Extending the Relativity of Time

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Abstract. More than 100 years ago, Einstein's special relativity demonstrated that time is a relative notion. The observed rate of a moving clock differs from the rate of a stationary clock. In fact, the observed rate depends on the clock's velocity. All admissible velocities are bounded by the speed of light. These predictions of special relativity have been verified experimentally in several different ways. It is natural to ask whether acceleration also influences the observed rate of a moving clock in addition to the influence due to its velocity. Today, there are several existing experimental techniques to test whether acceleration influences the observed rate of a clock. We introduce here an extension of special relativity, which we call *extended relativity (ER)*, by assuming that acceleration effects the observed rate of a clock. We derive transformations between uniformly accelerated systems in ER. We show that ER predicts that there is a maximal acceleration. We obtain relativistic dynamics in ER. We show that Kundig's 1963 experiment indicates that acceleration does influence the rate of a clock, supporting the ER model and providing an estimate for the maximal acceleration. We will present an upcoming experiment which is designed to test whether acceleration influences the rate of a clock, and to determine the value of the maximal acceleration. A map for physics under ER will be presented. We will show how ER handles black-body radiation and some quantum properties of a Hydrogen-like atom.

1. Introduction

About 400 years ago, Galileo Galilei introduced the *Principle of Relativity*, which states that the laws of physics are the same in any inertial system. Let $(t, \mathbf{x})$ denote the space-time coordinates of an event A in an inertial frame K. It was clear in Galileo's day that the spatial coordinate $\mathbf{x}'$ of A in a second inertial frame K' will depend on t, $\mathbf{x}$ and the relative velocity $\mathbf{v}$ between the K and K'. But, what about the time component t' of the event A in K'? At the time of Galileo, there was no evidence to suggest that $t' \neq t$. Moreover, there was no mathematical model in Galileo's day to handle such an assumption. It was therefore assumed that $t' = t$. The ensuing space-time transformations are called *Galilean transformations* and are the basis for classical physics.

About 120 years ago, the validity of the Galilean transformations came into question. The results of the Michelson-Morley experiment could not be explained using the velocity addition based on Galilean transformations. It was also shown that the Maxwell equations are not covariant with respect to these transformations, while they *are* covariant under different transformations, called the *Lorentz transformations*. Albert Einstein, based on the principle of relativity and the constancy of speed of light, developed *special relativity* in

which the transformation between inertial systems are Lorentz transformations. Under these transformations, the observed rate of a moving clock depends on its velocity. This is called the *time dilation* of a moving clock.

This raises the following question: "Does the observed rate of a moving clock also depend on its **acceleration**?" Currently, it is assumed that acceleration does not influence the observed rate of the clock. This is called the Clock Hypothesis or Clock Postulate. Till now this assumption was natural, as there was no physical evidence for this influence and no model to describe it. See a discussion of this question in article [19] "Does a clock's acceleration affect its timing rate?".

By the equivalence principle, gravitation can be interpreted as acceleration. In Einstein's General Relativity, gravitational fields are well described by the metric tensor on spacetime. Time dilation in general relativity is expressed by this metric. However, the time dilation of a clock due to gravitation cannot be observed, since both standard clocks and processes are affected in the same way. But, the difference in the time dilation between two different points could be observed and is called the gravitational redshift, see [33] pp.79-80. In general relativity the time dilation within an accelerated system with respect to an inertial lab system is assumed to be the same as in the instantaneous comoving inertial system. For uniformly accelerated systems, this is a part of the Hypothesis of Locality introduced by Mashhoon [25, 26] and of the Weak Hypothesis of Locality used in [18]. In these systems, the usual gravitational redshift occurs.

In this paper, we will show that there is a model for a relativity theory in which the observed time of a moving clock is also influenced by the acceleration of the clock. We will show that this theory implies the existence of a *maximal acceleration*. In special relativity, the time dilation can be measured by the transverse Doppler shift. Similarly, in this model, the additional time dilation predicted by this theory could be measured by the additional Doppler shift due to the acceleration of the clock. We will provide evidence supporting this extension of special relativity and also provide a numerical estimate of the value of the maximal acceleration. We call our extension of relativity, in which the observed time depends also on the acceleration, **Extended Relativity** (ER). We will describe feasible experiments to test ER and present a new map for physics under ER. Finally, we will show that some quantum effects can be understood within the framework of ER.

2. Kinematics of accelerated systems in ER

In this section, we will study accelerated systems within the the framework of Special Relativity. To study such accelerated systems, A. Einstein introduced the Clock Hypothesis, which states that the "rate of an accelerated clock is identical to that of the instantaneously comoving inertial clock." Not all physicists agree with this hypothesis. L. Brillouin ([6] p.66) wrote that "we do not know and should not guess what may happen to an accelerated clock." If we assume the validity of the Clock hypothesis, then the space-time transformation between accelerated systems are well known, see [28] and others.

In [14], we presented a systematic approach for transformations between accelerated systems *without* assuming the Clock Hypothesis. Our approach to describing transformations between two uniformly accelerated systems is based on the symmetry which follows from the general principle of relativity.

2.1. proper velocity - time description of events

The first step for describing transformations between two uniformly accelerated systems is to introduce a new *proper velocity - time description* of events. Proper velocity (*p-velocity* in short) is defined as

$$\mathbf{u} = \gamma(v)\mathbf{v} = \frac{d\mathbf{r}}{d\tau},$$

where $\mathbf{v} = d\mathbf{r}/dt$, $\gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}$ and τ is the proper time of the moving object. The proper velocity is also the canonically conjugate variable to the position in the relativistic phase-space.

The relativistic *acceleration* $\mathbf{g}$, which appears in the relativistic dynamic equation, is defined (see [32] p.71) to be the derivative of p-velocity with respect to time t:

$$\mathbf{g} = \frac{d\mathbf{u}}{dt} . \tag{1}$$

This acceleration coincides [18] with the acceleration in the comoving frame. Note that if an object moves with this constant acceleration, then its p-velocity satisfies the equation

$$\frac{d^2\mathbf{u}}{dt^2} = 0 . \tag{2}$$

We will say that an object is *uniformly accelerated* if its acceleration is constant, or equivalently, satisfies (2). If the velocity of a uniformly accelerated object is parallel to the acceleration, then it moves with the well-known *hyperbolic motion* (see [28], [32] and [9]).

In the *p-velocity-time description*, an event is described by the time at which the event occurred and the p-velocity $\mathbf{u} \in R^3$ of the event. The evolution of an object in a system can be described by the p-velocity $\mathbf{u}(t)$ of the object at time t. The line $(t, \mathbf{u}(t))$ replaces the world-line of special relativity in this description. To obtain the position of the object at time t, we have to know the initial position of the object and then integrate its ordinary velocity (which is readily computed from the p-velocity) with respect to time. The following Table 1 shows the parallels between the space-time transformations for inertial systems and the p-velocity-time transformations between uniformly accelerated systems.

Table 1. Parallelism between the space-time transformations for inertial systems and the p-velocity-time transformations between uniformly accelerated systems.

Systems	Relative motion	Relative motion eqn.	Event descr.
inertial	uniform velocity	$\frac{d^2\mathbf{r}}{dt^2} = 0$	$(t, \mathbf{r})$
accelerated	constant acceleration	$\frac{d^2\mathbf{u}}{dt^2} = 0$	$(t, \mathbf{u})$

To obtain the Lorentz transformations in special relativity, it is essential that the relative position of the origins of the frames connected with the two inertial systems depends linearly on time. This linear map expresses the relative velocity between the systems. For two uniformly accelerated systems, if we assume that the systems are comoving, meaning having zero relative velocity at time $t = 0$, then the uniform acceleration between the systems, defined by (1), is a linear map from time to p-velocities.

Let T denote the transformation mapping the time and p-velocity (t, u) of an event in a uniformly accelerated system K_g to the time and p-velocity (t', u') of the same event measured in the uniformly accelerated system K_0. The situation is analogous to that of the space-time transformations between two inertial systems. In that case, the relative motion of one system with respect to the other is described by a uniform velocity, which is a linear map from time to space (or a line in the space-time continuum). For uniformly accelerated systems, the relative motion of one system with respect to the other is described by a uniform *acceleration*, which is a linear map from time to p-velocities (or a line in the p-velocity-time continuum). Since the space-time transformation between two inertial systems is linear, we will assume that the p-velocity-time transformation T between two uniformly accelerated systems is also *linear*. This

assumption can be also justified by use of relativistic dynamics, as follows. Uniformly accelerated motion in an inertial system is described by a straight line in the p-velocity-time continuum and corresponds to a relativistic motion under a constant force. Any straight line in the p-velocity-time continuum in a uniformly accelerated system corresponds to relativistic motion under a constant force in this system and in any other uniformly accelerated system. Hence, the transformation T maps lines to lines and is, therefore, linear.

2.2. General proper velocity - time transformations between accelerated systems

To define the symmetry operator between two uniformly accelerated systems, we will use an extension of the principle of relativity, which we will call the *General Principle of Relativity*. This principle, as formulated by M. Born (see [4], p. 312), states that the "laws of physics involve only relative positions and motions of bodies. From this it follows that no system of reference may be favored *a priori* as the inertial systems were favored in special relativity." The principle of relativity from special relativity states that there is no preferred *inertial* system, and, therefore, the notion of rest (zero velocity) is a relative notion. From the general principle of relativity, it follows that there is no preference for inertial (zero acceleration) systems. Hence, when considering accelerated systems, we no longer give preference to free motion (zero force) over constant force motion. This makes all uniformly accelerated systems equivalent.

