

Noncommutative spectral geometry: a short review

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Abstract. We review the noncommutative spectral geometry, a gravitational model that combines noncommutative geometry with the spectral action principle, in an attempt to unify General Relativity and the Standard Model of electroweak and strong interactions. Despite the phenomenological successes of the model, the discrepancy between the predicted Higgs mass and the current experimental data indicate that one may have to go beyond the simple model considered at first. We review the current status of the phenomenological consequences and their implications. Since this model lives by construction at high energy scales, namely at the Grand Unified Theories scale, it provides a natural framework to investigate early universe cosmology. We briefly review some of its cosmological consequences.

1. Introduction

Given the plethora of precise cosmological and astrophysical data and the measurements obtained by particle physics experiments reaching constantly higher energy scales, we are presently in a position to falsify early universe cosmological models. In return, by comparing the theoretical predictions against current data, we are able to constrain the fundamental theories upon which our cosmological models were based. A fruitful such example is the cosmological model based on Noncommutative Geometry [1, 2] and the Spectral Action principle, leading to the Noncommutative Spectral Geometry (NCSG), a theoretical framework that can provide [3] a purely geometric explanation for the Standard Model (SM) of strong and electroweak interactions. This model lives by construction in high energy scales, the Grand Unified Theories (GUTs) scale, offering a natural framework to study early universe cosmology [4]–[11]. Hence, instead of postulating a Lagrangian upon which we will build our cosmological model, we will adopt the one dictated by NCSG, with the constraints imposed by NCSG itself, and within the framework of this gravitational theory we will address some cosmological issues. Clearly, this is a solid approach, since the cosmological model is inspired and controlled from a fundamental theory. In return, by comparing the predictions of the model against high energy physics measurements and cosmological data, we will be able to constrain some of the free parameters of NCSG and/or its basic element, namely the choice of the algebra.

In this presentation, we will first briefly review [12] some elements of NCSG and then discuss some cosmological consequences of the model. Our purpose is not to give a full and detailed analysis, but rather highlight some results and open questions and guide the reader through this powerful interplay between mathematics, high energy physics, cosmology and astrophysics ¹.

¹ We refer the reader to the contribution by Sakellariadou, Stabile and Vitiello [13], presented in the same meeting, for a discussion on firstly, the physical meaning of the choice of the geometry and its relation to quantisation, and



Let us clarify that the approach we will follow here is *a priori* distinctive from the noncommutative approach based upon $[x^i, x^j] = i\theta^{ij}$, where θ^{ij} is an anti-symmetric real $d \times d$ matrix, used to implement fuzziness of space-time (with d the space-time dimensionality). One finds in the literature that noncommutative space is often of Moyal type, involving noncommutative tori or Moyal planes. Note however that the Euclidean version of Moyal noncommutative field theory is compatible with the spectral triples formulation of noncommutative geometry. This statement holds for compact noncommutative spaces, while it has been argued [15] that the compactness restriction is merely a technical one and noncompact noncommutative spin geometry can be built, implying that the classical background of noncommutative field theory can be recast in the spectral triple approach developed by Connes to describe noncommutative spaces.

2. Elements of noncommutative spectral geometry

It is reasonable to argue that the notion of geometry, as we are familiar with, loses its meaning at very high energy scales, namely near and above Planck scale. The simple classical picture and the notion of a continuous space should cease to be valid as quantum gravity effects turn on. Thus, according to one such school of thought, one may argue that at sufficiently high energy scales, spacetime becomes discrete and coordinates no longer commute. The noncommutative spacetime can be thus seen as a quantum effect of gravity, an approach that may shed some light on the regularisation of quantum field theory. One can, to a first approximation, consider the simplest class of noncommutative spaces (almost commutative), which are not incompatible with low energy physics (namely, today's physics) and study their consistency with experimental and observational data within the realm of high energy physics and cosmology, respectively. If such a programme passes successfully this nontrivial test, as a second step, one should attempt to construct less trivial noncommutative spaces whose limit is this simple but successful case studied first. All current studies remain at present within the first step of this promising programme.

