Hybrid dynamics as a constrained quantum system

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Abstract. Hamiltonian formulation of quantum dynamics and nonlinear constraints are used to derive dynamical equations of a hybrid classical-quantum system. Starting with a compound quantum system in the Hamiltonian formulation, conditions for classical behavior are imposed on one of its subsystems and the corresponding hybrid dynamical equations are derived. The dynamical equations for hybrid systems in pure and in mixed states indicate that the hybrid systems have properties that are not exhausted by those of quantum and classical systems.

1. Introduction

Classical and quantum theories have developed different formalisms to successfully describe interactions between systems belonging to their respective domains. Correlations between quantum objects are kinematically captured by direct product structure of Hilbert spaces adopted to the linear evolution. On the other hand compound classical systems are described on the Cartesian product of the component's phase spaces. In this work the framework of Hamiltonian dynamical systems is used to treat the hybrid quantum-classical systems and to develop a description of the interactions within such systems which is consistent with the main physically justified requirements. The Hamiltonian framework was used in [1] to develop a description of the hybrid classical-quantum systems by treating both, quantum and classical, formally as Hamiltonian systems described in the Hamiltonian language. The coupling between the systems is introduced somewhat *ad hoc* as if both systems were classical, just because they are both described in the framework of the Hamiltonian dynamical systems. This assumption about the treatment of compound systems is not trivially obvious. For example, such treatment of coupling between two quantum systems, both separately described in the Hamiltonian framework, would be incorrect. In this paper we start with the total compound quantum system in the geometric Hamiltonian framework [2]. The next step is to consider a classical limit of one of the component systems. To this purpose we utilize our recently developed theory of general quantum constraints within the Hamiltonian approach [3], and the corresponding description of the classical limit [4, 5]. The Hamiltonian form of the derived evolution equations of the hybrid system turns out to be the same as the one postulated in Ref. [1] and therefore satisfies a list of standard requirements collected and tested in [1]. We also study the evolution of general ensembles of hybrid system, and demonstrate that, consistently with the nonlinear pure states evolution, the ensembles evolve in such a way that initially equivalent mixed states of the quantum degrees of freedom become nonequivalent. Non-unitary evolution of the quantum degrees of freedom in a hybrid system suggests that the hybrid systems, if existent, must be considered as conceptually independent class and not as such whose properties are exhausted by the properties of quantum and of classical systems.

2. Quantum dynamics as a Hamiltonian dynamical system

Schrödinger dynamical equation on $\mathcal{H}$ generates a Hamiltonian dynamical system on an appropriate symplectic manifold [6, 7, 8, 9]. In fact the Hilbert space $\mathcal{H}$ is viewed as a real manifold $\mathcal{M}$. In general $\mathcal{M}$ is an infinite dimensional Euclidean manifold. The point from $\mathcal{M}$ corresponding to the vector $|\psi\rangle \in \mathcal{H}$ is denoted by X_ψ. The real manifold $\mathcal{M}$ has Riemannian and symplectic structure. The scalar product on $\mathcal{H}$ is decomposed into it's real and imaginary parts

$$\langle\psi_1|\psi_2\rangle = G(\psi_1,\psi_2) + i\Omega(\psi_1,\psi_2). \tag{1}$$

G is Riemannian metric on $\mathcal{M}$ and Ω is symplectic form on $\mathcal{M}$.

$\mathcal{M}$ associated with the Hilbert space $\mathcal{H}$ can be viewed as a phase space of a Hamiltonian dynamical system.

Quantum observables $\hat{A}$ are represented by functions of the form $A(\psi) = \langle\psi|\hat{A}|\psi\rangle$. Only functions of this form, i.e. bilinear, have physical interpretation of quantum observables.