From the general principle of relativity, it is logical to assume that *the transformations between the descriptions of an event in two uniformly accelerated systems depend only on the relative motion between these systems.* Consider now two uniformly accelerated systems K_g and K_0, with a constant acceleration g between them. We choose reference frames in such a way that the description of relative motion of K_g with respect to K_0 coincides with the description of relative motion of K_0 with respect to K_g. The above principle implies that the transformation T mapping the description of an event in system K_g to the description of the same event in system K_0 will coincide with the transformation $\widetilde{T} = T^{-1}$ from system K_0 to K_g. This implies that T is a symmetry, or $T^2 = Id$.

The following derivation of the explicit form of the transformations between K_0 and K_g follows [10], where it is done for inertial systems. Similar derivations were obtained by several authors. The choice of the reference frames is as follows. We choose the origins O of K_g and O' of K_0 of the p-velocity axes to be the same at $t = 0$. We also synchronize the clocks positioned at the origins of the frames at time $t = 0$. We chose the direction of the first p-velocity axis in each system in such a way that the relative acceleration of the second system will be opposite to the direction of this axis. The corresponding axis will thus be reversed, as in Figure 1. The other two axes are chosen to be parallel.
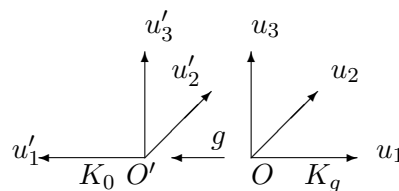

Figure 1. Symmetric uniformly accelerated systems

Note that with this choice of the axes, the acceleration g of O' in K_g is equal to the acceleration of O in K_0, and, thus, the p-velocity-time transformation problem is fully symmetric with respect to K_g and K_0. We will denote this transformation by S_g, since it is a symmetry and depends only on the acceleration **g** between the systems. By our choice of the p-velocity axes, the systems are symmetric with respect to the second and third coordinates. We may thus assume $u'_2 = u_2$

and $u'_3 = u_3$. For the time and the first p-velocity coordinates, S_g has the form

$$
\begin{pmatrix} t' \\ u'_1 \end{pmatrix} = S_g \begin{pmatrix} t \\ u_1 \end{pmatrix} = \begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} \begin{pmatrix} t \\ u_1 \end{pmatrix}.
$$

The world-line of the origin O in K' is $(S_{00}t, S_{10}t)$. From (1), the relative acceleration of system K with respect to K' is $g = S_{10}/S_{00}$. If we denote $\tilde{\gamma} = S_{00}$, we get

$$
S_g = \tilde{\gamma} \begin{pmatrix} 1 & \kappa \\ g & \alpha \end{pmatrix}
$$

for some constants α, κ. Now, from the symmetry $S_g^2 = I$, we have $\alpha = -1$ and $\tilde{\gamma} = \frac{1}{\sqrt{1+\kappa g}}$. Thus,

$$
S_g = \frac{1}{\sqrt{1 + \kappa g}} \begin{pmatrix} 1 & \kappa \\ g & -1 \end{pmatrix}.
$$

Now we have two choices:

- If the observed time does not depend on the acceleration, then $\tilde{\gamma} = 1$ and $\kappa = 0$. This is true if the clock hypothesis is valid. In this case, the p-velocity-time transformations are Galilean.

- If the observed time depends on the acceleration, then $\tilde{\gamma} \neq 1$ and $\kappa \neq 0$. This is the assumption of extended relativity (ER).

To define the p-velocity-time transformations between accelerated systems under ER, it remains only to define the value of κ. To do this, we introduce a metric $\mathrm{diag}(\mu^2, -1, -1, -1)$ on p-velocity-time $(t, \mathbf{u})$. We are looking for a μ which will make the symmetry S_g *self-adjoint*. In order for S_g to be self-adjoint, we must have

$$
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot S_g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot S_g \begin{pmatrix} 0 \\ 1 \end{pmatrix},
$$

where $\cdot$ is the product corresponding to our metric. This identity yields $-g = \mu^2 \kappa$. A satisfactory μ exists if $\kappa < 0$.

With the above choice of μ, the self-adjoint symmetry S_g becomes an *isometry*. This means that the interval $d\tilde{s}^2 = \mu^2 dt^2 - |du|^2$ is conserved under the transformation S_g. Consider a uniformly accelerated particle whose acceleration a has magnitude $|a| = \mu$ in system K. The interval $d\tilde{s}^2 = 0$ in system K, and thus the interval is also zero in system K'. This implies that there is a unique acceleration magnitude, μ which is conserved between K and K'. From the generalized principle of relativity, this unique acceleration magnitude, which we will denote $\mu(g)$, can depend only on the magnitude of the relative acceleration g between the accelerated systems.

The following argument shows that the unique acceleration magnitude $\mu(g)$ is a universal constant. Consider three uniformly accelerated systems K, K' and K''. Assume that the relative acceleration of K' with respect to K is $\mathbf{g}$ and that the relative acceleration of K'' with respect to K' is also $\mathbf{g}$. From the above argument, an acceleration with magnitude $\mu(g)$ in K will have the same magnitude in K'. Similarly, an acceleration with magnitude $\mu(g)$ in K' will have the same magnitude in K''. Hence, an acceleration with magnitude $\mu(g)$ in K will have the same magnitude in K''. But, the relative acceleration between K and K'' is $2\mathbf{g}$. Thus, $\mu(g) = \mu(2g)$. Repeating this argument, we see that this unique acceleration magnitude is independent of the relative acceleration between the systems. Thus, we have shown that in ER (if the observed time also depends on the acceleration), there is a unique universal acceleration magnitude such that

any uniform acceleration with this magnitude in one uniformly accelerated system, will have the same magnitude in any other uniformly accelerated system. We will call it the **maximal acceleration** and denote its magnitude by a_m.

From the above arguments, $a_m = \mu$, which implies that $\kappa = -g/a_m^2$. Thus, the proper velocity-time transformation between two uniformly accelerated systems (for parallel axes, by replacing our previous u_1 with $-u_1$) in ER is

$$
\begin{aligned}
t' &= \tilde{\gamma}(t + gu_1/a_m^2) \\
u_1' &= \tilde{\gamma}(gt + u_1) \\
u_2' &= u_2 \\
u_2' &= u_2,
\end{aligned}
\tag{3}
$$

with

$$
\tilde{\gamma} = \frac{1}{\sqrt{1 - g^2/a_m^2}}.
\tag{4}
$$

This is a Lorentz-type transformation.

The existence of a maximal acceleration for massive objects has already been predicted by Caianiello, based on the time-energy uncertainty relation in quantum mechanics (see Caianiello [7], Papini and Wood [30] and Papini *et al.* [29] and references therein). Note that the acceleration used by Caianiello is the proper acceleration, defined as $\mathbf{g} = d\mathbf{u}/d\tau$ and not $\mathbf{g} = d\mathbf{u}/dt$, as defined in (1). In ER, the proper acceleration is unbounded, just as the proper velocity is unbounded. The estimate of the Caianiello maximal acceleration in Scarpetta [31] is $a_m = 5 \cdot 10^{50} g$, which is too large to have an effect in real physical processes. Boundedness of the proper acceleration excludes black-holes, since on the surface of a black-hole the proper acceleration has no bound. But in ER black-holes may exist, see end of next section for the meaning of the horizon in Schwarzschild universe.

The clock hypothesis has been tested and was found to be valid to great accuracy. See, for example, the experiment of Bailey J. *et al* [1] for measurements of the time dilation for muons. This paper claims that the experiment supports the validity of the clock hypothesis for accelerations $10^{18} g$. But the acceleration mentioned there is the proper acceleration and the magnitude of our acceleration in this experiment is significantly smaller then the claimed value, So, this experiment does not contradict the estimate that we will obtain later. For a long time, B. Mashhoon argued against the Clock Hypothesis and developed nonlocal transformations for accelerated observers (see the review article [24] and references therein). Our approach treats the problem differently.

3. Relativistic Dynamics in ER

Relativistic Dynamics in ER is a dynamics extending classical dynamics and preserving two limitations: the velocity $\mathbf{v} = d\mathbf{r}/dt$ is bounded by c, and the acceleration $\mathbf{g} = d\mathbf{u}/dt$ is bounded by a_m.

To derive the dynamics equations of ER, we will follow [13] and ideas which helped to change the first order dynamic system in classical mechanics to a corresponding system in special relativity. A first-order dynamic system in classical mechanics for the motion of an object of mass m under a force $\mathbf{F}$ can be written as

$$
\left\{
\begin{array}{l}
\frac{d\mathbf{x}}{dt} = \mathbf{v} \\
\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m}
\end{array}
\right\}.
\tag{5}
$$

Einstein's relativistic dynamics equation in special relativity is $\frac{dp}{dt} = F$, or $m_0 \frac{d\gamma(v)v}{dt} = F$, where m_0 is the rest-mass of the object [8]. Thus, in special relativity the corresponding dynamic

system is

$$
\left\{
\begin{array}{l}
\gamma \frac{d\mathbf{x}}{dt} = \mathbf{u} \\
\frac{d\mathbf{u}}{dt} = \frac{\mathbf{F}}{m_0}
\end{array}
\right\},
\tag{6}
$$

where $\mathbf{u}$ is the p-velocity.

