Constants of the motion, universal time and the Hamilton-Jacobi function in general relativity

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Abstract.
In most text books of mechanics, Newton's laws or Hamilton's equations of motion are first written down and then solved based on initial conditions to determine the constants of the motions and to describe the trajectories of the particles. In this essay, we take a different starting point. We begin with the metrics of general relativity and show how they can be used to construct by inspection constants of motion, which can then be used to write down the equations of the trajectories. This will be achieved by deriving a Hamiltonian-Jacobi function from the metric and showing that its existence requires all of the above mentioned properties. The article concludes by showing that a consistent theory of such functions also requires the need for a universal measure of time which can be identified with the "worldtime" parameter, first introduced by Steuckelberg and later developed by Horwitz and Piron.

Max Born in his book "Natural Philosophy of Cause and Chance" gives a derivation of Newton's laws of gravity from Kepler's laws of planetary motion noting that it "is the basis on which [his] whole conception of causality in physics rests"([1]). In the spirit of that insight, this essay will explore the metrics of general relativity and show how it is possible to use them to derive both the constants of the motion and the particle trajectory in a gravitational field. We will also show that a consistent development of the above mentioned approach requires the use of a universal time parameter which can be identified with the "worldtime" defined by Steuckelberg ([5])and further developed by Horwitz et al.([2]).

The key to this development will be rewriting the metric as an exact differential associated with the Hamiltonian-Jacobi function. Such exact differentials can always be constructed by noting that the inner product of any gradient vector $\nabla\psi(s)$ with a tangent vector to a curve $\mathbf{ds}$ is always exact. More precisely, using spinor notation, a local tetrad can be constructed at any point on the curve ([4]), with one-form $\tilde{ds} = \gamma^a dx_a$ and its dual $\tilde{\partial}_s\psi = \gamma^a \frac{\partial\psi}{\partial x_a}$ such that

$$\frac{\tilde{ds}}{d\tau}.\tilde{\partial}_s\psi = \frac{1}{2}\left\{\frac{\tilde{ds}}{d\tau}, \tilde{\partial}_s\psi\right\} + \frac{1}{2}\left[\frac{\tilde{ds}}{d\tau}, \tilde{\partial}_s\psi\right] \tag{1}$$

$$= \frac{d\vec{\psi}}{d\tau} + \frac{\mathbf{ds}}{d\tau} \wedge \frac{\partial\vec{\Psi}}{\partial s}. \tag{2}$$

Equations (1) and (2) can be identified by noting that the anti-commutator and commutator relationships in $\frac{\tilde{ds}}{d\tau}.\tilde{\partial}_s\psi$ associated with the spinor tetrad, define a dot product and a cross product respectively. Historically, equation (2) was first discovered by Grossman in 1841 and then independently rediscovered by Clifford.

We say that $\psi(\tau) \equiv W$ is a Hamilton-Jacobi function if (1) is true and $\left[\frac{\tilde{ds}}{d\tau}, \tilde{\partial}_s W\right] = 0$, or equivalently W is a Hamilton Jacobi function whenever $dW = p_a^* dx^a$ is an exact differential, where $p_a^* = \frac{\partial W(s)}{\partial x^a} = W' p_a$, and $p_a = \frac{\partial s}{\partial x}$. Usually, p_o is denoted by $-H$ and $dW = p_1^* dx^1 + p_2^* dx^2 + p_3^* dx^3 - H^* dt$. In particular, if s is a parameter denoting the length of a smooth curve then [4]

$$dW = p_1^* dx^1 + p_2^* dx^2 + p_3^* dx^3 - H^* dt \ \text{ iff } \ ds = p_1 dx^1 + p_2 dx^2 + p_3 dx^3 - H dt. \tag{3}$$

It follows that Hamilton-Jacobi functions can be constructed at will starting from the metric. Consider

$$ds^2 = g_{ij} dx^i dx^j \tag{4}$$

this is equivalent to

$$\frac{ds}{d\tau} ds = g_{ij} \frac{dx^i}{d\tau} dx^j \qquad \tau \text{ a parameter.} \tag{5}$$

Now choose W such that $dW = \frac{ds}{d\tau} ds$ then W will be a Hamilton-Jacobi function with

$$p_j^* \equiv \frac{\partial W(s)}{\partial x^j} = g_{ij} \frac{dx^i}{d\tau}, \qquad \tau \text{ a parameter.} \tag{6}$$

It is easy to check that this is consistent, and that for all i, j

$$p_i^* = p_j'^* \frac{\partial x'^j}{\partial x^i} \qquad \text{and} \qquad \frac{\partial p_j^*}{\partial x^i} = \frac{\partial p_i^*}{\partial x^j}. \tag{7}$$

In the case of a specified space (for example defined by a Schwarzschild or a Robertson-Walker metric), the parameter τ can in principle be conveniently chosen to be the proper time associated with a unit rest mass along a specified geodesic passing through the origin and to which all other time parameters can then be referred, although it does not have to be so. It can be identified with the "worldtime" defined by Horwitz et al.in [2]. Indeed, in the case of a particle moving with constant speed along a geodesic there is an affine relationship between it and the proper time along the curve. However, in the case of non-geodesic motion no such affine connection will exist in general between the proper time and the worldtime. In practice, the choice of this universal time is not so clear. A particle in motion within the Schwarzschild space of the earth, is in turn also moving within the Schwarzschild metric of the sun which in turn is to a first approximation moving within a solar system that is part of a specific galaxy moving along a geodesic associated with the Robertson-Walker metric. The best that one can hope to do is invoke the principle of equivalence and establish a universal time parameter with respect to some standard "laboratory" frame defined with respect to the fixed stars. We shall postpone further discussion until the next section.