The Poisson bracket of two functions relates to the commutator between corresponding observables

$$\{A_1, A_2\} = \frac{1}{i\hbar}\langle[\hat{A}_1, \hat{A}_2]\rangle. \tag{2}$$

The Schrödinger evolution generated by a Hamiltonian $\hat{H}$

$$i\hbar|\dot{\psi}\rangle = \hat{H}|\psi\rangle, \tag{3}$$

is equivalent to the Hamilton's equations on $\mathcal{M}$

$$\dot{X}^a_\psi = \Omega^{ab}\nabla_b H(X_\psi). \tag{4}$$

2.1. *Constrained quantum systems*

The Hamiltonian framework for quantum systems opens up a possibility to treat nonlinear constraints [3, 10]. In particular, the formalism of nonlinear constraints provides a natural framework to study the classical limit of a quantum system [4, 5].

Constraints are generally given by a set of k independent functional equations

$$f_l(X) = 0,\ l = 1, 2, \dots, k. \tag{5}$$

which define a submanifold Γ of $\mathcal{M}$

Equations of motion of the constrained system are obtained using the method of Lagrange multipliers. In the Hamiltonian form:

$$\dot{X} = \Omega(\nabla X, \nabla H_{tot}), \qquad H_{tot} = H + \sum_{l=1}^{k}\lambda_l f_l, \tag{6}$$

that should be solved together with the equations of the constraints (5).

The Lagrange multipliers λ_l are to be determined from the following conditions

$$\begin{aligned} 0 = \dot{f}_l &= \Omega(\nabla f_l, \nabla H_{tot}) && (7)\\ &= \Omega(\nabla f_l, \nabla H) + \sum_{m=1}^{k}\lambda_m\Omega(\nabla f_l, \nabla f_m). && (8) \end{aligned}$$

If one does not know the structure of Γ than one can follow Dirac's procedure. For this, the constraints have to be regular. The case of interest here involves precisely the irregular constraints that cannot be easily replaced, in the general case, by an equivalent set of regular constraints. However, one often knows that the manifold determined by the constraints Γ is a symplectic submanifold of $\mathcal{M}$. In this case one knows that: $H_{tot}|_\Gamma = H|_\Gamma$, i.e. the constrained system is a Hamiltonian system on Γ and it's Hamiltonian is simply $H|_\Gamma$.

2.2. *Example: System of oscillators*

The dynamical algebra is $\otimes_i h_{4,i}$ represented in the Hilbert space $\mathcal{H} = L_2(R^n)$. The fundamental observables are represented by $2n$ operators $(\hat{Q}_i, \hat{P}_i)$, $i = 1, 2, \ldots n$, satisfying $[\hat{Q}_i, \hat{P}_j] = i\delta_{i,j}$ and the Hamiltonian is

$$\begin{aligned}\hat{H} &= \sum_{i=1}^{n} \frac{1}{2m_i}\hat{P}_i^2 + V(\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_n) \\ &= \sum_{i=1}^{n} \frac{1}{2m_i}\hat{P}_i^2 + \frac{m_i\omega_i^2}{2}\hat{Q}_i^2 + \ldots, \end{aligned} \tag{9}$$

The symplectic phase space $\mathcal{M}$ of the Hamiltonian formulation of the quantum oscillators system is the space of fields on $\mathbf{R}^n$ with the symplectic coordinates given by $\phi(x_1, \ldots, x_n), \pi(x_1, \ldots, x_n)$ $(x_i \in \mathbf{R})$ where $\langle x|\psi\rangle = \phi(x) + i\pi(x) \in \mathcal{H}$. By X_ψ we denote an element from $\mathcal{M}$.

For notational simplicity we continue with only one nonlinear oscillator. General case is commented later.

The constraints are given by the following equation

$$f(X) = f_q(X) + f_p(X) = 0 \tag{10}$$

where

$$f_q(X) = (\Delta\hat{Q})^2 - \frac{\hbar}{2m\omega} = 0, \tag{11}$$

$$f_p(X) = (\Delta\hat{P})^2 - \frac{m\omega\hbar}{2} = 0, \tag{12}$$

The constraint implies an equivalence relation on $\mathcal{M}$ (or on $\mathcal{H}$).

$$X_1 \sim X_2 \Leftrightarrow q(X_1) = q(X_2) \wedge p(X_1) = p(X_2) \tag{13}$$

where $q(X_\psi) = \langle\psi|\hat{Q}|\psi\rangle \ldots$.