In special relativity, the time dilation (due to the velocity) factor γ enters in the first equation. Since $\gamma(\mathbf{v}(\mathbf{u})) = \sqrt{1 + u^2/c^2}$, we can rewrite this system as

$$
\left\{
\begin{array}{l}
\frac{d\mathbf{x}}{dt} = \frac{\mathbf{u}}{\sqrt{1 + u^2/c^2}} \\
\frac{d\mathbf{u}}{dt} = \frac{\mathbf{F}}{m_0}
\end{array}
\right\}.
$$

Note that the first equation ensures that the magnitude of the velocity $\frac{d\mathbf{x}}{dt}$ during the evolution does not exceed c, as required in special relativity.

For the corresponding system in ER, in order to preserve the limitation of the acceleration $d\mathbf{u}/dt$ by a_m, we introduce the time dilation (due to the acceleration) factor $\tilde{\gamma}$, defined by (4), in the second equation. This leads to

$$
\left\{
\begin{array}{l}
\gamma \frac{d\mathbf{x}}{dt} = \mathbf{u} \\
\tilde{\gamma} \frac{d\mathbf{u}}{dt} = \frac{\mathbf{F}}{m_0}
\end{array}
\right\}.
$$

As above, we can rewrite this system as

$$
\left\{
\begin{array}{l}
\frac{d\mathbf{x}}{dt} = \frac{\mathbf{u}}{\sqrt{1 + |\mathbf{u}|^2/c^2}} \\
\frac{d\mathbf{u}}{dt} = \frac{\mathbf{F}}{\sqrt{m_0^2 + |\mathbf{F}|^2/a_m^2}}
\end{array}
\right\}.
\tag{7}
$$

We will call this system the *relativistic dynamics equation in ER*.

The classical Hamiltonian for motion under a conserved force $F(x)$ with potential $V(x)$ is $H_{cl}(x, p) = \frac{p^2}{2m} + V(x)$, where x is the object's position, and $p = mv$ is the object's momentum. The Hamiltonian is the energy of an object, written as a function on the *phase space* (x, p), and remains constant throughout the motion. We will change the phase space to (x, v), where v is the velocity of the object and change the Hamiltonian to *energy per unit mass*:

$$
H_{cl}(x, v) = \frac{v^2}{2} + V(x) = \int_0^v u\, du - \int_0^x a(y)\, dy,
\tag{8}
$$

where $V(x)$ is now the potential per unit mass, and $a(x) = F(x)/m$ is the acceleration at the point x. Note that the first integral, which represents the kinetic energy, depends on the velocity, while the second integral, which represents the potential energy, depends on a similar way on the acceleration. Under this modification, we can rewrite the system (5) as a Hamiltonian system

$$
\left\{
\begin{array}{l}
\frac{dx}{dt} = \frac{\partial H_{cl}}{\partial u} \\
\frac{du}{dt} = -\frac{\partial H_{cl}}{\partial x}.
\end{array}
\right\}.
\tag{9}
$$

Note that this Hamiltonian system is symmetric in x and u, as required by *Born Reciprocity* [5], which states that the "laws of nature are symmetrical with regard to space and momentum."

The symmetry becomes even more explicit in the case of the *classical harmonic oscillator*. For analysis of the harmonic oscillator, we use an inertial frame in which the attractive center is placed at rest at the origin. In this frame, the acceleration is $a(x) = -\frac{k}{m}x$. The motion of the oscillator is characterized by the *natural frequency* of the oscillator, defined as $\omega = \sqrt{\frac{k}{m}}$.

Substituting this into equation (8), we obtain the Hamiltonian for the classical harmonic oscillator:

$$H_{cl}(x, u) = \frac{u^2}{2} + \frac{(\omega x)^2}{2} = \int_0^u v\,dv + \int_0^{\omega x} y\,dy\,. \tag{10}$$

Note that both the kinetic energy and the potential energy are quadratic expressions in the variables u and ωx, respectively. Such expressions are natural in classical mechanics, which uses Euclidean geometry.

In special relativity, we have two candidates for the velocity component of the classical phase space: the usual velocity $v = dx/dt$ and the proper velocity $u = \gamma(v)v$. System (6) suggests using a phase space of (x, u), position and p-velocity. The Lagrangian formulation [20] of relativistic dynamics also implies that the canonical momentum in special relativity is $p = m_0 u$ and, for unit mass, should be u. Relativistic dynamics replaces the first term in (8) by the Einstein formula for the kinetic energy $E_k = mc^2 = m_0 c^2 \gamma$. Thus, using the connection between v and u, the Hamiltonian per unit rest-mass in special relativity is

$$H_{sr}(x, u) = c^2 \gamma(v(u)) + V(x) = c^2 \sqrt{1 + \frac{u^2}{c^2}} + V(x). \tag{11}$$

This Hamiltonian reproduces the dynamics equation (6) and can be derived from the relativistic Lagrangian.

Note that this Hamiltonian breaks the symmetry in x and u. This can be seen explicitly for the harmonic oscillator. The Hamiltonian for the harmonic oscillator in special relativity is

$$H_{sr}(x, u) = c^2 \sqrt{1 + \frac{u^2}{c^2}} + \frac{\omega^2 x^2}{2}\,. \tag{12}$$

The kinetic energy E_k is now expressed by a hyperbola $\frac{E_k^2}{c^4} - \frac{u^2}{c^2} = 1$, which is natural for special relativity, which uses hyperbolic geometry. The asymptotes of this hyperbola are $E_k = cu$ for $u \to \infty$ and $E_k = -cu$ for $u \to -\infty$. Nevertheless, the second term (the potential energy) is still a parabola, as in the classical case.

The broken symmetry and Born reciprocity is restored by the relativistic dynamics in ER. The relativistic dynamics equation (7) in ER suggests the following Hamiltonian:

$$H_{er}(\mathbf{x}, \mathbf{u}) = c^2 \sqrt{1 + |\mathbf{u}|^2/c^2} + U(\mathbf{x}), \tag{13}$$

with

$$\mathbf{a}(t) = \frac{d\mathbf{u}}{dt} = -\frac{\partial U(\mathbf{x})}{\partial x_j} = \frac{F_j(\mathbf{x})}{\sqrt{1 + |\mathbf{F}(\mathbf{x})|^2/(ma_m)^2}}\,. \tag{14}$$

For the harmonic oscillator, the Hamiltonian becomes

$$H_{er}(x, u) = c^2 \sqrt{1 + \frac{u^2}{c^2}} + \frac{a_m^2}{\omega^2} \sqrt{1 + \frac{\omega^4 x^2}{a_m^2}}\,. \tag{15}$$

This Hamiltonian has the same type of Born Reciprocity as in (10) for the classical harmonic oscillator. Note that the minimal energy of an oscillator $E_0 = m_0(c^2 + a_m^2/\omega^2)$ is chosen to be non-zero, in order to have a well-defined behavior at infinity for both kinetic and potential energies.

The relativistic dynamics equation (7) in ER can be rewritten as the usual Hamilton system:

$$\left\{ \begin{array}{l} \frac{d\mathbf{x}}{dt} = \frac{\partial H_{er}}{\partial \mathbf{u}} \\ \frac{d\mathbf{u}}{dt} = -\frac{\partial H_{er}}{\partial \mathbf{x}} \end{array} \right\}\,. \tag{16}$$

The solutions of equation (7) automatically satisfy the two limitations: the velocity $\mathbf{v} = d\mathbf{r}/dt$ is bounded by c, and the acceleration $\mathbf{g} = d\mathbf{u}/dt$ is bounded by a_m. The classical mechanics dynamics is obtained from this equation when $c \to \infty$ and $a_m \to \infty$, while the special relativity dynamics is obtained from this equation when $a_m \to \infty$.

If the mass of the moving object is positive, then its acceleration magnitude $|\mathbf{a}|$ is strictly less then a_m. On the other hand, any zero-mass particle, like a photon, accelerates with the maximal acceleration a_m. Note that in special relativity, a zero-mass particle has infinite acceleration, if this acceleration is defined properly [18]. This implies that the electromagnetic radiation which is generated by zero-mass particles is described in the p-velocity-time continuum by plane waves $f(\omega t - \tilde{\mathbf{k}}u)$, with $|\tilde{\mathbf{k}}|/\omega = 1/a_m$.

Consider now the freely falling object in a gravitational field outside a spherically symmetric mass M. If we use the classical description of the gravitation, the acceleration $\mathbf{g}$ due to the gravitation will be $\mathbf{g} = GM/r^2$, where r is the distance from the object to the center of the mass M. So, from (7) the dynamics equation for a freely falling object in ER will be

$$\frac{d\mathbf{u}}{dt} = \frac{\mathbf{g}}{\sqrt{1 + g^2/a_m^2}} \,.$$

In the Schwarzschild universe, the acceleration generated by the gravitation force is $\tilde{\mathbf{g}} = \mathbf{g}/\sqrt{1 - 2gr/c^2}$, see [32] p.230. So, from (7) the ER dynamics equation of a freely falling object in Schwarzschild universe is

$$\frac{d\mathbf{u}}{dt} = \frac{\tilde{\mathbf{g}}}{\sqrt{1 + \tilde{g}^2/a_m^2}} = \frac{g}{\sqrt{|1 - 2gr/c^2| + g^2/a_m^2}} \,.$$

This means that at the usual horizon $r = 2GM/c^2 = c^2/(2g)$ of this model, the magnitude of the acceleration is a_m, and $\tilde{\gamma}$, defined by (4), is infinite, implying that time stops at this horizon. As mentioned earlier, any massive object cannot reach the maximal acceleration in Minkowski space. But in the Schwarzschild universe each object at the horizon becomes light-like.