1. Curves and constants of the motion

For the purpose of this article we shall confine ourselves to working with a four dimensional pseudo-Riemannian manifold and/or Minkowski space, although many of the observations could easily be extended to higher dimension spaces. Also, for what follows, we will continue to work with an arbitrary parameter τ but will simplify notation by using p for p^*. Our interest is to classify Hamilton-Jacobi functions $W = W(s)$ associated with specific forms of the metric $ds = p_1 dx^1 + p_2 dx^2 + p_3 dx^3 - H dt$, where x^i are generalized coordinates. Three specific cases arise:

(1) All of the p_i are constant.
(2) Some of the p_i are constant
(3) None of the p_i are constant.
All three cases apply to all metrics. However, in practise the specific solutions chosen and the coordinate system used usually reflect the problem under investigation. In either case, the simplest possible solutions occur when all the p_i are constants, although these are not usually the only solutions and identifying them as constants depends on the coordinate system used. For example, the differential

$$\begin{aligned} dW(s) &= \gamma(s)(2xydx + x^2dy) && (8)\\ &= p_x dx + p_y dy && (9) \end{aligned}$$

is an exact differential for all parameterizations $x = x(s)$, $y = y(s)$,where $\gamma(s) = \gamma(x^2y)$ is smooth. Also, if we require that they are both constant, they pick out a very specific family of curves (one for each k) associated with $x^2 = 2kxy$. On the other hand, rewriting the above metric in polar coordinates gives

$$\begin{aligned} dW(s) &= \gamma(s)(2xydx + x^2dy) && (10)\\ &= \gamma(s)[(3r^2\cos^2\theta\sin\theta dr + (r^3\cos^3\theta - 2r^3\cos\theta\sin^2\theta)d\theta] && (11)\\ &= p_r dr + p_\theta d\theta && (12) \end{aligned}$$

Once again, it is easy to check that $\frac{\partial p_r}{\partial\theta} = \frac{\partial p_\theta}{\partial r}$. However, the previous requirement that

$$p_x = 2\gamma xy = k, \qquad \text{a constant,}$$

now precludes that

$$p_r = 3\gamma r^2\cos^2\theta\sin\theta = \frac{3}{2}k\cos\theta$$

is constant, except in the trivial case when r and θ are both constants and vice versa. So the choice of coordinate systems is important.

Secondly, it is worth recalling that if $\dot{p}_i = 0$ for all i then this implies motion along a geodesic only if the coordinate system (x^i) forms a tetrad. In other cases, motion will be along some other type of curve. This leads to the following definitions and lemma [6](p44).

Definition 1 *If $s = s(\tau)$ where τ is a parameter and*

$$2f_i = \frac{d}{d\tau}\left(\frac{\partial\dot{s}^2}{\partial\dot{x}^i}\right) - \frac{\partial\dot{s}^2}{\partial x^i}$$

then $f_i = f_i(\tau)$ is called the acceleration tensor with respect to τ.

Lemma 1 *Let $ds^2 = g_{ij}dx^idx^j$, with ds an exact differential and $p_i(\tau) = g_{ij}\dot{x}^j = \frac{\partial s}{\partial x^i}\dot{s}$. If in phase space $\frac{\partial\dot{s}^2}{\partial x^i} = 0$ then $\dot{p}_i(\tau) = 0$ iff $f_i = 0$.*

Proof:

$$2f_i \equiv \frac{d}{d\tau}\left(\frac{\partial\dot{s}^2}{\partial\dot{x}^i}\right) - \frac{\partial\dot{s}^2}{\partial x^i}. \qquad (13)$$

But $\frac{\partial\dot{s}^2}{\partial x^i} = 0$ implies $\frac{d}{d\tau}\left(\frac{\partial\dot{s}^2}{\partial\dot{x}^i}\right) = 2\dot{p}_i$ from the definition of p_i. Therefore, $\dot{p}_i = 0$ iff $f_i = 0$. The result follows.$\square$

In practice, as we shall see below, the lemma above permits one to solve for those curves and determine those potentials for which momentum is conserved. It also allows us in special cases to read off the constants of the motion by inspection. Also, in the event that not all p_i are constant, then as noted above, the requirement that s be a Hamilton-Jacobi curve in a given coordinate system allows one to determine all possible motions not involving spin or vortex motion.