The manifold determined by the constraint is the symplectic submanifold $\Gamma \subset \mathcal{M}$ if fact $\Gamma = \mathcal{M} / \sim$, where $\sim$ is the equivalence relation (13). Each equivalence class contains a single coherent state $|p, q\rangle$. The manifold is parameterized by $\{p, q\}$.

The constrained system defined by the Hamiltonian (9) and the constraints (10) preserve the equivalence classes and preserve the dispersions of the fundamental quantum observables $\hat{P}, \hat{Q}$. The constrained system on Γ represents the coarse-grained description of the (quantum) nonlinear oscillator.

Γ is symplectic and therefore $H_{tot}|_\Gamma = H|_\Gamma$ i.e. the Hamiltonian of the constrained system on Γ is simply $H(p, q) = \langle p, q|\hat{H}|p, q\rangle$.

H_{tot} preserves constant and minimal quantum fluctuations of fundamental observables, while the evolution with H can in general make them quite large.

We turn to the macro-limit. The constrained system satisfies

$$\langle V(\hat{Q})\rangle_\alpha = V(q) + \sum_{k=1}^{\infty} \frac{(\Delta\hat{Q})_\alpha^{2k}}{2^k k!} V^{(2k)}(q), \tag{14}$$

where $q = \langle\hat{Q}\rangle_\alpha$ and $(\Delta\hat{Q})_\alpha = \sqrt{\hbar/(2m\omega)}$. Thus, the total Hamiltonian in a point $\alpha \equiv (q, p)$ on the constrained manifold is

$$\begin{aligned}H_{tot} &= \frac{p^2}{2m} + V(q) + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k} \\ &\equiv h_{cl} + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \frac{\hbar^k V^{(2k)}(q)}{(2m\omega)^k}. \end{aligned} \tag{15}$$

For the system with more than one oscillators, that might be nonlinear and interacting, the condition that $\Delta\hat{Q}_i$ and $\Delta\hat{P}_i$ are simultaneously minimal implies that each of the oscillators is always in some pure H_4 coherent state $|\alpha_i(t)\rangle$. Thus, the total state $|\psi(t)\rangle$ is always given by the tensor product of the single oscillator's pure coherent states $|\psi(t)\rangle = \otimes_i|\alpha_i(t)\rangle$.

To summarize: We see that the classical system emerges because of:
a) the coarse-grained description of the quantum system and then
b) the macroscopic limit.
It is important to note that the two factors, i.e. the coarse-graining and the macro-limit, are independent and both are necessary.

3. Hybrid systems

3.1. Derivation of pure states evolution equations

The Hilbert space of a bipartite quantum-quantum system is defined as $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}$. The corresponding phase space is obtained as $\mathcal{M} \neq \mathcal{M}_1 \times \mathcal{M}_2$; The corresponding point from $\mathcal{M}$ has complex canonical coordinates $(\psi(x), \psi^*(x))$ where $\psi(x) \in \mathcal{H}$.

The Poisson bracket of two functions on $\mathcal{M}$ is given by

$$\{f_1, f_2\}_{\mathcal{M}} = \frac{1}{i\hbar}\int dx \left(\frac{\delta f_1}{\delta\psi(x)}\frac{\delta f_2}{\delta\psi^*(x)} - \frac{\delta f_2}{\delta\psi(x)}\frac{\delta f_2}{\delta\psi^*(x)}\right). \tag{16}$$

For simplicity we consider the first system to be given by the $k-$fold product of the Heisenberg algebras, that is by the basic operators $(\hat{Q}_1, \ldots, \hat{Q}_k,\, \hat{P}_1, \ldots, \hat{P}_k) \equiv (\hat{Q}, \hat{P})$.