The largest accelerations observed in astronomy are the accelerations on neutron stars, where the acceleration is of order $10^{12} m/s^2$. Since the ER correction of the dynamics is of order $g^2/a_m^2 \leq 10^{-14}$ it will be hard to observe this correction in astronomy.

4. Testing Maximal Acceleration

4.1. Doppler type shift for an accelerated source

Extended relativity predicts an additional longitudinal Doppler shift of a source accelerating in the direction of the radiation. The observation of such a shift can be used to test extended relativity and to determine the value of the maximal acceleration.

Consider a source radiating at frequency ω with a "wave vector" $\tilde{\mathbf{k}}$ in the x-direction. We want to determine the frequency ω' of this radiation as observed in a system moving with acceleration $\mathbf{g}$ in the x-direction with respect to the source. By use of the p-velocity-time transformation (3) between these systems, we get

$$f(\omega t - \tilde{k}u) = f\left(\omega\tilde{\gamma}(t' + \frac{gu'}{a_m^2}) - \tilde{k}\tilde{\gamma}(gt' + u')\right) = f(\omega' t' - \tilde{k}' u').$$

Thus,

$$\omega' = \tilde{\gamma}(\omega - \tilde{k}g) = \omega \frac{1 - \frac{|\tilde{k}|}{\omega}g}{\sqrt{1 - \frac{g^2}{a_m^2}}} = \omega \frac{1 - \frac{g}{a_m}}{\sqrt{1 - \frac{g^2}{a_m^2}}}.$$

If $\frac{g}{a_m} \ll 1$, we get

$$\omega' = \omega\left(1 - \frac{g}{a_m}\right). \tag{17}$$

4.2. Kündig's experiment

After the discovery of the Mössbauer effect in 1958, quantitative measurements of relativistic time dilation were carried out in the 1960s based on this effect. The experiments reported full agreement with the time dilation predicted by Einstein's theory of relativity. In these experiments, the Mössbauer source was placed at the center of a fast rotating disk, and an absorber was placed at the rim of the disk. In the analyses of these experiments, it was assumed that the absorption line of the rotating absorber stays the same as at rest, and is only shifted by the time dilation factor. As it was shown in [15], this assumption is wrong.

Kündig's experiment (1963) [22] measured the transverse Doppler shift for a rotating disk by means of the Mössbauer effect, see Figure 2. In (only) this experiment, the absorption line of the
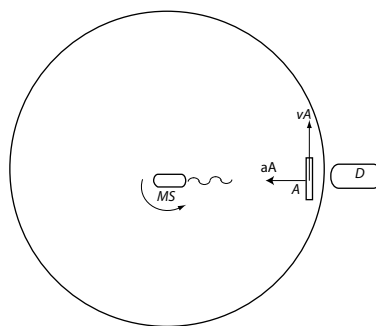
Figure 2. The rotating disk of the Kündig's experiment

rotating absorber was obtained by putting the source on a transducer. Thus, currently, Kündig's experiment is the only proper experiment to measure the transverse second order Doppler shift by Mössbauer spectroscopy using a rotating absorber.

For this experiment, Special relativity predicts a transverse Doppler effect of the value due to the time dilation of the absorber $\frac{\triangle E}{E} \approx -\frac{R^2\omega^2}{2c^2} = -b\frac{R^2\omega^2}{2c^2}$. The same value was obtained by Kündig analyzing the problem in the frame attached to the rotating disk and the accelerated absorber and treating the problem by the principle of equivalence and the general theory of relativity. The centrifugal force acting on the absorber is interpreted as a gravitational force with the potential $\Phi = -\frac{1}{2}R_A^2\omega^2$. The time of the absorber is slowed down by the gravitational potential in this analysis. Kündig in [22] claimed that the observed value of $b = 1.0065 \pm 0.011$, which is in full agreement with special relativity's prediction.

But Kholmetskii *et al* (2008) [21] found an error in the data processing of the results of Kündig's experiment. After the correction of the error, the correct value in this experiment is $b = 1.192 \pm 0.03$. Since the accuracy in the experiment was about 1%, this shows a significant deviation from special relativity's prediction. But, as we see from Figure 2, in this experiment there is also an acceleration between the source and the absorber. In [11], this deviation was explained by use of the additional longtitudal Doppler shift (17) due to the acceleration between the source and the absorber.

In Kündig's experiment, the acceleration $a = R\omega^2$ toward the source and $R = 9.3cm$. Hence, the additional shift with respect to that predicted by special relativity is $\frac{2c^2}{a_mR} \approx 0.192$ This gives an approximate value of $a_m = 10^{21}\frac{cm}{s^2}$ for the maximal acceleration. Notice that the calculated value of b is independent of the speed of rotation. This agrees approximately with the data in [21].

These observations lead us to the following conclusions:

- The observed rate of a clock also depends on the acceleration of the clock.

- The correcting time dilation, due to the acceleration, defines the value of the maximal acceleration to be $a_m \approx 10^{21} \frac{cm}{s^2}$.
- The value of the maximal acceleration is independent of the rotational velocity.

Note that Kholmetskii's correction of the result of the experiment can be interpreted as a change of the gravitational potential $\Phi = -\frac{1}{2}R_A^2\omega^2$ in the the frame attached to the disk to $\tilde{\Phi} = -\frac{1}{2}R_A^2\omega^2(1 + c^2\omega^2/a_m^2)$.

4.3. Future tests of maximal acceleration

An experiment is currently in process by the author and research teams from Hebrew University and Ben-Gurion University in Israel for testing ER and determining the value of the maximal acceleration. In the experiment, a Mössbauer source MS will be mounted, as usual, on a transducer. A semicircular absorber A will be placed on a disk of radius about 6 cm. The detector D will be diametrically opposed to the source, as shown in the Figure 3. Two collimators C will be placed between the source and the detector restricting the width of the measured radiation ray and insuring that the center of the disk is in this ray. The disk with the absorber will be rotated with a high-speed vibrationless spindle with several angular velocities ω, ranging up to 60,000 rpm. We will separate the counting of the detector for the times when the acceleration of the absorber is in the direction of the radiation, as in case (a) of Figure 3, and the times when the acceleration of the absorber is opposing the direction of the radiation, as in case (b) of Figure 3. Two absorption curves will be obtained for each case.
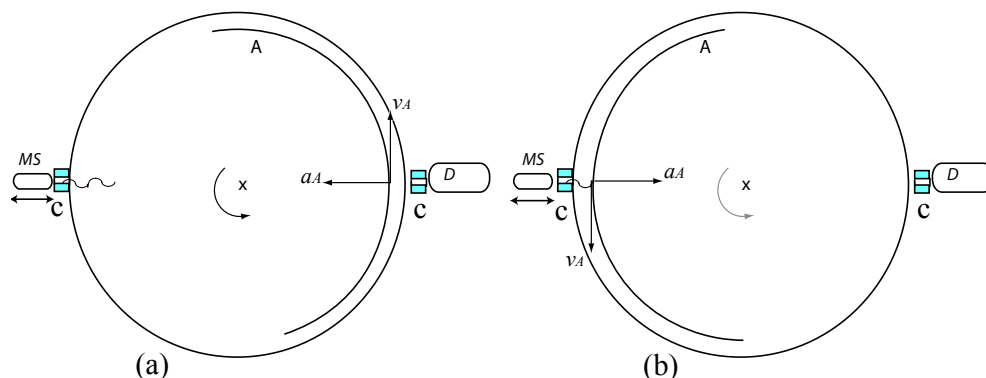

Figure 3. The proposed experiment, two positions of the absorber.

The radiation from the source MS undergoes two shifts, one due to the known transversal Doppler shift, and an expected second shift due to the acceleration of the absorber. The first shift should be the same for both cases, while the second shift changes sign from case (a) to (b). If the Clock Hypothesis is valid, no shift between the two absorption curves should be observed. On the other hand, if such a shift is observed and is significant, it will prove that the acceleration influences the observed rate of the clock and provide support for the validity of ER. Moreover, this experiment will enable us to calculate the value of the maximal acceleration.