The choice of the noncommutative space, followed by Alain Connes and his collaborators, was such that at low energy scales one recovers the SM action. There is an intrinsic difference between this approach and other ones that attempt to capture the effects of quantum gravity, which can no longer be switched off once we reach Planck energy scale. More precisely, in the NCSG approach one does not postulate the physics at very high energy scales, but instead one is guided by low energy physics. Hence, within NCSG the SM is considered as a phenomenological model which dictates the geometry of space-time so that the Maxwell-Dirac action functional leads to the SM action. To be more specific, in the framework of NCSG we are following here, gravity and the SM fields are put together into matter and geometry on a noncommutative space made from the product of a four-dimensional commutative manifold by a noncommutative internal space. Combining noncommutative geometry with the spectral action principle, and choosing the smallest finite dimensional algebra that can account for the SM particles, Connes and his collaborators have obtained a purely geometric explanation for the SM Lagrangian coupled to gravity. Thus, the model we will briefly present here, has been tailored in order to give the SM of particle physics. This implies that if this model leads to small discrepancies with respect to the data, one could then first examine a larger algebra (that could accommodate particles beyond the SM sector), before considering large deviations from commutative spaces.

Noncommutative spectral geometry is composed by a two-sheeted space, made from the product of a smooth four-dimensional manifold \mathcal{M} (with a fixed spin structure), by a discrete noncommutative space \mathcal{F} composed by only two points. The internal space \mathcal{F} has dimension 6 to allow fermions to be simultaneously Weyl and chiral, while it is discrete to avoid the infinite tower of massive particles that are produced in string theory. The noncommutative nature of \mathcal{F} is given

secondly, the relation of NCSG to the gauge structure of the theory and to dissipation, summarising the results of Ref. [14].

by a spectral triple, introduced by Connes as an extension of the notion of Riemannian manifold to noncommutative geometry. The real spectral triple is given by $(\mathcal{A}_{\mathcal{F}}, \mathcal{H}_{\mathcal{F}}, D_{\mathcal{F}})$. The algebra $\mathcal{A}_{\mathcal{F}} = C^\infty(\mathcal{M})$ of smooth functions on \mathcal{M} is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}_{\mathcal{F}}$ of Euclidean fermions; it is essentially the algebra of coordinates. The operator $D_{\mathcal{F}}$ is the Dirac operator $\not{D}_{\mathcal{M}} = \sqrt{-1}\gamma^\mu \nabla_\mu^s$ on the spin Riemannian manifold \mathcal{M} . It is such that $J_{\mathcal{F}} D_{\mathcal{F}} = \epsilon' D_{\mathcal{F}} J_{\mathcal{F}}$, where $J_{\mathcal{F}}$ is an anti-linear isometry of the finite dimensional Hilbert space, with the properties $J_{\mathcal{F}}^2 = \epsilon$, $J_{\mathcal{F}} \gamma_{\mathcal{F}} = \epsilon'' \gamma_{\mathcal{F}} J_{\mathcal{F}}$, with $\gamma_{\mathcal{F}}$ the chirality operator and $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$. The operator $D_{\mathcal{F}}$ is a linear self-adjoint unbounded operator and corresponds to the inverse of the Euclidean propagator of fermions; it is given by the Yukawa coupling matrix which encodes the masses of the elementary fermions and the Kobayashi–Maskawa mixing parameters.

The spectral geometry is then given by the product rules:

$$\mathcal{A} = C^\infty(\mathcal{M}) \oplus \mathcal{A}_{\mathcal{F}} \quad , \quad \mathcal{H} = L^2(\mathcal{M}, S) \oplus \mathcal{H}_{\mathcal{F}} \quad , \quad \mathcal{D} = \mathcal{D}_{\mathcal{M}} \oplus 1 + \gamma_5 \oplus \mathcal{D}_{\mathcal{F}} \quad ,$$

where $L^2(\mathcal{M}, S)$ is the Hilbert space of L^2 spinors and $\mathcal{D}_{\mathcal{M}}$ is the Dirac operator of the Levi-Civita spin connection on the four-dimensional manifold \mathcal{M} . Note that the chirality operator is $\gamma = \gamma_5 \oplus \gamma_{\mathcal{F}}$ and the anti-unitary operator on the complex Hilbert space is $J = J_{\mathcal{M}} \oplus J_{\mathcal{F}}$, with $J_{\mathcal{M}}$ the charge conjugation.

Since in the following we only consider the noncommutative discrete space \mathcal{F} , we omit the subscript \mathcal{F} to keep the notation lighter.

Assuming the algebra \mathcal{A} to be symplectic-unitary, it reads [17]

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) \quad , \quad (1)$$

with $k = 2a$; \mathbb{H} is the algebra of quaternions, which encodes the noncommutativity of the manifold. The first possible value for k is 2, corresponding to a Hilbert space of four fermions. This choice is however ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions in each of the three generations; the number of generations is a physical input in the theory. The model developed by Connes and his collaborators is the minimal one ($k = 4$) that can account for the Standard Model.

