

Inner symmetries of relativistic systems, the Laplace-Runge-Lenz symmetry, and the relativistic centre-of-mass

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Abstract. The talk concerns the inner symmetries of composite relativistic systems, their (generic) relation with the Laplace-Runge-Lenz (LRL) symmetry and the definition of the relativistic centre-of-mass. Global Lorentz-Poincaré symmetry implies the existence of LRL symmetry, in a naturally generalized sense, as part of the inner symmetry of these systems, and in a manner which is independent of the internal interaction. The corresponding LRL vectors form the internal moments associated with the Lorentz boost, which, in turn, determines the centre-of-mass.

1. Introduction

As the title suggests, the talk concerns the intrinsic relationship between the inner symmetries of composite relativistic systems, the Laplace-Runge-Lenz (LRL) symmetry, and the definition of the relativistic centre-of-mass (CM). The prime motive for the exploration that led to these results was the wish to find yet another way to solve and describe relativistic dynamics. The idea is that if Kepler-Coulomb systems, which are the non-relativistic limit of gravitational or electromagnetic systems, are endowed with the LRL symmetry (at least for 2-body systems) which completes the dynamical analysis of the system [1, 2, 3, 4], then perhaps the corresponding relativistic systems are endowed with a generalization of the LRL symmetry [5, 6, 7] which enables a better insight towards the full solution of the relativistic dynamics. For many years this issue puzzled me; recently many pieces of the puzzle came together and fell naturally and amazingly into their place.

The purpose of the talk is to show :

- (i) That integration of the relativistic CM coordinate is not unique (as opposed to Newtonian systems), and the LRL vector appears naturally as part of the process. This is the fundamental observation upon which the talk is based.
- (ii) The rôle of the LRL symmetry as inducing configuration-(or shape-)transformations which are energy preserving.
- (iii) Since the definition of the relativistic CM involves the Lorentz boost, it follows that the LRL symmetry is naturally associated with Lorentz-Poincaré symmetry.
- (iv) The LRL symmetry thus being an integral part of the internal symmetry of relativistic systems.



The following presentation is based on, and extends, recent publications by the present author [8, 9, 10]. The systems considered are Lorenz-Poincaré-symmetric, endowed with total linear momentum P^μ and total angular momentum $J^{\mu\nu}$ which are conserved and serve as generators of the corresponding global transformations. The metric convention is $g_{\mu\nu} = \text{diag}(-c^2, 1, 1, 1)$. The relativistic invariant mass of the system is $M = \sqrt{-P^2}/c^2$.

2. The centre of mass

As a starting point, let us recall the way we arrive at the definition of the Newtonian CM in a general reference frame. The CM velocity, defined from the total linear momentum and total mass, is transformed into a derivative

$$\vec{V}_{CM} \equiv \frac{\vec{P}}{M_o} = \frac{\sum_a m_a \vec{v}_a}{M_o} = \frac{d}{dt} \left(\frac{\sum_a m_a \vec{x}_a}{M_o} \right), \quad (1)$$

($M_o \equiv \sum_a m_a$), leading to the identification and definition of the expression in the brackets as the CM coordinate,

$$\vec{X}_N \equiv \frac{\sum_a m_a \vec{x}_a}{M_o} \quad (2)$$

Since any integration is determined up to an arbitrary additive constant, it might well be asked, why not add such a constant also to Eq.(2) :

$$\vec{X}'_N = \frac{\sum_a m_a \vec{x}_a}{M_o} + \vec{C} \quad (3)$$

The argument is, of course, that if the particles are all located together at one point, say $\vec{x}_1 = \vec{x}_2 = \dots = \vec{x}_o$, then also \vec{X}_N should coincide with the same point, requiring \vec{C} to vanish. This requirement rules out any c-number \vec{C} , but what if \vec{C} is not a c-number but a vector observable which is an integral-of-the-motion that vanishes when the particles' coordinate coincide ? Then a non-zero \vec{C} cannot be ruled out. Newtonian dynamics lives very happily with definition (2), without the need for any addition. The case is essentially different in relativistic dynamics.