Following is a list of other possible tests of ER and the maximal acceleration:

(i) Additional testing the Doppler shift due to the acceleration of a rotating disk by means of the Mössbauer effect

(ii) Muon-life experiment- improved (Bailey 77)- testing time dilation due to acceleration

(iii) Tests of the relativistic ER dynamics - Synchrotron

(iv) Tests of the relativistic ER dynamics by particle scattering

(v) Short-pulse laser acceleration-testing Doppler shift due to acceleration

(vi) Theoretical prediction of Quantum Mechanical effects based on ER

5. Extended relativity modification of physics

Extended relativity and the existence of a maximal acceleration change the entire map of physics. With the development of Einstein's theory of relativity and the development of quantum mechanics, the map of physics of the 20th century looks as follows. The areas of physics are determined by the magnitude of their velocities with respect to the maximal velocity (the speed of light c) and by their size. Classical mechanics is valid for large-sized objects and velocities significantly smaller than the speed of light. Relativistic dynamics is for large objects with speeds close to the speed of light. Quantum mechanics (non-relativistic) deals with small-sized objects with velocities significantly smaller than the speed of light, see Figure 4.
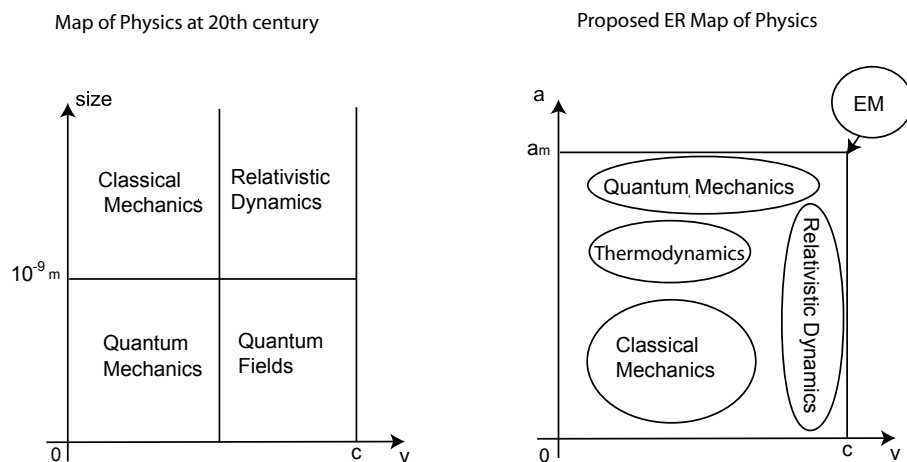

Figure 4. Two maps for physics

If the forthcoming experiments prove the validity of ER and the maximal acceleration turns out to be of the order of our current estimate, then the map of physics will need to be changed. First, we suggest to replace the size axis in the map by the acceleration. This will define a bounded domain of all ER admissible velocities and accelerations. This domain is the tangent domain to the relativistic phase-space. Classical mechanics is valid for both velocities and accelerations which are far from their maximal values. Relativistic dynamics is valid for velocities close to the speed of light and accelerations which are far from the maximal acceleration.

Quantum mechanics is not a model for small-sized objects, but for systems with accelerations close to the maximal one. For example, to describe the motion of electrons (small-sized objects) in a synchrotron, relativistic dynamics is used, and not quantum mechanics, since the velocities are close to the speed of light. On the other hand, the accelerations in the quantum region are

extremely high. In [13] we asked "Can ER incorporate *quantum* phenomena?" Some partial results will be presented in Subsection 5.2.

Thermodynamics in this map is positioned between classical and quantum mechanics. Obviously, the velocity of thermal vibrations is far from the speed of light. Because of the high frequency of these oscillations, their acceleration can get very large. Indeed, for vibrations with small values of the natural frequency ω, the classical model describes well the radiation curves. But for large ω, only by assuming Planck's hypothesis, can one obtain the experimentally derived radiation curves. Note also that this hypothesis plays an important role for quantum mechanics. In the next subsections, we show that ER can explain the differences in thermal vibrations for small and large frequencies. Moreover, in this model, there is no "ultra-violet catastrophe" ([3] p.255).

Finally, electromagnetic fields (EM) are composed of photons which are zero-mass particles. As explained at the end of Section 3, in ER such particles move with maximal acceleration and reach velocities close to the speed of light in a very short time after emission. Thus, we positioned electromagnetism in the corner of our domain.

5.1. Quantum-like behavior of the harmonic oscillator in extended relativity
Following [12], we will describe now the motion of the harmonic oscillator in ER. The ER dynamic system for the harmonic oscillator is

$$\left\{ \begin{array}{l} \frac{d\mathbf{x}}{dt} = \frac{\mathbf{u}}{\sqrt{1+|\mathbf{u}|^2/c^2}} \\ \frac{d\mathbf{u}}{dt} = \frac{\omega^2 \mathbf{x}}{\sqrt{1+\frac{\omega^4 x^2}{a_m^2}}} \end{array} \right\}. \tag{18}$$

The Hamiltonian of this oscillator was given in (15). The effective potential energy $V_{eff} = \frac{a_m^2}{\omega^2}\sqrt{1+\frac{\omega^4 x^2}{a_m^2}}$ is now expressed by a hyperbola $\frac{V_{eff}^2}{\alpha^4} - \frac{(\omega x)^2}{\alpha^2} = 1$, with $\alpha = a_m/\omega$. The asymptotes of this hyperbola are $V = a_m x$ for $x \to \infty$ and $V = -a_m x$ for $x \to -\infty$ and are independent of the natural frequency ω of the harmonic oscillator. We normalized the potential to have the same limit at infinity and not by setting the value to zero when $x = 0$, as it is done for the classical harmonic oscillator. The value $V_{eff}(0) = a_m^2/\omega^2$ of the potential at the origin tends to zero as ω becomes extremely large, as shown in Figure 5. This will happen in
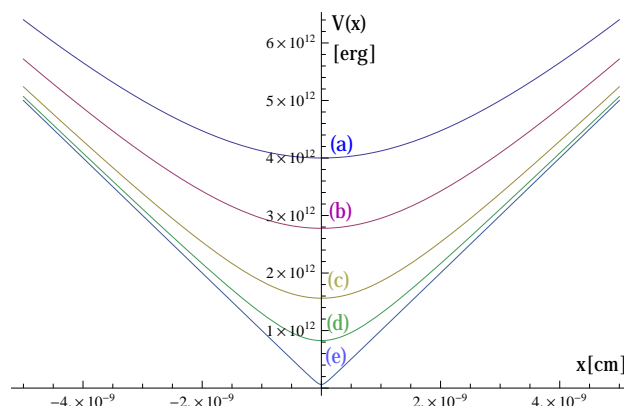

Figure 5. Effective potential $V_{eff}(x)$ for different ω :
(a)$5 \cdot 10^{14} s^{-1}$, (b)$6 \cdot 10^{14} s^{-1}$,(c)$8 \cdot 10^{14} s^{-1}$,(d)$11 \cdot 10^{14} s^{-1}$, (e)$40 \cdot 10^{14} s^{-1}$

the quantum region, where the ratio between the forces acting on the particles to their masses is extremely large.

From the Taylor expansion of V_{eff}, we see that if $\omega^4 x^2/a_m^2 \ll 1$, then $V_{eff} \approx \frac{\omega^2 x^2}{2} + const$. This is the classical potential (per unit mass) for the harmonic oscillator. When $\omega^4 x^2/a_m^2 \gg 1$, on the other hand, which occurs for extremely large ω, the potential energy of an ER harmonic oscillator is approximately

$$V_q(x) \approx a_m|x|\,,\tag{19}$$

where the subscript q means "quantum."

We will solve first the evolution equation of the ER harmonic oscillator with this potential. Let A denote the *amplitude* of the vibrations. Assume that at time $t = 0$, the position of the oscillator was at $x(0) = -A$. At this time, the velocity was $u(0) = 0$. Denoting the total energy by E, the free energy is

$$E - E_0 = m_0 V_q(x(0)) = m_0 a_m A\,.\tag{20}$$

To solve the evolution of motion with potential $V_q(x)$, we start with the second equation of (18):

$$a(t) = \frac{du}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V_q}{\partial x} = \begin{cases} a_m, & x < 0; \\ -a_m, & x > 0. \end{cases}\tag{21}$$

Thus, the acceleration $a(t)$ is a *square wave*, as shown in Figure 6. We interpret $a(t)$ as a *digitization* of the standard acceleration signal of the classical harmonic oscillator.
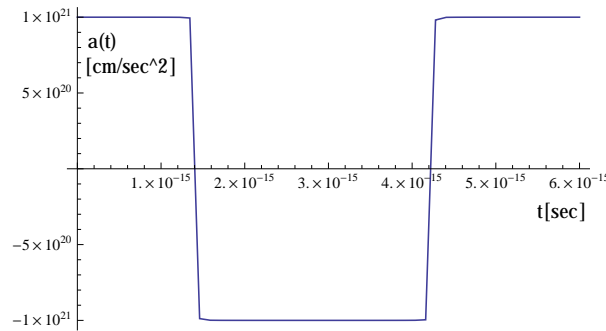

Figure 6. Square wave $a(t)$ for $A = 10^{-9}cm$

Note that this curve differs significantly from the acceleration curve of the classical harmonic oscillator. Since the radiation of the oscillating charge depends on its acceleration, the radiation in our model for extremely large ω will differ significantly from the radiation of a classical harmonic oscillator charge.

Equation (21) implies that $u(t) = a_m t + u_0$ for $x(t) < 0$, and $u(t) = -a_m t + u_0$ for $x(t) > 0$. After time $\frac{1}{4}T$, which is one fourth of the period T, the position of the oscillator will be $x(\frac{1}{4}T) = 0$. During the time interval $[0, \frac{1}{4}T]$, the proper velocity is $u(t) = a_m t$. Similarly, on the interval $[\frac{1}{4}T, \frac{3}{4}T]$, the proper velocity is $u(t) = a_m(\frac{1}{2}T - t)$. Figure 7 shows the function $u(t)$, which is called a *triangle wave*.