To obtain the NCSG action we apply the spectral action principle to the product geometry $\mathcal{M} \times \mathcal{F}$. Thus, the bare bosonic Euclidean action is simply

$$\text{Tr}(f(D_{\mathcal{A}}/\Lambda)) \quad , \quad (2)$$

where $D_{\mathcal{A}} = D + \mathcal{A} + \epsilon' J \mathcal{A} J^{-1}$ are uni-modular inner fluctuations, f is a cutoff function and Λ fixes the energy scale. This action can be seen *à la* Wilson as the bare action at the mass scale Λ . The fermionic term can be included in the action functional by adding $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space \mathcal{H} of the quarks and leptons.

Using heat kernel methods, the trace $\text{Tr}(f(D_{\mathcal{A}}/\Lambda))$ can be written in terms of the geometrical Seeley-de Witt coefficients a_n , which are known for any second order elliptic differential operator, as $\sum_{n=0}^{\infty} F_{4-n} \Lambda^{4-n} a_n$, where the function F is defined such that $F(\mathcal{D}_{\mathcal{A}}^2) = f(\mathcal{D}_{\mathcal{A}})$. To be more precise, the bosonic part of the spectral action can be expanded in powers of Λ in the form [18, 19]

$$\text{Tr} \left(f \left(\frac{\mathcal{D}_{\mathcal{A}}}{\Lambda} \right) \right) \sim \sum_{k \in \text{DimSp}} f_k \Lambda^k \int |\mathcal{D}_{\mathcal{A}}|^{-k} + f(0) \zeta_{\mathcal{D}_{\mathcal{A}}(0)} + \mathcal{O}(1) \quad , \quad (3)$$

where f_k are the momenta of the smooth even test (cutoff) function which decays fast at infinity:

$$\begin{aligned} f_0 &\equiv f(0) \quad , \\ f_k &\equiv \int_0^\infty f(u) u^{k-1} du \quad , \quad \text{for } k > 0 \quad , \\ f_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0) \quad . \end{aligned}$$

The noncommutative integration is defined in terms of residues of zeta functions $\zeta_{\mathcal{D}_A}(s) = \text{Tr}(|\mathcal{D}_A|^{-s})$ at poles of the zeta function and the sum is over points in the dimension spectrum of the spectral triple. For a four-dimensional Riemannian geometry, the trace $\text{Tr}(f(\mathcal{D}_A/\Lambda))$ can be expressed perturbatively as [20]–[21]

$$\text{Tr}(f(\mathcal{D}_A/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots . \quad (4)$$

Since the Taylor expansion of the cutoff function vanishes at zero, the asymptotic expansion of Eq. (4) reduces to

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 . \quad (5)$$

Hence, the cutoff function f plays a rôle only through its three momenta f_0, f_2, f_4 , which are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively. More precisely, the first term in Eq. (5) which is in Λ^4 gives a cosmological term, the second one which is in Λ^2 gives the Einstein-Hilbert action functional, and the third one which is Λ -independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric.

As it has been shown in Ref. [3], the NCSG summarised above, offers a purely geometric approach to the SM of particle physics, where the fermions provide the Hilbert space of a spectral triple for the algebra and the bosons are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry. More precisely, the computation of the asymptotic expression for the spectral action functional results to the full Lagrangian for the Standard Model minimally coupled to gravity, with neutrino mixing and Majorana mass terms; supersymmetric extensions have been also considered [22].

3. Phenomenological consequences

We will briefly review the phenomenological consequences of the NCSG model. We assume that the function f is well approximated by the cutoff function and ignore higher order terms. Normalisation of the kinetic terms implies

$$\frac{g_3^2 f_0}{2\pi^2} = \frac{1}{4} \quad \text{and} \quad g_3^2 = g_2^2 = \frac{5}{3} g_1^2 ,$$

which coincide with the relations obtained in GUTs, while

$$\sin^2 \theta_W = \frac{3}{8} , \quad (6)$$

which is also found for the $SU(5)$ and $SO(10)$ groups. Assuming the validity of the big desert hypothesis, the running of the couplings $\alpha_i = g_i^2/(4\pi)$ with $i = 1, 2, 3$ is obtained via the RGE.