The relativistic 3-D CM velocity $\vec{V}_{CM} = \vec{P}/P^o$ may be expanded in a general inertial reference frame in powers of $1/c^2$ as

$$\vec{V}_{CM} = \frac{\sum_a m_a \vec{v}_a}{M_o} + O(c^{-2}) = \frac{d}{dt} \left[\frac{\sum_a m_a \vec{x}_a}{M_o} + O(c^{-2}) \right] \quad (4)$$

so that integration for the CM-coordinate itself yields

$$\vec{X}_{CM} = \frac{\sum_a m_a \vec{x}_a}{M_o} + O(c^{-2}) = \vec{X}_N + O(c^{-2}) \quad (5)$$

The phenomenon that is revealed and discussed in the following, being the main theme of this presentation, an effect that is uniquely relativistic and disappears in the non-relativistic limit, is that the process of integration leading from Eq.(4) to Eq.(5) is not unique : While the integration of the Newtonian expression (1) involves only the kinematic relation $\vec{v}_a = d\vec{x}_a/dt$, the integration of the $O(c^{-2})$ -term in Eq.(4) requires, as becomes evident in the following, the dynamical equations of motion. As a consequence, the $O(c^{-2})$ -term in Eq.(5) is not an integral of the motion, and may be obtained in more than one form.

As an illustration, let us compute the CM coordinate of a 2-body electromagnetic system in the Post-Newtonian approximation. To keep things relatively simple let us consider the system

in the CM reference frame ($\vec{P} = \vec{p}_1 + \vec{p}_2 = 0$). The total energy (keeping terms to order c^{-2}) is [11]

$$E = c^2 P^0 \approx M_o c^2 + \frac{\vec{p}^2}{2\mu} - \left(\frac{1}{m_1^3} + \frac{1}{m_2^3} \right) \frac{p^4}{8c^2} + \frac{\kappa}{r} + \frac{\kappa}{2m_1 m_2 c^2 r} \left[\vec{p}^2 + \frac{(\vec{p} \cdot \vec{r})^2}{r^2} \right] \quad (6)$$

with $M_o = m_1 + m_2$, $\mu = m_1 m_2 / M_o$, $\vec{r} = \vec{x}_1 - \vec{x}_2$ and $\vec{p} = \vec{p}_1 = -\vec{p}_2$. The time derivative of the Newtonian CM coordinate (2) certainly does not vanish, so that, inserting the particles' velocities $\vec{v}_a = \partial H / \partial \vec{p}_a$, it becomes

$$\frac{d\vec{X}_N}{dt} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{M_o} \approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \left[\left(\frac{\vec{p}^2}{\mu} + \frac{\kappa}{r} \right) \vec{p} + \frac{\kappa (\vec{p} \cdot \vec{r})}{r^3} \vec{r} \right] \quad (7)$$

Since the rhs of Eq.(7) is already $O(c^{-2})$, we may use the Newtonian equation of motion to express $d\vec{X}_N/dt$ as a total derivative, and it turns out that this may be performed in two distinct ways :

$$\frac{d\vec{X}_N}{dt} \approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \frac{d}{dt} \left[\left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} \right] \quad (8a)$$

$$\approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \frac{d}{dt} [(\vec{p} \cdot \vec{r}) \vec{p}] \quad (8b)$$

with the immediate solutions

$$\vec{X}_N \approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \left[\left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} \right] + \vec{C}_1 \quad (9a)$$

$$\approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} [(\vec{p} \cdot \vec{r}) \vec{p}] + \vec{C}_2 \quad (9b)$$

A-priori, both modes of integration have the same privileges, without any preference of one over the other. Evidently, the two constants of integration cannot vanish simultaneously, and, since the integration involved the equations of motion, their difference must be a dynamical constant of the motion. The surprising result is that the difference between these constants is

$$\vec{C}_2 - \vec{C}_1 = \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \left[\left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} - (\vec{p} \cdot \vec{r}) \vec{p} \right] = \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \vec{K} \quad (10)$$

where

$$\vec{K} = \left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} - (\vec{p} \cdot \vec{r}) \vec{p} = \vec{p} \times \vec{\ell} + \frac{\mu\kappa}{r} \vec{r} \quad (11)$$

is the classical LRL vector of the corresponding Kepler-Coulomb system !

The significance of this result is, of course, in demonstrating that the classical (Newtonian) LRL vector emerges naturally in a relativistic calculation involving the Lorentz boost. This state of affairs was originally observed by Dahl already years ago [12], but apparently passed almost unnoticed by the physics community. The purpose of the present talk is to discuss the meaning and applications of this observation to fully relativistic systems.