It is known (see for example: http://mathworld.wolfram.com /FourierSeriesTriangleWave) that the Fourier Series of a triangle wave $u(t)$ is

$$u(t) = \frac{2a_m T}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left(\frac{2\pi(2k+1)t}{T}\right).\tag{22}$$

We will call the frequency of the leading term in this decomposition the *effective frequency* and denote it by ω_e. Thus
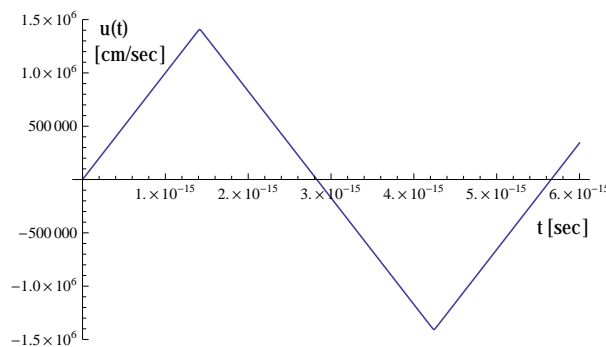
$$\omega_e = \frac{2\pi}{T},\tag{23}$$

Figure 7. Triangle wave $u(t)$ for $A = 10^{-9} cm$

and we can rewrite (22) as

$$u(t) = \frac{4a_m}{\pi \omega_e} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \sin\left((2k+1)\omega_e t\right) . \tag{24}$$

For the acceleration, one gets:

$$a(t) = \frac{du(t)}{dt} = \frac{4a_m}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \cos\left((2k+1)\omega_e t\right) . \tag{25}$$

This shows that the spectrum of these waves is supported at the points $\omega_e(2k+1)$, for $k = 0, 1, 2, \ldots$. This is similar to the spectrum of the quantum harmonic oscillator, which is known to be $\frac{\hbar \omega}{2}(2k+1)$, see [23] p.186. This indicates the connection of this new type of motion with the motion of the harmonic oscillator in the quantum region.

We use now the first Hamiltonian equation (9), which is also valid in ER, to calculate the displacement $x(t)$ during the time interval $[0, \frac{1}{4}T]$:

$$\frac{dx}{dt} = \frac{\partial H}{\partial u} = \frac{u(t)}{\sqrt{1 + u(t)^2/c^2}} = \frac{a_m t}{\sqrt{1 + (a_m t)^2/c^2}}. \tag{26}$$

Integrating this equation, we get $x(t) = \frac{c^2}{a_m}\left(\sqrt{1 + \frac{(a_m t)^2}{c^2}} - 1\right) - A$.

Note that this graph is close to the displacement of the classical harmonic oscillator. This is due to the fact that the velocities are well below the speed of light and, therefore, $x(t)$ approximately equals to the integral of $u(t)$. From (24), we remark that the contributions of the non-leading terms in the Fourier decomposition will be relatively small.

Substituting $x(\frac{1}{4}T) = 0$, we get $A = \frac{c^2}{a_m}\left(\sqrt{1 + \frac{(a_m T)^2}{16c^2}} - 1\right)$. From this, we can calculate the period T of oscillations:

$$T^2 = 16\frac{A^2}{c^2} + 32\frac{A}{a_m}. \tag{27}$$

The following Table 2 summarizes the behavior of the ER harmonic oscillator in the classical and quantum regions.

Note that the period T of the ER harmonic oscillator in the quantum region depends on the *amplitude* A of the motion and not on the natural frequency ω of the oscillator. This is in

Table 2. ER harmonic oscillator in two regions

ER oscillator.	classical region	quantum region
$a(t)$	$A\omega^2 \cos\omega t$	square wave (Figure 6)
$u(t)$	$A\omega \sin\omega t$	triangle wave (Figure 7)
$x(t)$	$-A\cos\omega t$	Figure 4
T	$2\pi/\omega$	$\sqrt{16\frac{A^2}{c^2} + 32\frac{A}{a_m}}$
$E - E_0$	$m_0 A^2 \omega^2/2$	$m_0 A a_m$
spectrum	$\{\omega\}$	$\frac{2\pi}{T}(2k+1) : k = 0, 1, 2, 3...$

contrast to the classical region, where T depends on ω and not on A. The period T can also be expressed in terms of the free energy by substituting $A = (E - E_0)/(m_0 a_m)$.

There is a difference in the energy spectrum of the quantum oscillator and the spectrum of the ER harmonic oscillator. The basic energy quanta in the quantum harmonic oscillator depend on the natural frequency of the corresponding classical harmonic oscillator. This makes it difficult to explain the fact that the blackbody spectrum is independent of the materials of which the walls are composed ([2] p. 20). On the other hand, the quanta of radiation of the ER harmonic oscillator depend only on the energy, which depends, in turn, on the temperature.

We have shown that the ER harmonic oscillator exhibits two significantly different behaviors. When ω is small, the oscillator behaves classically, while for large values of ω, the behavior is *quantum-like*. In the classical area, the spectrum of position, velocity, and acceleration oscillations consists of a single point ω, an inherent parameter of the oscillator. In the *quantum* area, on the other hand, the spectrum is similar to the spectrum of the energy of a *quantum harmonic oscillator*. In the *border area*, the classical signals become *digitized*. In Figure 8, we see the transition of the acceleration $a(t)$ from the classical to the quantum region.
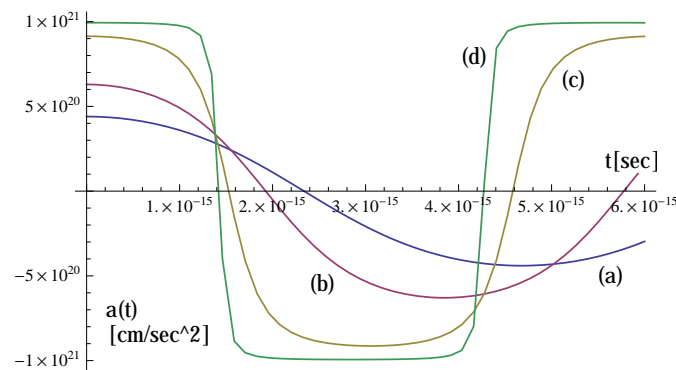

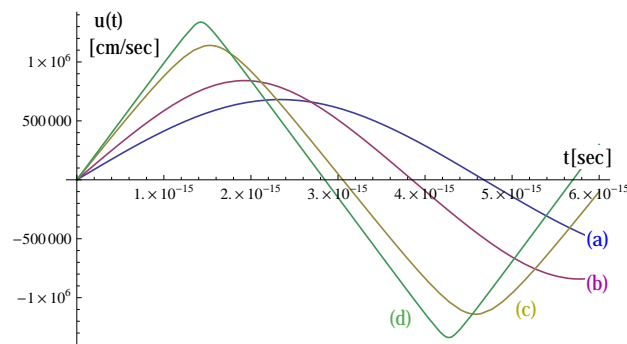**Figure 8.** Evolution of $a(t)$ from the classical to the quantum region for ω : (a) $7 \cdot 10^{14} s^{-1}$, (b) $9 \cdot 10^{14} s^{-1}$, (c) $15 \cdot 10^{14} s^{-1}$, (d) $30 \cdot 10^{14} s^{-1}$

Figure 9 features the transition of the velocity $u(t)$ from the classical to the quantum region.

In Figure 10 we show the effective oscillation frequency ω_e as a function of the natural oscillator frequency $\omega = \sqrt{k/m}$ for oscillations with amplitude $A = 10^{-9} cm$. For the classical harmonic oscillator $\omega_e = \omega$, while for the ER harmonic oscillator this identity holds for small ω, while for large ω, ω_e approaches some limiting frequency, which could be calculated from (23) and (27).

Figure 9. Evolution of $u(t)$ from the classical to the quantum region for ω : (a) $7 \cdot 10^{14} s^{-1}$,(b) $9 \cdot 10^{14} s^{-1}$, (c) $15 \cdot 10^{14} s^{-1}$, (d) $30 \cdot 10^{14} s^{-1}$
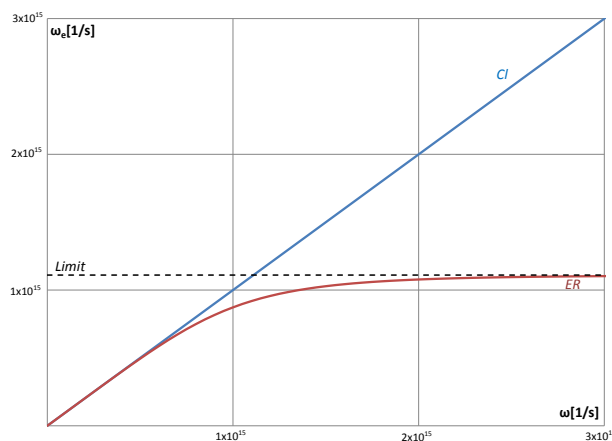

Figure 10. The effective oscillation frequency ω_e and the natural oscillator frequency ω for classical and ER harmonic oscillators

For an oscillator with a given ω, the signal will become digitized as the energy increases, which can be expressed by an increase of the amplitude A. In Figure 11, we show the transition of the acceleration $a(t)$ from the classical to the quantum region for an oscillator with $\omega = 10^{15} s^{-1}$, as the amplitude increases.

To understand the transition between the classical and quantum regions of the ER harmonic oscillator, we will compare it with the classical harmonic oscillator, as shown in Figure 12. We observe how gradually the signal gets digitized and the effective frequency ω_e of the ER harmonic oscillator becomes smaller than the corresponding frequency ω of the classical harmonic oscillator.