Considering only one-loop corrections, hence the β -functions are $\beta_{g_i} = (4\pi)^{-2} b_i g_i^3$ with $i = 1, 2, 3$ and $b = (41/6, -19/6, -7)$, it was shown [3] that the gauge couplings and the Newton constant do not meet at a point, the error being within just few percent. The lack of a unification scale implies that the big desert hypothesis is only approximately valid and new physics are expected between the unification and today's energy scales. Phrasing it differently, the lack of a unique unification scale implies that even though the function f can be approximated by the cutoff function, there exist small deviations. On the positive side, the NCSG model leads to the correct representations of the fermions with respect to the gauge group of the SM, the gauge bosons appear as inner fluctuations along the continuous directions while the Higgs doublet appears as part of the inner fluctuations of the metric. Spontaneous Symmetry Breaking mechanism for the electroweak symmetry arises naturally with the negative mass term without any tuning. The see-saw mechanism is obtained and the 16 fundamental fermions (the number

of states on the Hilbert space) are recovered. At unification scale $\Lambda \sim 1.1 \times 10^{17}$ GeV, with $g \sim 0.517$, the Renormalisation Group Equations (RGE) lead to a top quark mass of ~ 179 GeV.

However, in zeroth order, the model predicts a heavy Higgs mass of ~ 170 GeV, which is ruled out by current experimental data. One has though to keep in mind that the Higgs mass is sensitive to the value of unification scale, as well as to deviations of the spectral function from the cutoff function we have considered. Hence, the Higgs mass should be determined by considering higher order corrections and incorporating them to the appropriate RGE. One may argue that the reason for which the top quark mass is consistent with experimental data while the predicted Higgs mass is ruled out, is simply because the top quark mass is less sensitive to the ambiguities of the unification scale than the Higgs mass is. This may indeed be the case since the bosonic part of the action is given by an infinite expansion assuming convergence of higher order terms.

In a more drastic modification of the NCSG model at hand, one may consider a bigger algebra than the one considered so far, which was chosen so that it leads to the Standard Model particles. Thus, one may argue that the discrepancy between the predicted Higgs mass and the experimental constraints may be resolved by considering models beyond the SM. One may construct [23] such a model based on a minimal spectral triple which contains the SM particles, but it has also new vector-like fermions and a new U(1) gauge subgroup. In the model presented in Ref. [23] it also appears a new complex scalar field that couples to the right-handed neutrino, the new fermions and the standard Higgs field. For the case of a nonzero vacuum expectation value the new scalar and the Higgs field mix and the mass eigenstates may consist of a light scalar particle with $m_{H_1} \sim 120$ GeV and a heavy particle with $m_{H_2} \leq 170$ GeV.

More recently, Chamseddine and Connes have argued [24] that including a real scalar singlet, strongly coupled to the Higgs doublet, they can accommodate a Higgs mass of order 125 GeV. This singlet field is associated with the Majorana mass of the right-handed neutrino. Even though this singlet is responsible for the breakdown of the symmetry of the discrete space, it has been neglected in the original calculations [3] of the phenomenological consequences of the spectral action. Note that as we have shown in Ref. [7] this scalar singlet cannot play the rôle of the inflaton field and, at the same time, provide the seeds of temperature anisotropies.

Nevertheless, it is fair to say that even the simplest version of NCSG predicts a Higgs mass of the correct order of magnitude, which is certainly a nontrivial result. Finally, let me note that this approach to unification does not provide any explanation of the number of generations, nor leads to constraints on the values of the Yukawa couplings.

4. Cosmological consequences

Since the NCSG gravitational model lives by construction at high energy scales, it provides a natural framework to investigate early universe cosmology [4]-[11]. We will review some of these consequences in what follows.

Within NCSG it is natural to obtain the bosonic action in Euclidean signature. It reads [3]

$$\begin{aligned} \mathcal{S}^E = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \gamma_0 + \tau_0 R^* R^* + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} \right. \\ \left. + \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \frac{1}{2} |D_\mu \mathbf{H}|^2 - \mu_0^2 |\mathbf{H}|^2 - \xi_0 R |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \right) \sqrt{g} d^4x, \end{aligned} \quad (7)$$

where

$$\begin{aligned}\kappa_0^2 &= \frac{12\pi^2}{96f_2\Lambda^2 - f_0\mathfrak{c}} \quad , \quad \alpha_0 = -\frac{3f_0}{10\pi^2} \quad , \\ \gamma_0 &= \frac{1}{\pi^2} \left(48f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d} \right) \quad , \quad \tau_0 = \frac{11f_0}{60\pi^2} \quad , \\ \mu_0^2 &= 2\Lambda^2 \frac{f_2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \quad , \quad \xi_0 = \frac{1}{12} \quad , \quad \lambda_0 = \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2} ;\end{aligned}\tag{8}$$