To get a first clue for the meaning of the constants \vec{C}_1 and \vec{C}_2 , we recall that in the standard approach to the issue of the relativistic CM coordinate it is defined [13, 14] (in the CM reference frame) from the Lorentz boost $\vec{N} = J^{io}$ via

$$\vec{X}_{CM} = \frac{\vec{N}}{M} \quad (12)$$

where M is the invariant relativistic total mass of the system. With the PN Lorentz boost [11] (in the CM reference frame)

$$\vec{N} \approx M\vec{X}_N + \frac{m_2 - m_1}{2m_1m_2c^2} \left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} \quad (13)$$

and the post-Newtonian total mass

$$M = \frac{E}{c^2} \approx M_o + \frac{\vec{p}^2}{2\mu c^2} + \frac{\kappa}{rc^2}, \quad (14)$$

the standard approach yields, via Eq.(12),

$$\vec{X}_{CM} = \frac{\vec{N}}{M} \approx \vec{X}_N + \frac{m_2 - m_1}{2m_1m_2M_o c^2} \left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) \vec{r} = \vec{C}_1 \quad (15)$$

However, the second solution in (9b), which is a-priori valid just as the first one, suggests an alternative definition for the CM coordinate,

$$\vec{X}_{CM} = \frac{\vec{N}}{M} + \frac{m_1 - m_2}{2m_1m_2M_o c^2} \vec{K} \approx \vec{X}_N + \frac{m_2 - m_1}{2m_1m_2M_o c^2} (\vec{p} \cdot \vec{r}) \vec{p} = \vec{C}_2 \quad (16)$$

The different constants of integration correspond, therefore, to different possible definitions of the CM coordinate.

The LRL vector is an internal vector, which manifests the internal configuration of the system. The standard approach as in Eq.(15) restricts the CM coordinate to dependence only upon the global configuration of the system (via the dependence upon \vec{N} and M , or more generally upon P^μ and $J^{\mu\nu}$, alone). Eq.(16), on the other hand, opens the door for a broader definition, in which the CM coordinate depends also upon the internal configuration of the system.

3. Universality of the Laplace-Runge-Lenz symmetry

The above demonstration, of the appearance of the LRL vector in the computation of the post-Newtonian CM coordinate, calls for further inquiry regarding the appearance of the LRL symmetry in fully relativistic systems. Therefore, our intention is to explore the proposition that a generalization of this vector appears, in a similar fashion, in the computation of the CM coordinate for fully relativistic systems.

As a requisite, we need first to explain and define what is meant by LRL symmetry and its generalization to relativistic systems.

The LRL vector \vec{K} (Eq.(11)) is a constant of the motion in classical Kepler-Coulomb systems with Hamiltonian

$$H = \frac{\vec{p}^2}{2\mu} + \frac{\kappa}{r} \quad (17)$$

On energy hyper-surfaces $H = E$ in phase-space, the internal angular momentum $\vec{\ell} = \vec{r} \times \vec{p}$ and \vec{K} generate together, with the Poisson brackets (PB)

$$\begin{aligned} \{\ell^i, \ell^j\} &= \varepsilon^{ijk} \ell^k, & \{K^i, \ell^j\} &= \{\ell^i, K^j\} = \varepsilon^{ijk} K^k \\ \{K^i, K^j\} &= -2\mu E \varepsilon^{ijk} \ell^k \end{aligned} \quad (18)$$

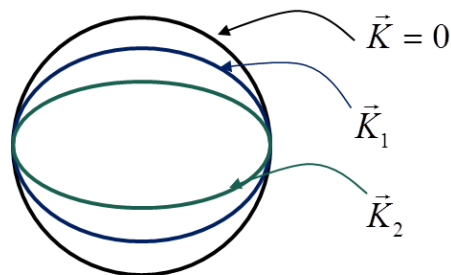
a closed Lie-Poisson algebra which is $o(4)$ or $o(3,1)$ according to the sign of E [2, 3, 4]. Let \vec{u} be an arbitrary constant unit vector and χ a dimensionless parameter, so that $\delta\mathcal{G} = (\vec{K} \cdot \vec{u}) \delta\chi$ is a

generator of the infinitesimal transformations induced by \vec{K} . As is evident from Eq.(18), these transformations change \vec{K} and $\vec{\ell}$, remaining on the same energy hyper-surface via the relation

$$\vec{K}^2 - 2\mu E \vec{\ell}^2 = \mu^2 \kappa^2 \quad (19)$$

The shape of the orbits is characterized by \vec{K} via the eccentricity $\varepsilon = |\vec{K}| / (\mu |\kappa|)$, while their spatial orientation is determined by the direction of \vec{K} . Elliptic closed orbits, for instance, with the same energy but differing by their value of $|\vec{K}|$ or $|\vec{\ell}|$ have the same major axis, and it is the minor axis that changes (Fig. 1) :

$$2a = \frac{|\kappa|}{|E|} \quad , \quad 2b = \sqrt{\frac{2\vec{\ell}^2}{\mu |E|}} \quad (20)$$



$$(|\vec{K}_2| > |\vec{K}_1| > 0)$$

Figure 1. Elliptic configurations corresponding to different LRL vectors.