Since thermal vibrations can be represented as harmonic oscillators, the results of this paper may have applications to *blackbody radiation*. Blackbody radiation curves split into two regions: a classical region, corresponding to small values of ω, and a second region, in which ω is large. Note also that for large ω, the distances between the spectrum lines of the ER harmonic oscillator are an integer multiple of a constant. This is reminiscent of Planck's assumption. Since Relativistic Dynamics ($a_m = \infty$) does *not* predict different behaviors for different values of ω, our results provide more support, both to our conjecture that there *exists* a maximal acceleration a_m, and to our estimated *value* of a_m.
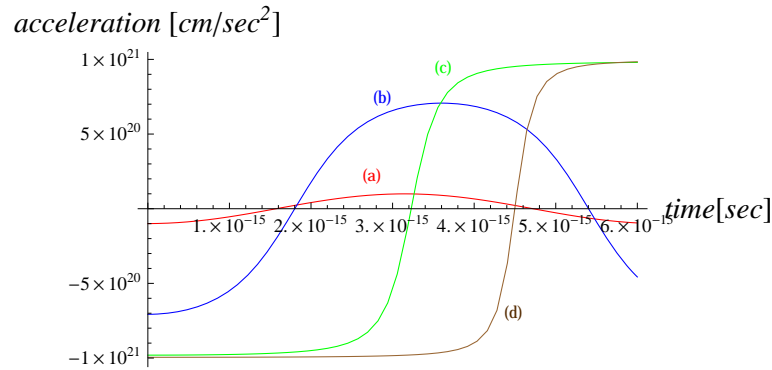
Figure 11. Evolution of $a(t)$ from the classical to the quantum region for $\omega = 10^{15}s^{-1}$ and amplitude A: (a) 10^{-10}cm,(b) 10^{-9}cm, (c) $5 \cdot 10^{-9}$cm, (d) 10^{-8}cm.
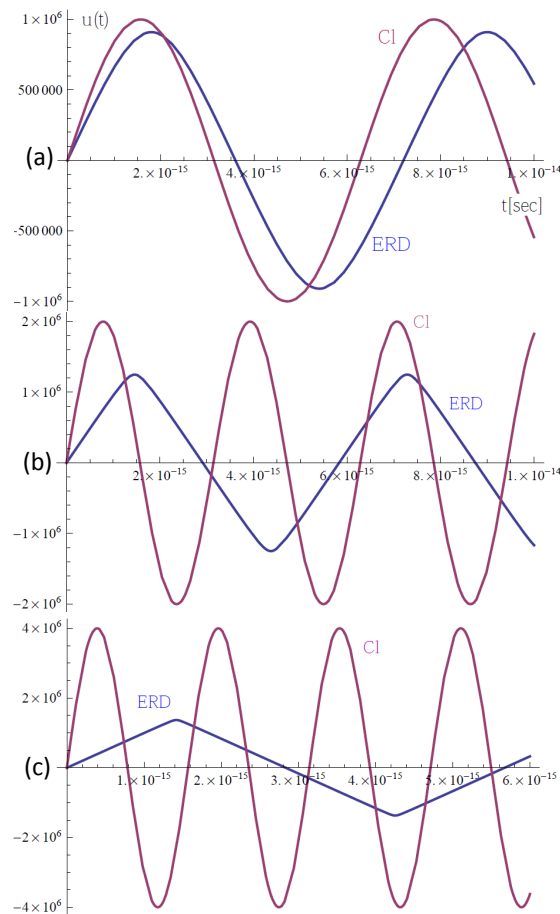

Figure 12. Velocity $u(t)$ for classical and ER harmonic oscillators for (a) $\omega = 10^{15}$, (b) $\omega = 2 \cdot 10^{15}$ and (c) $\omega = 4 \cdot 10^{15}$

Thus far, we have considered vibrations of the harmonic oscillator in ER without introducing *radiation*. In reality, when we consider thermal vibrations of atoms in a solid, the atom has to be considered positively charged moving in a negatively charged surrounding. In this case, the

accelerated charged atom will radiate. The Abraham-Lorentz force caused by radiation is given by $\mathbf{F}_{rad} = \frac{2}{3}\frac{q^2}{c^3}\dot{\mathbf{a}}$. Thus, if the harmonic oscillator is in the quantum region, which is the usual state for high temperatures, most of the time the atom will not radiate, since the $|\mathbf{a}(t)| = a_m$. The radiation will be strong for the time when the acceleration changes the direction, which is the equilibrium position. So, the radiation will come in spikes. After each spike the oscillator will lose its energy, which implies that the following spike will be of a lower intensity and less sharp. This can be understood from Figure 11. In order to obtain blackbody radiation curves, we plan to continue the study of the electromagnetic radiation of the ER harmonic oscillator.

5.2. ER dynamics Hydrogen atom model

Here we present the results published in [16] Consider a system of two particles, a proton with mass $m_p = 1.7 \cdot 10^{-27} kg$ and an electron with mass $m_e = 9 \cdot 10^{-31} kg$. Denote the position and the proper velocity of the proton by $\mathbf{r}_p, \mathbf{u}_p$ and of the electron by $\mathbf{r}_e, \mathbf{u}_e$. At this point, we will restrict ourselves only to the Coulomb force, ignoring the interaction of the particles with the fields. The force of the proton acting on the electron is thus $\mathbf{F_1} = k(\mathbf{r}_p - \mathbf{r}_e)/|\mathbf{r}_p - \mathbf{r}_e|^3$, with $k = 2.3 \cdot 10^{-28} Nm^2$, while the electric force of the electron acting on the proton is $\mathbf{F_2} = k(\mathbf{r}_e - \mathbf{r}_p)/|\mathbf{r}_e - \mathbf{r}_p|^3 = -\mathbf{F_1}$.

Typical distances between the proton and the electron are of order $0.5A = 0.5 \cdot 10^{-10} m$. For both the proton and the electron, we have

$$\left|\frac{F_2}{a_m m_p}\right|^2 \approx 35 \gg 1, \quad \left|\frac{F_1}{a_m m_e}\right|^2 \approx 10^8 \gg 1. \tag{28}$$

Thus, the second equation of system (7) for both particles becomes

$$\frac{d\mathbf{u}_p}{dt} \approx a_m \frac{\mathbf{r}_e - \mathbf{r}_p}{|\mathbf{r}_e - \mathbf{r}_p|}, \ \frac{d\mathbf{u}_e}{dt} \approx a_m \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|}, \tag{29}$$

which implies that the magnitude of the acceleration of each particle will be close to the maximal one.

To estimate the velocities of the particles, we can use the connection of the acceleration and the velocity in circular motion. Since, in this case, $v^2 = aR$, in our system we will have $v \approx 10^{4.5} m/s \ll c$. Thus, in the first equation of system (7), we can ignore the denominator, and this equation becomes $\mathbf{u} \approx d\mathbf{r}/dt$. Substituting this into the second equation, we get an approximation of the ER dynamics equation for the particles in a hydrogen-like atom:

$$\frac{d^2\mathbf{r}_p}{dt^2} \approx a_m \frac{\mathbf{r}_e - \mathbf{r}_p}{|\mathbf{r}_e - \mathbf{r}_p|}, \ \frac{d^2\mathbf{r}_e}{dt^2} \approx a_m \frac{\mathbf{r}_p - \mathbf{r}_e}{|\mathbf{r}_p - \mathbf{r}_e|}. \tag{30}$$

As usual for a two-body problem, in order to solve the system (30) of two particles, we decompose their motion into the motion of a "center of mass" and the motion of each particle with respect to the "center of mass." For our system, the average of these two equations gives

$$\frac{d^2(\mathbf{r}_p + \mathbf{r}_e)/2}{dt^2} \approx 0. \tag{31}$$

This implies that the point $\mathbf{R} := (\mathbf{r}_p + \mathbf{r}_e)/2$ moves freely, as a "center of mass." Thus, in this model, the definition of the "center of mass" differs from the definition in classical mechanics.

In a classical model for a hydrogen-like atom, the center of mass is positioned at or close to the proton, and the electron moves around the more or less stationary proton. Hence, classically, there should be a significant magnetic field for the atom. In our model, however, both the

electron and the proton move around the the new "center" with similar trajectories. They thus produce the same magnetic field, but of opposite signs, due to their opposite charges. Thus, the total magnetic moment of our atom will be almost zero.

We introduce $\mathbf{r} = (\mathbf{r}_p - \mathbf{r}_e)/2$, so that $\mathbf{r}_p = \mathbf{R} + \mathbf{r}$ and $\mathbf{r}_e = \mathbf{R} - \mathbf{r}$. Subtracting the second equation of (30) from the first and dividing by 2, we obtain

$$\frac{d^2\mathbf{r}}{dt^2} \approx -a_m \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{32}$$

which we call the "radial equation" describing the relative motion of the particles with respect to the center of mass. This is a typical equation for motion in a central field and can be solved by known methods. We will solve this equation by the method of [27], Chapter 14.

The solution of equation (32) is in a stable plane generated by the initial vectors $\mathbf{r}(0), \dot{\mathbf{r}}(0)$. Without loss of generality, we may assume that this plane is the x, y-plane. We complexify this plane by identifying the point (x, y) with the complex number $\zeta = x + iy = r(t)e^{i\varphi(t)}$. With this notation, $\frac{\mathbf{r}}{|\mathbf{r}|} = e^{i\varphi(t)}$, and equation (32) becomes

$$\ddot{r}e^{i\varphi} + 2i\dot{r}\dot{\varphi}e^{i\varphi} + i\ddot{\varphi}re^{i\varphi} - r\dot{\varphi}^2 e^{i\varphi} = -a_m e^{i\varphi}.$$

Dividing by $e^{i\varphi}$, we have

$$\ddot{r} + 2i\dot{r}\dot{\varphi} + i\ddot{\varphi}r - r\dot{\varphi}^2 = -a_m. \tag{33}$$

The imaginary part of this equation is

$$2\dot{r}\dot{\varphi} + \ddot{\varphi}r = 0 \quad \Rightarrow \quad \frac{d}{dt}(r^2\dot{\varphi}) = 0,$$

implying that the angular momentum $r^2\dot{\varphi}$ in the center of mass system is conserved. Denote the angular momentum, defined from the initial conditions, by c_1. Then, the equation

$$\dot{\varphi} = c_1/r^2 \tag{34}$$

uniquely defines $\varphi(t)$ if we solve first the equation for $r(t)$ and use the initial conditions.