\mathbf{H} is a rescaling $\mathbf{H} = (\sqrt{af_0}/\pi)\phi$ of the Higgs field ϕ to normalise the kinetic energy. The geometric parameters $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ correspond to the Yukawa parameters (which run with the RGE) of the particle physics model and the Majorana terms for the right-handed neutrinos. Extrapolations to lower energy scales are possible through RGE analysis, however at low (today's) energy scales nonperturbative effects can no longer be neglected. Thus, any results based on the asymptotic expansion and on RGE analysis can only be valid for early universe cosmology. Note also that the relations in Eq. (8) are tied to the scale of the asymptotic expansion; there is no reason for these constraints (boundary conditions) to hold at scales below the unification scale Λ .

To apply the NCSG action in cosmology we must express it in Lorentzian signature. Assuming that one can perform a Wick rotation in imaginary time, the Lorentzian version of the gravitational part of the asymptotic expression for the bosonic sector of the NCSG action reads [3]

$$\mathcal{S}_{\text{grav}}^{\text{L}} = \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tau_0 R^* R^* \xi_0 R |\mathbf{H}|^2 \right) \sqrt{-g} \, d^4x \quad ,\tag{9}$$

leading to the equations of motion [4]:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{1}{B^2}\delta_{\text{cc}} \left[2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} + C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = \kappa_0^2 \delta_{\text{cc}} T_{\text{matter}}^{\mu\nu} \quad ,$$

where $B^2 \equiv -(4\kappa_0^2\alpha_0)^{-1}$. The nonminimal coupling between the Higgs field and the Ricci curvature scalar is captured by the parameter δ_{cc} , defined by $\delta_{\text{cc}} \equiv [1 - 2\kappa_0^2\xi_0\mathbf{H}^2]^{-1}$.

In the low energy weak curvature regime, the nonminimal coupling between the background geometry and the Higgs field can be neglected, leading to $\delta_{\text{cc}} = 1$. In this regime, noncommutative corrections do not occur at the level of a Friedmann-Lemaître-Robertson-Walker (FRLW) background, since in this case the modified Friedmann equation reduces to its standard form [4]. Any modifications to the background equation will be apparent at leading order only for anisotropic and inhomogeneous models. For instance, consider the Bianchi type-V model, for which the space-time metric, in Cartesian coordinates, reads

$$g_{\mu\nu} = \text{diag} \left[-1, \{a_1(t)\}^2 e^{-2nz}, \{a_2(t)\}^2 e^{-2nz}, \{a_3(t)\}^2 \right] \quad ;\tag{10}$$

$a_i(t)$ with $i = 1, 2, 3$ are arbitrary functions, denoting the scale factors in the three spatial coordinates and n is an integer. Defining $A_i(t) = \ln a_i(t)$ with $i = 1, 2, 3$, the modified Friedmann

equation reads [4]:

$$\begin{aligned}
 \kappa_0^2 T_{00} = & -\dot{A}_3 \left(\dot{A}_1 + \dot{A}_2 \right) - n^2 e^{-2A_3} \left(\dot{A}_1 \dot{A}_2 - 3 \right) \\
 & + \frac{8\alpha_0 \kappa_0^2 n^2}{3} e^{-2A_3} \left[5 \left(\dot{A}_1 \right)^2 + 5 \left(\dot{A}_2 \right)^2 - \left(\dot{A}_3 \right)^2 - \dot{A}_1 \dot{A}_2 - \dot{A}_2 \dot{A}_3 - \dot{A}_3 \dot{A}_1 - \ddot{A}_1 - \ddot{A}_2 - \ddot{A}_3 + 3 \right] \\
 & - \frac{4\alpha_0 \kappa_0^2}{3} \sum_i \left\{ \dot{A}_1 \dot{A}_2 \dot{A}_3 \dot{A}_i + \dot{A}_i \dot{A}_{i+1} \left(\left(\dot{A}_i - \dot{A}_{i+1} \right)^2 - \dot{A}_i \dot{A}_{i+1} \right) \right. \\
 & + \left(\ddot{A}_i + \left(\dot{A}_i \right)^2 \right) \left[-\ddot{A}_i - \left(\dot{A}_i \right)^2 + \frac{1}{2} \left(\ddot{A}_{i+1} + \ddot{A}_{i+2} \right) + \frac{1}{2} \left(\left(\dot{A}_{i+1} \right)^2 + \left(\dot{A}_{i+2} \right)^2 \right) \right] \\
 & \left. + \left[\ddot{A}_i + 3\dot{A}_i \ddot{A}_i - \left(\ddot{A}_i + \left(\dot{A}_i \right)^2 \right) \left(\dot{A}_i - \dot{A}_{i+1} - \dot{A}_{i+2} \right) \right] \left[2\dot{A}_i - \dot{A}_{i+1} - \dot{A}_{i+2} \right] \right\} \quad (11)
 \end{aligned}$$