In a hybrid classical-quantum system the total quantum fluctuations of the first subsystem, that is the sum of dispersions of the basic observables $(\hat{Q}, \hat{P})$ must be preserved minimal during the evolution:

$$F(X_\psi) = \sum_{i=1}^{k}((\Delta\hat{Q}_i)^2_\psi + (\Delta\hat{P}_i)^2_\psi) - min = 0. \tag{17}$$

The evolution of the fully quantum composite system must be modified in such a way that the constraint is respected.

The manifold $\bar{\Gamma}$ of the constraint is a nonlinear symplectic submanifold of $\mathcal{M}$ locally isomorphic with the Cartesian product $\Gamma_1 \times \mathcal{M}_2$. Γ_1 is the manifold of the standard Heisenberg algebra minimal uncertainty coherent states of the first subsystem, denoted by $|\alpha\rangle$ or $|q,p\rangle$, and $\mathcal{M}_2 \sim \mathcal{H}_2$ is the quantum phase space of the second subsystem.

At each point $|\mathcal{C}\rangle\rangle$ of $\bar{\Gamma}$ given by

$$|\mathcal{C}\rangle\rangle = |\alpha\rangle|\omega_2\rangle \equiv |q,p\rangle|\omega_2\rangle, \tag{18}$$

there are local symplectic coordinates $(q, p, \omega_2(x_2), \omega_2^*(x_2))$ expressed in terms of $|\mathcal{C}\rangle\rangle$ as $q = \langle\langle\mathcal{C}|\hat{Q}|\mathcal{C}\rangle\rangle$, $p = \langle\langle\mathcal{C}|\hat{P}|\mathcal{C}\rangle\rangle$ and $\omega_2(x_2) = \langle x_2|\langle q,p|\mathcal{C}\rangle\rangle$.

The constrained manifold $\bar{\Gamma}$ is symplectic. Therefore, the constrained system is Hamiltonian with the Hamilton's function given by the original Hamilton's function $\langle\langle\psi|\hat{H}|\psi\rangle\rangle$ evaluated on the constrained manifold. Therefore, the dynamics is generated by the Poisson bracket on $\mathcal{M}$ and the Hamiltonian

$$\begin{aligned} H_t &= \langle\langle\mathcal{C}(\psi)|\hat{H}|\mathcal{C}(\psi)\rangle\rangle = \langle\langle\psi|q,p\rangle\langle q,p|\hat{H}|q,p\rangle\langle q,p|\psi\rangle\rangle \\ &\equiv \langle\langle\psi|\hat{H}_\alpha(q,p)|\psi\rangle\rangle, \end{aligned} \tag{19}$$

where $\hat{H}_\alpha(q,p) \equiv |q,p\rangle\langle q,p| \otimes \langle q,p|\hat{H}|q,p\rangle$. In fact the constrained evolution of an arbitrary function-observable $A(\psi) = \langle\langle\psi|\hat{A}|\psi\rangle\rangle$ on the constrained manifold is obtained by reducing the following equation

$$\begin{aligned}\dot{A}(\psi) &= \{A(\psi), H_t\}_{\mathcal{M}} \\ &= \frac{1}{i\hbar}\int dx \left(\frac{\delta A(\psi)}{\delta\psi(x)}\frac{\delta H_t}{\delta\psi^*(x)} - \frac{\delta H_t}{\delta\psi(x)}\frac{\delta A(\psi)}{\delta\psi^*(x)}\right) \qquad (20)\end{aligned}$$

on the constrained manifold $\bar{\Gamma}$.

For example, before reduction on $\bar{\Gamma}$ the dynamical equation for $q = \langle\langle\psi|\hat{Q}|\psi\rangle\rangle$ and $p = \langle\langle\psi|\hat{P}|\psi\rangle\rangle$ are given by

$$\dot{q} = \frac{1}{i\hbar}\langle\langle\psi|[\hat{Q},\hat{H}_\alpha]|\psi\rangle\rangle + \frac{\partial H_t}{\partial p}, \qquad (21)$$

$$\dot{p} = \frac{1}{i\hbar}\langle\langle\psi|[\hat{P},\hat{H}_\alpha]|\psi\rangle\rangle - \frac{\partial H_t}{\partial q}. \qquad (22)$$