Hence, while $\vec{\ell}$ is responsible for internal rotations, \vec{K} -induced transformations are responsible for shape or configuration-changing, together composing the full internal symmetry of the system.

Although many still regard the LRL vector and the symmetry associated with it as 'accidental', corresponding only to Newtonian Kepler-Coulomb systems, it is known already since the mid-sixties that all rotationally invariant systems are endowed with a generalization of the LRL vector [5, 6, 7]. Consequently, generalized (constant) LRL vectors are definable for all rotationally symmetric systems, including systems with open orbits. The generalization of the LRL symmetry does not depend upon the type of interaction, implying its universality for all rotationally symmetric systems.

These properties of the LRL symmetry are well summarized by the following proposition [9]: The assumption of internal rotational symmetry implies the existence of an internal angular momentum $\vec{\ell}$, the generator of internal rotations (so that $\{\ell^i, V^j\} = \varepsilon^{ijk} V^k$ for all 3-vectors \vec{V}). Then, if \vec{K} is a constant vector so that :

- (i) the scalar $\vec{\ell} \cdot \vec{K}$ is \vec{K} -invariant, in the sense that $\{\vec{K}, \vec{\ell} \cdot \vec{K}\} = 0$,
- (ii) \vec{K}^2 , being a constant scalar observable, is expressible as some function $\vec{K}^2 = F(H, \vec{\ell}^2, \mathcal{A})$ only of H , ℓ^2 (which is always the case for 2-body systems), and possibly on some observables \mathcal{A} that are also \vec{K} -invariant, $\{\vec{K}, \mathcal{A}\} = 0$,

then the self PB of \vec{K} are necessarily of the form

$$\{K^i, K^j\} = -\frac{\partial(\vec{K}^2)}{\partial(\vec{\ell}^2)} \varepsilon^{ijk} \ell^k \quad (21)$$

The classical LRL vector (11) with $\vec{K}^2 = \mu^2 \kappa^2 + 2\mu E \vec{\ell}^2$ and the self PB (18) is an immediate manifestation of Eq.(21). Any internal vector observable \vec{K} that satisfies the conditions of the proposition may be regarded as a generalized Laplace-Runge-Lenz vector. Since these conditions are very broad, generalized LRL vectors are not unique; rather, for every system it is a whole family of vectors that satisfy the LRL conditions. Therefore, the LRL property does not refer to a specific vector, but it is the property of the whole system, stemming from its rotational symmetry.

The remarkable feature about Eq.(21) is its universality : It is based only upon the rotational symmetry of the system, and is independent of any particular form of the interaction. Also, it does not require any particular recipe for the computation of the PB (except for the general properties of Poisson brackets), and is therefore suitable also for systems which lack canonical or phase-space structure,

4. Relativistic Laplace-Runge-Lenz symmetry

Carrying-over the LRL symmetry to relativistic systems is now straight-forward. Even in the absence of clear and unique definition of canonical phase-space variables in classical relativistic systems, Poisson brackets $\{\cdot, \cdot\}$ may be postulated satisfying the standard rules of Lie-Poisson algebras, namely [15],

- (i) Antisymmetry : $\{\mathcal{A}, \mathcal{B}\} = -\{\mathcal{B}, \mathcal{A}\}$
- (ii) Jacoby identity : $\{\mathcal{A}, \{\mathcal{B}, \mathcal{C}\}\} + \{\mathcal{B}, \{\mathcal{C}, \mathcal{A}\}\} + \{\mathcal{C}, \{\mathcal{A}, \mathcal{B}\}\} = 0$
- (iii) Product ("Leibnitz") rule : $\{\mathcal{A}, \mathcal{B}\mathcal{C}\} = \{\mathcal{A}, \mathcal{B}\}\mathcal{C} + \{\mathcal{A}, \mathcal{C}\}\mathcal{B}$
- (iv) Derivative rule: $\{\mathcal{A}, f(\mathcal{B})\} = \{\mathcal{A}, \mathcal{B}\} f'(\mathcal{B})$

If $\delta\mathcal{G}$ is the generator of an infinitesimal space-time transformation and $\delta\mathcal{A}$ is the variation of an observable \mathcal{A} under that transformation, then the variation should define the PB via $\delta\mathcal{A} = \{\mathcal{A}, \delta\mathcal{G}\}$. With well-defined and conserved total 4-linear momentum P^μ and 4-angular momentum $J^{\mu\nu}$, the fundamental PB are those of the Lorentz-Poincaré Lie-Poisson algebra,

$$\begin{aligned} \{P^\mu, P^\nu\} &= 0, & \{P^\mu, J^{\nu\lambda}\} &= g^{\mu\lambda} P^\nu - g^{\mu\nu} P^\lambda \\ \{J^{\mu\nu}, J^{\lambda\rho}\} &= g^{\mu\rho} J^{\nu\lambda} - g^{\mu\lambda} J^{\nu\rho} + g^{\nu\lambda} J^{\mu\rho} - g^{\nu\rho} J^{\mu\lambda} \end{aligned} \quad (22)$$

Besides these relations, no specific canonical structure is assumed. The PB of any observable which is constructed from P^μ and $J^{\mu\nu}$ are easily computed from the fundamental PB (22), and the PB of other observables with P^μ and $J^{\mu\nu}$ are deduced from their transformation properties.