To find $r(t)$, substitute (34) into the real part of equation (33) to get $\ddot{r} - \frac{c_1^2}{r^3} + a_m = 0$. Multiplying this equation by $2\dot{r}$ and integrating by t, we get

$$\dot{r}^2 = c_2 - c_1^2 r^{-2} - 2a_m r,$$

with c_2 defined by the initial conditions. This equation shows that the radial part can be regarded as motion in one dimension in an effective field

$$U_{eff} = \frac{c_1^2 m}{2r^2} - ma_m r,$$

where $\frac{c_1^2 m}{2r^2}$ is called the centrifugal energy, and $U(r) = ma_m r$ is the potential energy.

Since

$$\dot{r} = \pm\sqrt{c_2 - c_1^2 r^{-2} - 2a_m r}, \tag{35}$$

in order that solutions will exist, the values of r must be restricted by

$$c_2 r^2 - c_1^2 + 2a_m r^3 \geq 0. \tag{36}$$

The solutions of this inequality are $r_1 < r < r_2$, where r_1 and r_2 are the two positive roots of the cubic polynomial in (36), which always exist since this polynomial is negative at 0, negative towards ∞, and has at least one non-negative value at the initial state.

Thus, the solutions of the radial equation (32) of Extended Relativistic Dynamics for a hydrogen-like atom are obtained by solving the first-order differential equation (35) and then (34). It is known that only for central fields with potential energy proportional to r^2 or $1/r$ all finite motions take place in closed paths. The classical electromagnetic field is of this type, but under our dynamics, $U(r) = m a_m r$ is not. Hence, in general, our solution oscillates between the two radial values r_1 and r_2 and is not a closed path, see Figure 13. We can get a circular path
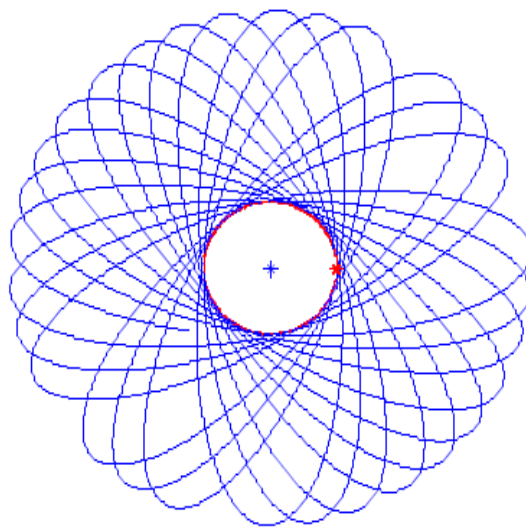

Figure 13. Electron and proton trajectories in ER dynamics Hydrogen like atom

at the minimum of the effective potential $r = \sqrt[3]{c_1^2/a_m}$, corresponding to the initial velocity $v = \sqrt{r a_m}$, when the velocity is perpendicular to **r**. For certain discrete values, we can get closed path after n periods.

The frequency of these oscillations can be estimated from the fact that the magnitude of the acceleration is approximately a_m, and in approximately circular motion, we have $a = R\omega^2$. These considerations yield a frequency of $\nu \approx 10^{14} s^{-1}$. This implies that during one measurement time, the particle will cover a whole area in the annulus $r_1 < r < r_2$.

6. Summary and Discussion

Based on the symmetry following from the general principle of relativity, we have shown in Section 2 that if the observed time of an accelerated clock differs from the observed rate of a comoving inertial clock, then there is a universal maximal acceleration, which we denote by a_m. A relativity theory in which the observed time depends also directly on the acceleration we call **Extended Relativity** ER. We presented a systematic approach for transformations between two uniformly accelerated systems in ER. These transformations (3) are of Lorentz type. In Section 3 we presented an ER dynamics equation (7) which is an extension of the relativistic dynamics and in which all admissible solutions have a speed bounded by c, the speed of light, and an acceleration bounded by a_m, the maximal acceleration. We also obtained an ER Hamiltonian (13).

In Section 4, based on the results of the previous section, we have shown that ER predicts an additional Doppler shift due to the acceleration. We have shown that in Kündig's experiment (1963), which is the only proper experiment which measured time dilation by Mössbauer spectroscopy using a rotating absorber, this additional Doppler shift was observed. We described feasible experiments to test ER and to measure the value of the maximal acceleration.

An ER map for physics is proposed in Section 5. Extended relativity may provide a unifying framework for physics. It may also provide a new model for thermodynamics and quantum mechanics. We have shown that in ER, the harmonic oscillator has different behavior for different energies. For low energies its behavior is classical, while for high energies its behavior becomes quantum-like. Such energies do occur in thermal vibrations. In this model there is no "ultra-violet catastrophe." The model explains photon creation in thermal vibrations. For a hydrogen-like atom, we have shown that the classical electromagnetic force would generate accelerations above the maximal one. Thus, at the quantum level, ER dynamics differs significantly from relativistic dynamics. We obtained the first approximation of the solution for such systems, ignoring the interaction of the particles with the field. In a typical time that can be measured, the particle covers a whole area. This may provide an indication of the probabilistic description of particles in Quantum Mechanics. We have shown that in our model, the expression for the center of mass differs from the classical one. In our model, the total magnetic moment of a hydrogen atom is almost zero, which is not so in the classical (non-quantum) model. This observation also reveals the importance of the notion of symmetric velocity, which was introduced in Chapter 2 of [10]. This velocity is the relativistic half of the regular velocity. In our model, the velocity of both particles with respect to the new "center of mass" is the symmetric velocity of the velocity of the electron in the classical model. It is known that the transformations of the symmetric velocities are conformal [17]. Conformal transformations play an important role in the quantum region.

This is only the first step in analyzing the hydrogen-like atom by use of Extended Relativistic Dynamics. We plan to improve our model by: 1. Considering the next approximations of the model. 2. Incorporating the interaction of the charges with the fields. 3. Taking into the consideration the spin of the proton and the electron.

Acknowledgments

References
[1] Bailey J et al 1977 *Nature*, **268**, 301-305
[2] Bohm D 1951 *Quantum Theory* (Dover Pub.)
[3] Born M 1962 *Atomic Physics* (Blackie & Son, London)
[4] Born M 1965 *Einstein's Theory of Relativity* (Dover Publications, New York)
[5] Born M 1938 *Proc. Roy. Soc.* A, **165**, 291
[6] Brillouin L 1970 *Relativity Reexamined* (Academic Press, New York)
[7] Caianiello E R 1981 *Lett. Nuovo Cimento* **32** 65
[8] Einstein A 1905 *Ann. Phys.* **17** 891
[9] Franklin J 2010 *Eur. J. Phys.* **31** 291-298
[10] Friedman Y 2004 *Homogeneous Balls and their Physical Applications* (Birkhäuser, Boston)
[11] Friedman Y 2009 The maximal acceleration, arXiv:0910.5629
[12] Friedman Y 2012 Quantum-like behavior of the harmonic oscillator in extended relativity, to appear
[13] Friedman Y 2011 *Ann. Phys. (Berlin)* **523** 408
[14] Friedman Y, Gofman Yu 2010 *Phys. Scr.* **82** 015004
[15] Friedman Y, Nowik I 2012 *Phys. Scr.* **85** 065702

[16] Friedman Y, Resin E 2012 *Phys. Scr.* **86** 015002

[17] Friedman Y, Semon M 2005 *Phys. Rev.* E **72**, 026603

[18] Friedman Y, Scarr T *Covariant Uniform Acceleration and Time Dilation*, arXiv:1105.0492 and article in current ussue.

[19] Don Koks 1998 *Does a clock's acceleration affect its timing rate?* http://math.ucr.edu/home/baez/physics/Relativity/SR/clock.html

[20] Goldstein H, Poole C and Safko J 2002 *Classical Machanics* (Pearson , New Jersey)

[21] Kholmetskii A L, Yarman T and Missevitch O V, 2008 *Phys. Scr.* **77** 035302.

[22] Kündig W 1963*Phys. Rev.* **126** 2371 .

[23] Liboff R L 1980 *Introductory Quantum Mechanics* (Holden-Day Inc. San Francisco)

[24] Mashhoon B 2008 *Ann. Phys.* **17** 705.

[25] Mashhoon B 1990 Limitations of spacetime measurements *Phys. Lett.* A **143** p. 176-82

[26] Mashhoon B 1990 The hypothesis of locality in relativistic physics *Phys. Lett.* A **145** p. 147-53

[27] Landau L D and Lifshitz E M 1978 *Mechanics* 3rd ed. (Pergamon Press)

[28] Moller C 1972 *The Theory of Relativity* (Clarendon Press, Oxford)

[29] Papini G, Scarpetta G, Bozza V, Feoli A, Lambiase G 2002 *Phys. Lett.* A **300** 603

[30] Papini G and Wood W R 1992 *Phys. Lett.* A **170** 409

[31] Scarpetta G 1984 *Lett. Nuovo Cimento* **41** 51

[32] Rindler W 2001 *Relativity: Special, General, and Cosmological* (Oxford University Press, New-York)

[33] Weinberg S 1972 *Gravitation and Cosmology* (John Wiley & Sons, New York)