with $i = 1, 2, 3$; the t -dependence of the terms has been omitted for simplicity. Any term containing α_0 in Eq. (11) encodes a modification from the conventional case. The correction terms can be divided into two types. The first one contains the terms in braces in Eq. (11), which are fourth order in time derivatives and hence, for the slowly varying functions that are usually considered in cosmology, these corrections can be neglected. The second type, which appears in the third line in Eq. (11), occurs at the same order as the standard Einstein-Hilbert terms. However, since this correction term is proportional to n^2 , it vanishes for homogeneous versions of Bianchi type-V. In conclusion, the corrections to Einstein's equations can only be important for inhomogeneous and anisotropic space-times [4].

The coupling between the Higgs field and the background geometry can no longer be neglected once we reach energies of the Higgs scale. In this case, the nonminimal coupling of Higgs field to curvature leads to corrections to Einstein's equations even for homogeneous and isotropic cosmological models. To keep the analysis simpler, let us neglect the conformal term in Eq. (10), so that the equations of motion are [4]

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa_0^2 \left[\frac{1}{1 - \kappa_0^2 |\mathbf{H}|^2 / 6} \right] T_{\text{matter}}^{\mu\nu} . \quad (12)$$

Thus, $|\mathbf{H}|$ plays the rôle of an effective gravitational constant [4]. Alternatively, the nonminimal coupling of the Higgs field to the curvature can be seen, for static geometries, as leading to an increase of the Higgs mass [4]. In the presence of this nonminimal coupling, the cosmological model built upon the NCSG action share some similarities with chameleon gravity and dilatonic cosmology [4].

The nonminimal coupling between the Higgs field and the Ricci curvature may turn out to be crucial in early universe cosmology [5, 7]. Such a coupling has been introduced *ad hoc* in the literature, in an attempt to drive inflation through the Higgs field. However, the value of this coupling should be dictated by an underlying theory and cannot be tuned by hand for a purely phenomenological convenience. Actually, even if classically the coupling between the Higgs field and the Ricci curvature could be set equal to zero, a nonminimal coupling will be induced once quantum corrections in the classical field theory are taken into account. A large coupling between the Higgs field and the background geometry is plagued by pathologies [25, 26], but this is not the case for a small coupling, as in the NCSG case. It is therefore worth investigating whether the Higgs field could play the rôle of the inflaton within the NCSG context.

In a FLRW metric, the Gravity-Higgs sector of the asymptotic expansion of the spectral

action, in Lorentzian signature, reads

$$S_{\text{GH}}^{\text{L}} = \int \left[\frac{1 - 2\kappa_0^2 \xi_0 H^2}{2\kappa_0^2} R - \frac{1}{2} (\nabla H)^2 - V(H) \right] \sqrt{-g} d^4x , \quad (13)$$

where

$$V(H) = \lambda_0 H^4 - \mu_0^2 H^2 , \quad (14)$$

with μ_0 and λ_0 subject to radiative corrections as functions of energy. For large enough values of the Higgs field, the renormalised value of μ_0 and λ_0 must be calculated. At high energies the mass term in Eq. (14) is sub-dominant and can be neglected. It has been shown [7] that for each value of the top quark mass there is a value of the Higgs mass where the effective potential is about to develop a metastable minimum at large values of the Higgs field and the Higgs potential is locally flattened. Calculating [7] the renormalisation of the Higgs self-coupling up to two-loops, we have constructed an effective potential which fits the renormalisation group improved potential around the flat region. The analytic fit to the Higgs potential around the minimum of the potential is [7]:

$$V^{\text{eff}} = \lambda_0^{\text{eff}}(H) H^4 = [a \ln^2(b\kappa H) + c] H^4 , \quad (15)$$

where the parameters a, b are related to the low energy values of top quark mass m_t as [7]

$$\begin{aligned} a(m_t) &= 4.04704 \times 10^{-3} - 4.41909 \times 10^{-5} \left(\frac{m_t}{\text{GeV}} \right) + 1.24732 \times 10^{-7} \left(\frac{m_t}{\text{GeV}} \right)^2 , \\ b(m_t) &= \exp \left[-0.979261 \left(\frac{m_t}{\text{GeV}} - 172.051 \right) \right] . \end{aligned} \quad (16)$$