Since P^μ determines the orientation of the CM reference frame in space-time, we define internal observables by being :

- (a) Invariant under uniform translations, $\{\mathcal{A}, P^\mu\} = 0$;
- (b) If not scalars, all their components are confined to the 3D hyperplane perpendicular to P^μ .

The spatial internal angular momentum tensor $\ell^{\mu\nu}$, defined from P^μ and $J^{\mu\nu}$ via $\ell^{\mu\nu} \equiv \Delta_\lambda^\mu \Delta_\rho^\nu J^{\lambda\rho}$ with

$$\Delta_\nu^\mu \equiv \delta_\nu^\mu + \frac{P^\mu P_\nu}{M^2}, \quad (23)$$

is the generator of internal rotations in the CM reference frame. Dual to $\ell^{\mu\nu}$ is the vector (proportional to the Pauli-Lubanski vector) $\ell_\mu \equiv \varepsilon_{\mu\nu\lambda\rho} J^{\nu\lambda} P^\rho / (2M)$ with $\varepsilon_{1230} = +1$. The relativistic version of the LRL property is then

Let K^μ be an internal vector such that:

- (i) $\{K^\mu, \ell \cdot K\} = 0$
- (ii) $K^2 = F(M, \ell^2, \mathcal{A}) \quad (\{K^\mu, \mathcal{A}\} = 0)$

Then the self-PB of K^μ satisfy

$$\{K^\mu, K^\nu\} = -\frac{\partial(K^2)}{\partial(\ell^2)} \ell^{\mu\nu} \quad (24)$$

where $\ell^2 = \ell_\mu \ell^\mu = \frac{1}{2} \ell_{\mu\nu} \ell^{\mu\nu}$.

5. Relativistic centre-of-mass integration

The preceding results brought together suggest the following picture for fully relativistic systems. With constant total linear momentum, the space-time trajectory of the CM coordinate is expected to be a straight line in the direction of P^μ , and it may always be written as the centroid

$$X^\mu = X_o^\mu + \frac{P^\mu}{M} \tau \quad (25)$$

where τ is the CM proper time and X_o^μ is a constant four-vector, identified as the spatial CM coordinate. Appropriately fixing the zero of τ , X_o^μ may be assumed orthogonal to P^μ without loss of generality, $X_o \cdot P = 0$.

Once the CM coordinate [Eq.(25)] is assumed to be known, the total angular momentum $J^{\mu\nu}$ may always be split into combination of orbital (CM) and internal parts,

$$J^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu + j^{\mu\nu} = X_o^\mu P^\nu - X_o^\nu P^\mu + j^{\mu\nu} \quad (26)$$

so that $j^{\mu\nu}$ is the (constant) internal angular momentum relative to the centre-of-mass. Out of the six components of $j^{\mu\nu}$ three are independent of the CM coordinate, fully determined by $J^{\mu\nu}$ and P^μ alone, constituting the spatial internal angular momentum tensor $\ell^{\mu\nu}$. The remaining three components of $j^{\mu\nu}$ determine X_o^μ . It is convenient to define the vector

$$Q^\mu \equiv \frac{j^{\mu\nu} P_\nu}{M^2} \quad (27)$$

which incorporates these components. If X_o^μ is required to be formed of P^μ and $J^{\mu\nu}$ alone in a frame-independent manner then the unique possibility for it is the so-called centre-of-inertia [13, 14],

$$X_I^\mu \equiv -\frac{J^{\mu\nu} P_\nu}{M^2}, \quad (28)$$

which in the CM ref. frame is given by Eq.(12), and for which $Q^\mu = 0$. However, allowing a more general definition, X_o^μ is uniquely defined, inverting Eq.(26), in terms of P^μ , $j^{\mu\nu}$ and Q^μ ,

$$X_o^\mu = -\frac{(J^{\mu\nu} - j^{\mu\nu}) P_\nu}{M^2} = X_I^\mu + Q^\mu \quad (29)$$

Q^μ , certainly an internal vector, is thus regarded as the *shift vector*, shifting X_o^μ from the centre-of-inertia (28) to the value given by Eq.(29).

