) they are oscillatory with a phase difference of $\frac{\pi}{2}$ and c) when both u , and/or κ_1 tend to zero, $F_{\mathfrak{z}\kappa_1}(4u)$ tends to 1, while $G_{\mathfrak{z}\kappa_1}(4u)$ becomes infinite.

The final form of the wave function $\Psi(a, b)$ is

$$\Psi(a, b) = a^{\mathfrak{z}\kappa_1} S(ab), \quad (39)$$

so we can define a probability density of the form

$$\mu(ab) \Psi^*(a, b) \Psi(a, b) \propto u S^*(u) S(u). \quad (40)$$

4. Discussion

In this paper we used a method for the quantizations of *minisuperspace* actions which can be summarized in the following steps:

- (i) Go over to the Hamiltonian \bar{H} .
- (ii) Calculate the Noether symmetries as Killing fields of the metric $\bar{G}^{\alpha\beta}(q)$ and, trivially, symmetries of the constant potential.
- (iii) Identify the essential constants of the metric. Promote the allowed Q_I 's to operators according to (22) with $\mu = |\bar{G}|^{1/2}$.
- (iv) Promote the Hamiltonian constrain $\bar{\mathcal{H}}_c$ (which is a linear function of the Casimir invariant) via (21) to a Hermitian operator acting on the wave function.

We applied the above method for the case of static, spherically symmetric geometries. First, we begin from the Lagrangian (2) emanating from the line element (1). We find the simultaneous conformal Killing fields (8) of the supermetric and the potential, which define the three conserved charges (9). The unique Casimir invariant of their algebra is Q_{23} . In order to make it numerically proportional to the kinetic part of the Hamiltonian we are led to (11).

Our specific example (described in sections 2 and 3) has also been the subject of [24]. In that work, Vakili finds two of the Q_I 's but then he uses a linear combination of them in order to reproduce the essential constant M of the Schwarzschild metric. He thus reaches to the unique acceptable linear quantum operator equivalent to our \bar{Q}_1 . The clever choice of the lapse function, his equation (4), along with the somewhat unorthodox choice of factor ordering for the operators (see below his equation (55)) leads essentially to a constant potential (his equation (53)) and to the Laplacian operator (his equation (56)). As a result, the solution spaces found both by us and Vakili essentially coincide. Of course, our general theory given in [20] constitutes a systematic explanation of the various choices of his work.

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