In fact, for an arbitrary operator $\hat{A}_1$ acting only in $\mathcal{H}_1$ one has $\langle\langle\psi|[\hat{A}_1,\hat{H}_\alpha]|\psi\rangle\rangle|_{\bar{\Gamma}} = 0$. Therefore, the dynamical equations for the first system's coordinates and momenta are

$$\dot{q} = \frac{\partial H_t}{\partial p}, \quad \dot{p} = -\frac{\partial H_t}{\partial q}, \qquad (23)$$

where $H_t = \langle\langle\mathcal{C}(\psi)|\hat{H}|\mathcal{C}(\psi)\rangle\rangle$.

Dynamical equations for functions of the form

$$\omega_2(x_2) \equiv \langle x_2|\omega_2(\psi)\rangle = \langle x_2|\langle q,p|\psi\rangle\rangle \qquad (24)$$

are obtained as follows.

Starting again with the equation

$$\dot{\omega}_2(x_2) = \frac{1}{i\hbar}\int dx \left(\frac{\delta\omega_2}{\delta\psi(x)}\frac{\delta H_t}{\delta\psi^*(x)} - \frac{\delta H_t}{\delta\psi(x)}\frac{\delta\omega_2}{\delta\psi^*(x)}\right) \qquad (25)$$

and after somewhat lengthy calculation one obtains before the reduction on $\bar{\Gamma}$

$$\begin{aligned} i\hbar\,\dot{\omega}_2(x_2) &= \langle x_2|\langle q,p|\hat{H}|q,p\rangle|\omega_2\rangle \\ &+ \left(\frac{q}{2}\frac{\partial H_t}{\partial q} + \frac{p}{2}\frac{\partial H_t}{\partial p}\right)\omega_2(x_2) \\ &+ \frac{i}{\hbar}\langle x_2|\langle q,p|(\hat{p}-p/2)|\psi\rangle\rangle\langle\langle\psi|[\hat{q},\hat{H}_\alpha]|\psi\rangle\rangle \\ &- \frac{i}{\hbar}\langle x_2|\langle q,p|(\hat{q}-q/2)|\psi\rangle\rangle\langle\langle\psi|[\hat{p},\hat{H}_\alpha]|\psi\rangle\rangle. \qquad (26)\end{aligned}$$

Upon reduction on the constrained manifold $\bar{\Gamma}$ the last two terms are annulled and after dropping the pure phase term the relevant dynamical equations can be written in the form

$$i\hbar\,\dot{\omega}_2(x_2;\psi) = \langle x_2|\langle\alpha(\psi)|\hat{H}|\alpha(\psi)\rangle|\omega_2(\psi)\rangle. \qquad (27)$$

The equation has the form of a Schröedinger equation for the state vector $\omega_2(x_2;\psi) = \langle x_2|\langle q,p|\psi\rangle\rangle \in \mathcal{H}_2$, with the Hamiltonian operator $\langle\alpha(\psi)|\hat{H}|\alpha(\psi)\rangle$ acting on $\mathcal{H}_2$ and depending on $q = \langle\langle\psi|\hat{Q}|\psi\rangle\rangle$ and $p = \langle\langle\psi|\hat{P}|\psi\rangle\rangle$.

The Poisson bracket on $\bar{\Gamma}$ for arbitrary functions on $\bar{\Gamma}$ represented in the local coordinates $(q, p, \omega_2, \omega_2^*)$ is

$$\{f_1, f_2\}_{\bar{\Gamma}} = \sum_{i=1}^{k} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial f_1}{\partial p_i} \right) + \frac{1}{i\hbar} \int dx_2 \left(\frac{\delta f_1}{\delta \omega_2(x_2)} \frac{\delta f_2}{\delta \omega_2^*(x_2)} - \frac{\delta f_2}{\delta \omega_2(x_2)} \frac{\delta f_1}{\delta \omega_2^*(x_2)} \right). \quad (28)$$