In principle, the single particle trajectories may be parameterized each by a different timelike parameter, but for a common evolution picture a common parameter is required. Let σ be such a common evolution parameter. Then the single particle trajectories are $x^\mu = x_a^\mu(\sigma)$, with the particles' generalized velocities $\dot{x}_a^\mu \equiv dx_a^\mu/d\sigma$. Since the CM coordinate X^μ (25) is assumed to evolve in 4D space-time in the direction of P^μ , its projection $X_o^\mu = \Delta_\nu^\mu X^\nu$ onto the 3D CM-hyperplanes (perpendicular to P^μ , using the projection tensor (23)) should be constant, $\Delta_\nu^\mu dX^\nu/d\sigma = 0$. Introducing the Lorentz-covariant generalization of the Newtonian CM (2), the four-vector

$$X_N^\mu \equiv \frac{\sum_a m_a x_a^\mu}{M_o}, \quad (30)$$

then its 3D-projected derivative,

$$\Delta_\nu^\mu \frac{dX_N^\nu}{d\sigma} = \frac{\sum_a m_a \Delta_\nu^\mu \dot{x}_a^\nu}{M_o} = \frac{\sum_a m_a v_a^\mu}{M_o} \quad (31)$$

(with the spatial velocities $v_a^\mu \equiv \Delta_\nu^\mu \dot{x}_a^\nu$) vanishes in the nonrelativistic limit ($v_a/c \rightarrow 0$). Therefore, the time-varying part of $\Delta_\nu^\mu dX_N^\nu/d\sigma$ is purely relativistic, and the difference $R^\mu = \Delta_\nu^\mu X_N^\nu - X_o^\mu$ is an internal vector which vanishes in the nonrelativistic limit. Hence, in order to determine X_o^μ , we look for the internal four-vector R^μ which satisfies

$$\frac{dR^\mu}{d\sigma} = \Delta_\nu^\mu \frac{dX_N^\nu}{d\sigma} = \frac{\sum_a m_a v_a^\mu}{M_o} \quad (32)$$

and vanishes in the nonrelativistic limit. Eq.(32) is the exact relativistic analog of Eq.(1) and the full relativistic extension of Eq.(7). With its solution, the CM constant X_o^μ is given by

$$X_o^\mu = \Delta_\nu^\mu X_N^\nu - R^\mu \quad (33)$$

In the standard approach the CM constant X_o^μ is given by the centre-of-inertia X_I^μ (28), suggesting an immediate solution of Eq.(32) in the form

$$R_1^\mu = \Delta_\nu^\mu X_N^\nu - X_I^\mu = \Delta_\nu^\mu X_N^\nu + \frac{J^{\mu\nu} P_\nu}{M^2} \quad (34)$$

This is the relativistic extension of the post-Newtonian relation Eq.(15), and in the following is referred to as the trivial solution. But direct integration of Eq.(32) should yield also a different result, in analogy with Eq.(9b), which is denoted henceforth R_2^μ and referred to as the non-trivial solution, defining a different CM constant X_o^μ given by

$$X_{o,2}^\mu = \Delta_\nu^\mu X_N^\nu - R_2^\mu \quad (35)$$

Eliminating $\Delta_\nu^\mu X_N^\nu$ between the last two equations yields

$$X_{o,2}^\mu = R_1^\mu - R_2^\mu + X_I^\mu \quad (36)$$

so comparison with Eq.(29) implies

$$Q^\mu = R_1^\mu - R_2^\mu \quad (37)$$

The process of solving Eq.(32) for the nontrivial solution R_2^μ , identifying the shift vector (37), and constructing consequently the spatial CM component X_o^μ via Eq.(36), is referred to as *integration of the relativistic centre-of-mass* [10].

In the post-Newtonian case discussed above (Eq.(16)) the shift vector is

$$\vec{Q} \approx \frac{m_1 - m_2}{2m_1 m_2 M_o c^2} \vec{K} \quad (38)$$

with $\vec{X}_{o,2} = \vec{C}_2$. In the general case, the existence of nontrivial solutions R_2^μ implies the existence of corresponding nonzero vectors Q^μ , which are in fact LRL vectors. A couple of simply solvable relativistic 2-body systems [8, 10] which serve here as a model, indicate the shift vector in the form

$$Q^\mu = \frac{2(m_1 - m_2)}{M_o [M^2 - (m_1 - m_2)^2] c^2} K^\mu \quad (39)$$

with K^μ the (fully relativistic) LRL vector with the self PB

$$\{K^\mu, K^\nu\} = -\frac{(M^2 - M_o^2) [M^2 - (m_1 - m_2)^2] c^2}{4M^2} \ell^{\mu\nu} \quad (40)$$