The third parameter, c , encodes the appearance of an extremum and depends on the values for top quark mass and Higgs mass. The region where the potential is flat is narrow, thus to achieve a long enough period of quasi-exponential expansion, the slow-roll parameters, ϵ and η , must be slow enough to allow sufficient number of e-folds. In addition, the amplitude of density perturbations $\Delta_{\mathcal{R}}^2$ in the Cosmic Microwave Background (CMB) must be within the window allowed from the most recent experimental data. More precisely, inflation predicts that at horizon crossing (denoted by stars), the amplitude of density perturbations is related to the inflaton potential through $(V_*/\epsilon_*)^{1/4} = 2\sqrt{3\pi} m_{\text{Pl}} \Delta_{\mathcal{R}}^{1/2}$, where $\epsilon_* \leq 1$. Its value, as measured by WMAP7 [27], requires $(V_*/\epsilon_*)^{1/4} = (2.75 \pm 0.30) \times 10^{-2} m_{\text{Pl}}$, where m_{Pl} is the Planck mass.

Performing a systematic search in the parameter space using a Monte-Carlo chain, we have shown [7] that even though slow-roll inflation can be realised, the ratio of perturbation amplitudes is too large for any experimentally allowed values for the masses of the top quark and the Higgs boson. Note that running of the gravitational constant and corrections by considering the more appropriate de Sitter space-time, instead of the Minkowski geometry employed here, do not improve substantially the realisation of a successful slow-roll inflationary era [7].

Let us now proceed with the study of linear perturbations around a Minkowski background metric, which will allow us to constrain one of the three free parameters of the theory, namely the moment f_0 which is related to the coupling constants at unification. Note that we have to go beyond an FLRW space-time, since for a homogeneous and isotropic geometry the Weyl tensor vanishes, implying that the NCSG corrections to the Einstein equation vanish [4], rendering difficult to restrict B via cosmology or solar-system tests. It is worth noting that imposing a lower limit on B would imply an upper limit to the moment f_0 , and thus restrict particle physics at unification. To impose an upper limit to the moment f_0 , we will study the energy lost to gravitational radiation by orbiting binaries [8, 9]. Considering linear perturbations around a Minkowski background metric, the equations of motion read [9]

$$(\square - B^2) \square h^{\mu\nu} = B^2 \frac{16\pi G}{c^4} T_{\text{matter}}^{\mu\nu} , \quad (17)$$

where $T_{\text{matter}}^{\mu\nu}$ is taken to lowest order in $h^{\mu\nu}$. Since B plays the rôle of a mass, it must be real and positive, thus α_0 must be negative for Minkowski space to be a stable vacuum of the theory.

Consider the energy lost to gravitational radiation by orbiting binaries. In the far field limit, $|\mathbf{r}| \approx |\mathbf{r} - \mathbf{r}'|$ (\mathbf{r} and \mathbf{r}' denote the locations of observer and emitter, respectively), the spatial components of the general first order solution for a perturbation against a Minkowski background read [9]

$$h^{ik}(\mathbf{r}, t) \approx \frac{2GB}{3c^4} \int_{-\infty}^{t - \frac{1}{c}|\mathbf{r}|} \frac{dt'}{\sqrt{c^2(t-t')^2 - |\mathbf{r}|^2}} \mathcal{J}_1 \left(B \sqrt{c^2(t-t')^2 - |\mathbf{r}|^2} \right) \ddot{D}^{ik}(t') ; \quad (18)$$

D^{ik} is the quadrupole moment, defined as $D^{ik}(t) \equiv \frac{3}{c^2} \int x^i x^k T^{00}(\mathbf{r}, t) d\mathbf{r}$, and \mathcal{J}_1 a Bessel function of the first kind. In the $B \rightarrow \infty$ limit, the NCSG gravitational theory reduces to the standard General Relativity (GR), while for finite B gravity waves radiation contains massive and massless modes, which are both sourced from the quadrupole moment of the system.