The Hamiltonian form of the hybrid dynamics on the constrained phase space manifold $\bar{\Gamma}$ reads

$$\dot{q} = \{q, H_t\}_{\bar{\Gamma}}, \quad \dot{p} = \{p, H_t\}_{\bar{\Gamma}}, \quad (29)$$

$$\dot{\omega}_2 = \{\omega_2, H_t\}_{\bar{\Gamma}}, \quad \dot{\omega}_2^* = \{\omega_2^*, H_t\}_{\bar{\Gamma}}, \quad (30)$$

where the Hamilton's function $H_t(q, p, \omega_2(x_2), \omega_2^*(x_2))$ is given by $H_t = \langle\!\langle \mathcal{C}(\psi)|\hat{H}|\mathcal{C}(\psi)\rangle\!\rangle$ and $|\mathcal{C}\rangle\!\rangle = |q, p\rangle|\omega_2\rangle$.

The constrained dynamics which preserves minimal value of the quantum fluctuations of one of the subsystems is only the first step. The second step is the relevant macro-limit so that the minimal quantum fluctuations, still present in the corresponding coherent states, can be neglected when compared with actual values of the dynamical variables.

Therefore in the macro-limit the replacement

$$\langle\!\langle \mathcal{C}(\psi)|\hat{F}(\hat{Q}, \hat{P})|\mathcal{C}(\psi)\rangle\!\rangle \to F(q, p) \quad (31)$$

should be applied in the equations relevant for the first subsystem.

3.2. Ensembles of hybrid systems

For convenience, we denote the canonical coordinates in $\mathcal{M}_q$ of $\mathcal{M}_c \times \mathcal{M}_q$ by (x, y). Quantities defined exclusively as functions on $\mathcal{M}_q$ ($\mathcal{M}_c$) are referred on as quantum degrees of freedom or QDF.

Consider a general probability density $\rho(p, q, x, y)$ on the total hybrid phase space $\mathcal{M} = \mathcal{M}_c \times \mathcal{M}_q$[11].

In general, following the Hamiltonian formulation of the hybrid system dynamics, the evolution of $\rho(p, q, x, y; t)$ considered as a statistical ensemble on $\mathcal{M}$ is given by the Liouville equation

$$\frac{\partial}{\partial t}\rho(p, q, x, y; t) = \{H_t(p, q, x, y), \rho(p, q, x, y; t)\}_{\mathcal{M}}. \quad (32)$$

We shall argue that the most general statistical ensembles of hybrid systems need to be represented by general probability densities $\rho(p, q, x, y; t)$.

The density $\rho(p, q, x, y; t)$ generates a unique positive operator valued function (POVF):

$$\hat{\rho}(p, q; t) = \int_{\mathcal{M}_q} \rho(p, q, x, y; t)\hat{\Pi}(x, y) dM_q, \quad (33)$$

where $\hat{\Pi}(x, y)$ is the projection onto the vector form $\mathcal{H}_2$ corresponding to the point (x, y). $\hat{\rho}(p, q; t)$ can be called the hybrid statistical operator. It contains less information about the hybrid system state then the density $\rho(p, q, x, y; t)$.

The unconditional mixed state of the quantum subsystem of the hybrid in the state $\rho(p, q, x, y; t)$ is also uniquely obtained as

$$\hat{\rho}(t) = \int_{\mathcal{M}} \rho(p, q, x, y; t)\hat{\Pi}(x, y) dM. \quad (34)$$

Many hybrid ensembles, represented by different $\rho(p,q,x,y;t_0)$, have the quantum subsystem in the same conditional or unconditional mixed state. Each different $\rho(p,q,x,y;t_0)$ describes physically different ensembles of hybrid systems with the quantum subsystem in the same mixed state. The differences are manifested in the evolution of $\hat{\rho}(x,y;t)$ and $\hat{\rho}(t)$.