It is noted that Eqs.(39) and (40) are the same, for all the systems that were studied, regardless of the interaction. From Eqs.(39) and (40) then follows

$$\{Q^\mu, Q^\nu\} = -\frac{(M^2 - M_o^2) (m_1 - m_2)^2}{M^2 M_o^2 [M^2 - (m_1 - m_2)^2] c^2} \ell^{\mu\nu} \quad (41)$$

The last relation bears consequence regarding the long-standing issue of the canonicity of the relativistic CM coordinate : Using the fundamental PB (22) it may be shown that the self-PB of X^μ satisfy

$$\Delta_\lambda^\mu \Delta_\rho^\nu \{X^\lambda, X^\rho\} = \{Q^\mu, Q^\nu\} + \frac{\ell^{\mu\nu}}{M^2 c^2} \quad (42)$$

Combined with (41), it follows that $\{X^\mu, X^\nu\} \neq 0$, implying that X^μ cannot be canonical.

6. Applications

The scheme of integration of the relativistic CM has been already applied successfully to a number of models and systems, such as :

- Post-Newtonian extension of 2-body systems with arbitrary spherically-symmetric Newtonian potential [9]
- A fully relativistic two-body system without interactions) [8]
- A fully relativistic two-body system with special light-like, time-anti-symmetric, scalar-vector(EM-like) interaction (interaction retarded for one particle and advanced for the other) [10]

An interesting possible application stems from the fact that the CM integration scheme is not limited to 2-body systems, and, at least in principle, could be applied to systems with arbitrary number of particles or components. Therefore, it has the potential of defining a LRL-like vector which is an integral of motion for many-body systems. This has been done for post-Newtonian gravitational N-body systems with application to celestial mechanics [16], and for fully relativistic systems of many non-interacting bodies [10].

7. Extended Noether theorem

Finally, let us point out interesting associations that the foregoing results have with the extended Noether theorem regarding its application to the Lorentz boost. Consider a many-body system with Lagrangian $\mathcal{L}(\vec{x}_a, \vec{v}_a, t)$. Assume variation of the coordinates $\vec{x}_a \rightarrow \vec{x}_a + \delta\vec{x}_a$ with $\delta\vec{x}_a = \varepsilon \vec{\xi}_a(\vec{x}_b, \vec{v}_b, t)$, $\delta\vec{v}_a = \varepsilon \vec{\dot{\xi}}_a(\vec{x}_b, \vec{v}_b, t)$. Using Lagrange's equations of motion, the corresponding

variation of the Lagrangian may always be put in the form $\delta\mathcal{L} = \varepsilon d\left(\sum_a \vec{p}_a \cdot \vec{\xi}_a\right)/dt$. Then, if $\delta\mathcal{L} = 0$, the classical Noether theorem states that $F = \sum_a \vec{p}_a \cdot \vec{\xi}_a$ is conserved. If, however, $\delta\mathcal{L} \neq 0$ but there exists a function $\phi(\vec{x}_a, \vec{v}_a, t)$ so that it is possible to write $\delta\mathcal{L}$ as $\delta\mathcal{L} = \varepsilon d\phi(\vec{x}_a, \vec{v}_a, t)/dt$ while $\sum_a \vec{p}_a \cdot \vec{\xi}_a \neq \phi(\vec{x}_a, \vec{v}_a, t)$ then the *extended Noether theorem* states that $F = \sum_a \vec{p}_a \cdot \vec{\xi}_a - \phi(\vec{x}_a, \vec{v}_a, t)$ is a constant of the motion [17, 18].

Particular examples of Newtonian symmetries that require the extended theorem are [17] the Galilei transformation and the LRL symmetry in Kepler-Coulomb systems. Similarly, the symmetry associated with the PN Lorentz transformation, and the corresponding conservation of the Lorentz boost, cannot be dealt with under the terms of the classical Noether theorem, and require the extended theorem. The phenomenon that is then revealed, much similar to the integration of the relativistic CM, is that the integration that leads to the function $\phi(\vec{x}_a, \vec{v}_a, t)$ involves the dynamical equations of motion and is not unique; and again, the difference between the integrations involves the classical LRL vector.