Considering a binary pair of masses m_1, m_2 in circular (for simplicity) orbit in the (xy) -plane, the rate of energy loss is

$$-\frac{d\mathcal{E}}{dt} \approx \frac{c^2}{20G} |\mathbf{r}|^2 \dot{h}_{ij} \dot{h}^{ij} , \quad (19)$$

with [9]

$$\dot{h}^{ij} \dot{h}_{ij} = \frac{128\mu^2 |\rho|^4 \omega^6 G^2 B^2}{c^8} \times \left[f_c^2 \left(B|\mathbf{r}|, \frac{2\omega}{Bc} \right) + f_s^2 \left(B|\mathbf{r}|, \frac{2\omega}{Bc} \right) \right] , \quad (20)$$

$$f_s(x, z) \equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \sin \left(z \sqrt{s^2 + x^2} \right) , \quad (21)$$

$$f_c(x, z) \equiv \int_0^\infty \frac{ds}{\sqrt{s^2 + x^2}} \mathcal{J}_1(s) \cos \left(z \sqrt{s^2 + x^2} \right) . \quad (22)$$

The orbital frequency ω , defined in terms of the magnitude $|\rho|$ of the separation vector between the two bodies, is constant and equal to $\omega = |\rho|^{-3/2} \sqrt{G(m_1 + m_2)}$.

The integrals in Eqs. (21), (22), exhibit a strong resonance behavior at $z = 1$, which corresponds to the critical frequency [9]

$$2\omega_c = Bc , \quad (23)$$

around which strong deviations from the GR results are expected. This maximum frequency results from the natural length scale, given by B^{-1} , at which NCSG effects become dominant.

One can find in the literature several binary pulsars for which the rate of change of the orbital frequency is well-known and the predictions of GR agree with the data to a high accuracy. Since the magnitude of the NCSG deviations from GR must be less than the allowed uncertainty in the data, we are able to constrain [8] B , namely

$$B > 7.55 \times 10^{-13} \text{ m}^{-1} . \quad (24)$$

This constraint is not too strong but since it is obtained from systems with high orbital frequencies, future observations of rapidly orbiting binaries, relatively close to the Earth, could improve it by many orders of magnitude.

5. Conclusions

We have briefly described the prescription of Connes and collaborators in order to recover the action of the SM of particle physics from purely gravitational considerations. In this approach one recovers the Einstein plus Yang-Mills and Weyl actions including the spin-1 bosons and the part induced by the spin-0 Higgs fields. Thus, gravity and the electroweak and strong forces are described as purely gravitational forces on a unified noncommutative spacetime. The trick consists of employing spectral triples, consisting of an algebra (that is equivalent of a topological space), a Dirac operator (that corresponds to the metric on the topological space) and a Hilbert space of Dirac 4-spinors, on which the algebra is represented and on which the Dirac operator acts. In his approach, Connes combined spacetime with an internal space, composed by only 2 points, a construction that can be seen as a discrete Kaluza-Klein space where the product manifold of spacetime with extra spatial dimensions is replaced by the product of spacetime with discrete spaces represented by matrices.

The phenomenological consequences of this gravitational theory are in a very good agreement with current data, however the predicted Higgs mass seems to be ruled out. The reason for this discrepancy may be due to the ambiguities of the unification scale, the approximation of the spectral function by a cutoff one, or the assumption of convergence of the higher order terms in the infinite expansion of the bosonic part of the NCSG action. A more drastic approach is to consider larger algebras which predict particles beyond the SM sector. Even though these directions deserve further investigations, it is clear that the NCSG approach offers a beautiful explanation of the SM, the most successful particle physics model we have at hand, providing in addition a geometric explanation for the Higgs field which is otherwise introduced by hand.

The NCSG action offers a natural framework to build a cosmological model, which may allow us to explain some open questions in cosmology, as for instance through investigations on the rôle of scalar fields which arise naturally in NCSG and they do not have to be introduced by hand. Moreover, comparison of NCSG predictions against astrophysical data may allow us to constrain the free parameters of the theory.

Even though, as we have shown, the Higgs field can play the rôle of the inflaton field, in terms of providing a slow-roll period of fast expansion, it cannot provide the seed of perturbations leading to the observed CMB anisotropies and the large-scale structure. If one wants to decouple the inflaton from the curvaton field, then one has to provide such a field with the appropriate potential. In this sense, one may investigate the rôle of other scalar fields within the NCSG model. It is particularly encouraging that by studying gravity waves propagation we were able to constrain one of the free parameters of the theory, namely the one related to the coupling constants at unification.

In conclusion, NCSG offers a fruitful interplay between mathematics, gravitational theories, particle physics and cosmology, tracing another approach to the goal of unification.

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