In the purely quantum case all different $\rho(x,y;t_0)$ with the same first moment correspond to the physically equivalent quantum mixture $\hat{\rho}(t_0)$, and generate unique von Neumann evolution of $\hat{\rho}(t)$ which is obtained from the Liouville evolution of any such $\rho(x,y;t)$. Therefore, all such $\rho(x,y;t)$ are equivalent in the purely quantum case. In the hybrid case, different $\rho(p,q,x,y;t_0)$ which give the same $\hat{\rho}(p,q;t_0)$ (or $\hat{\rho}(t_0)$) generate different evolution of $\hat{\rho}(p,q;t)$ (or $\hat{\rho}(t)$) and thus must be considered as physically different.

The evolution equation for $\hat{\rho}(p,q)$ is

$$\begin{aligned}\frac{\partial\hat{\rho}(p,q;t)}{\partial t} &= \frac{1}{i\hbar}[\hat{H}_q+\hat{V}_{int}(p,q),\hat{\rho}(p,q;t)] + \{H_c(p,q),\hat{\rho}(p,q;t)\}_{p,q}\\ &+ \int_{\mathcal{M}_q}\{V_{int}(p,q,x,y),\rho(p,q,x,y;t)\}_{p,q}\,\hat{\Pi}(x,y)dM_q. \end{aligned} \tag{35}$$

The solution remains a well defined statistical operator on $\mathcal{H}$ for all t, which is a desirable property not shared by some other hybrid system theories.

The dynamical equation for $\hat{\rho}(t)$ is

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{1}{i\hbar}[\hat{H}_q,\hat{\rho}(t)] + \frac{1}{i\hbar}\int_{\mathcal{M}_c}[\hat{V}_{int}(p,q),\hat{\rho}(p,q;t)]dM_c\\ &+ \int_{\mathcal{M}_c}\{H_c(p,q),\hat{\rho}(p,q;t)\}_{p,q}dM_c\\ &+ \int_{\mathcal{M}}\{V_{int}(p,q,x,y),\rho(p,q,x,y;t)\}_{p,q}\,\hat{\Pi}(x,y)dM\\ &= \frac{1}{i\hbar}[\hat{H}_q,\hat{\rho}(t)] + \frac{1}{i\hbar}\int_{\mathcal{M}_c}[\hat{V}_{int}(p,q),\hat{\rho}(p,q;t)]dM_c. \end{aligned} \tag{36}$$

The first term on the right side of (36) generates the unitary part of the evolution and the second term does not preserve the norm of $\hat{\rho}$ and is responsible for non-unitary effects. Notice that the evolution of $\hat{\rho}(p,q;t)$ ($\hat{\rho}(t)$) cannot be expressed only in terms of $\hat{\rho}(p,q;t)$ ($\hat{\rho}(t)$), but irreducibly involves the probability density $\rho(p,q,x,y;t)$.

4. Conclusion

The evolution of QDF of a hybrid system is fundamentally different from the linear evolution of a quantum subsystem of a quantum system. The characteristic main features of the QDF evolution are expressed by the nonlinearity of the pure state evolution, or by the dynamically induced differences of $\hat{\rho}(t)$ with different initial convex mixture representations. It is well known that the linearity of the Schrödinger equation and the equivalence of different convex mixtures are both necessary in order to prevent superluminal communication in ordinary quantum mechanics of bipartite systems. If either of the two properties is violated, without further modification of the quantum formalism, superluminal communication between entangled parts of a bipartite system is possible. The nonlinear pure state evolution and the evolution dependence on the initially equivalent different ensembles appears quite naturally in the Hamiltonian description of hybrid systems, and in the same time the QDF of the hybrid might be in an entangled state [12]. Therefore, superluminal communication can be avoided only by some further modification of the scheme. It has been argued that the direct product might not be the natural type of coupling between systems with nonlinear evolution [13], and that nonlinear evolution might suggest non-standard computation of correlations [14]. Alternatively, one might consider the model of hybrid systems presented here as insufficient to describe fully the true features of coupled real quantum and real classical systems. One might try to incorporate back-reaction of the classical system on quantum fluctuations, like for example in [15], or one might explore the possibilities opened up by replacing a simple classical system by truly complex classical systems with many degrees of freedom [1, 12].

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