Thus we consider an infinitesimal Lorentz transformation from a simultaneity hyperplane in the CM reference frame to a simultaneity hyperplane in another inertial frame, moving with velocity $\delta\vec{V} = \varepsilon\vec{a}$ relative to the former. The corresponding variation vector is

$$\vec{\xi}_a(\vec{x}_a, \vec{v}_a, t) = \frac{1}{c^2} (\vec{x}_a \cdot \vec{a}) \vec{v}_a - \vec{a}t \quad (43)$$

Then, while

$$\sum_a \vec{p}_a \cdot \vec{\xi}_a = \frac{\vec{p}^2}{\mu c^2} \left[(\vec{X}_N \cdot \vec{a}) + \frac{m_2 - m_1}{M_o} (\vec{r} \cdot \vec{a}) \right], \quad (44)$$

the corresponding variation of the 2-body PN Lagrangian [11] may be written as a derivative $\delta\mathcal{L} = \varepsilon d\phi/dt$ in two different modes, with either

$$\phi_1 = - \left(M_o - \frac{\vec{p}^2}{2\mu c^2} + \frac{\kappa}{rc^2} \right) (\vec{X}_N \cdot \vec{a}) + \frac{m_2 - m_1}{2\mu M_o c^2} \left(\vec{p}^2 - \frac{\mu\kappa}{r} \right) (\vec{r} \cdot \vec{a}) \quad (45a)$$

or

$$\phi_2 = - \left(M_o - \frac{\vec{p}^2}{2\mu c^2} + \frac{\kappa}{rc^2} \right) (\vec{X}_N \cdot \vec{a}) + \frac{m_2 - m_1}{2\mu M_o c^2} \left[(\vec{r} \cdot \vec{p}) (\vec{p} \cdot \vec{a}) - \frac{2\mu\kappa}{r} (\vec{r} \cdot \vec{a}) \right], \quad (45b)$$

thus defining, as $F_i = \sum_a \vec{p}_a \cdot \vec{\xi}_a - \phi_i$, two constants of motion,

$$F_1 = M \left(\vec{X}_N \cdot \vec{a} \right) + \frac{m_2 - m_1}{2\mu M_o c^2} \left(p^2 + \frac{\mu\kappa}{r} \right) (\vec{r} \cdot \vec{a}) = \vec{N} \cdot \vec{a}, \quad (46a)$$

$$\begin{aligned} F_2 &= M \left(\vec{X}_N \cdot \vec{a} \right) + \frac{m_2 - m_1}{2\mu M_o c^2} \left[- (\vec{r} \cdot \vec{p}) (\vec{p} \cdot \vec{a}) + 2 \left(\vec{p}^2 + \frac{\mu\kappa}{r} \right) (\vec{r} \cdot \vec{a}) \right] = \\ &= \left(\vec{N} + \frac{m_2 - m_1}{2m_1 m_2 c^2} \vec{K} \right) \cdot \vec{a} \end{aligned} \quad (46b)$$

The constant F_1 is clearly related to the integration constant in Eq.(15), via $F_1 = M\vec{C}_1 \cdot \vec{a}$. The constant F_2 would be expected to be associated, in a similar way, to the integration constant in Eq.(16). Here, however, we encounter a bit of surprise, because while the expressions are similar, the coefficient of \vec{K} in Eq.(46b) has the opposite sign than that in Eq.(16). The reason for this ambiguity is not completely clear, and the relationship between the integration of the relativistic CM and the extended Noether theorem will be further studied in a separate publication.

8. Conclusion

In this lecture we have demonstrated and discussed the relativistic origin and application of the LRL symmetry. The existence of LRL symmetry as an internal symmetry is implied, via the Lorentz boost, from the global Lorentz-Poincare' symmetry, in a way that is independent of internal interactions.

The LRL symmetry is therefore an integral part of the inner symmetry of relativistic systems. The internal angular momentum $j^{\mu\nu}$, defined in Eq.(26), may be expressed in terms of $\ell^{\mu\nu}$ and the shift vector Q^μ as

$$j^{\mu\nu} = \ell^{\mu\nu} - Q^\mu P^\nu + Q^\nu P^\mu \quad (47)$$

We have seen that Q^μ is proportional to the LRL vector, which in itself is responsible for changing and determining the internal configuration. Thus, while $\ell^{\mu\nu}$ is responsible for spatial internal rotations, the other part of $j^{\mu\nu}$, $Q^\nu P^\mu - Q^\mu P^\nu$, is responsible for determination of the internal configurations or state of motion. Since the shift vector Q^μ appears in the mixed space-time components of $j^{\mu\nu}$, it may be regarded as the internal moment corresponding to the Lorentz boost, while $\ell^{\mu\nu}$ contains the internal moments corresponding to spatial rotations (these are regarded as internal moments, being computed relative to the CM frame, as opposed to global moments, computed relative to arbitrary inertial reference frames). These relations are summed up in the following table :

	rotations	+	change of configuration
Global space-time symmetry :	global rotations	+	Lorentz transformations
	\updownarrow		\updownarrow
Internal symmetry :	internal rotations	+	LRL

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