Example 1: As an application of the above theory, we begin by considering planar motion in Minkowski space with metric

$$ds^2 = dr^2 + r^2 d\theta^2 - c^2 dt^2. \tag{14}$$

This can be written with respect to a parameter τ by

$$\frac{ds}{d\tau}ds = \frac{dr}{d\tau}dr + r^2\frac{d\theta}{d\tau}d\theta - c^2\frac{dt}{d\tau}dt. \tag{15}$$

The requirement that $f_\theta = 0$ and $f_t = 0$ (see Def. 1) implies from the lemma that $p_\theta = r^2\dot{\theta}$ and $p_t = -c^2\dot{t}$ are constants of the motion along the curve $s(r, \theta, t) = h(r) + k_1\theta + k_2 t$ with respect to the parameter τ, where $s(\tau)$ is a Hamilton-Jacobi function, h a function and $\dot{p}_i(\tau) = \frac{\partial s}{\partial x^i}\dot{s}$.

Two cases are of particular interest:
(1) $\tau = s$ and $h(r) = kr$ (equivalent to $\dot{r} = constant \neq 0$)
(2) $\tau = s$ and $h(r) = k$ (equivalent to $\dot{r} = 0$).
In the first case

$$f_r(s) = -r\dot{\theta}^2 = -(r^2\dot{\theta})^2/r^3 = -k^2/r^3$$

defines an inverse cube law of motion. Moreover, both $\dot{r}$ and $r^2\dot{\theta}$ are non-zero constants and consequently are proportional to each other. It follows that in general the trajectory of a particle subjected to this force in Minkowski space obeys the equation $\dot{r} = \epsilon r^2\theta$ which is equivalent to $k_3 r - \epsilon r\theta = 1$. In the second case $r = constant$. This defines circular motion.

Finally, if in addition to $f_\theta = f_t = 0$, we restrict ourselves to purely geodesic motion given by taking $f_r = 0$, then it is easy to check that the only possible trajectory is given by $\theta = constant$ which is equivalent to straight line motion in $\Re^2$.

It should be clear that the form of the metric is key to associating specific curves with specific constants of the motion. In the above example, we derived the general form of the trajectories associated with the conservation of angular momentum $\dot{p}_\theta = r^2\theta$. However, one might seek curves in the same metric space associated with other constants. For example, if we define $\dot{\phi} = r\dot{\theta}$ then the metric takes the form

$$ds^2 = dr^2 + d\phi^2 - c^2 dt^2 \tag{16}$$

on the Minkowski manifold $M(r, \phi, t)$. The geodesic equations are given by $f_r = f_\phi = f_t = 0$ or equivalently $\dot{r} = k\dot{\phi}$. Substituting $r\dot{\theta}$ for $\dot{\phi}$ and solving for $r \neq constant$, gives the trajectory $r = Ae^{k\theta}$ on the manifold $M(r, \theta, t) \setminus \{(0, 0)\}$, while in the case $\dot{r} = 0$ we obtain the circle $r = r_o$, r_o a constant. In other words, the requirement that $r\dot{\theta}$ be a constant of the motion (and not $r^2\dot{\theta}$) determines the family of trajectories

$$\mathcal{F} = \{r = Ae^{k\theta} | A, k \text{ are constant}\} \bigcup \{r = r_o | r_o \text{ a constant}\}.$$

Example 2: As a second example consider the metric

$$\dot{s}ds = \frac{l\dot{r}}{\sin\theta}dr + r^2\dot{\theta}d\theta - c^2\dot{t}dt, \tag{17}$$

derivable from Kepler's first and second laws of planetary motion which state that planets move on ellipses given by $l/r = 1 + \epsilon\cos\theta$, with constant angular momentum. The Ricci curvature tensor $R_{ij} \neq 0$ and consequently the space is not flat. For the purpose of this essay, let us begin with the metric and require that $\dot{s}$ be an exact differential. Note that $p_t = -c^2\frac{\partial s}{\partial t}$ in terms of phase space is independent of r and θ. It follows from the lemma that $\dot{t}$ is a constant of the motion if $f_t = 0$. However, the lemma does not apply to both $p_r = \frac{l\dot{r}}{\sin\theta}$ and $p_\theta = r^2\dot{\theta}$, except in the case of $\dot{\theta} = 0$ (a geodesic) and therefore precludes these from being constants of the motion along other geodesics. However, there are other trajectories for which they are constants of the motion, determined by the equation $p_r = \epsilon p_\theta$ or equivalently that $\frac{l\dot{r}}{\sin\theta} = \epsilon r^2\dot{\theta}$. Integrating out gives the equation of a conic for an inverse square law of motion, which is Kepler's first law of motion.

Example 3: The same techniques can also be used to identifying the constants of the motion associated with all metrics in which the equations of motion obey the Hamilton-Jacobi equation. Consider the Schwarzschild space with a metric of the form

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{18}$$

For $f_t = 0$ and $f_\phi = 0$, it follows from the lemma that in phase space

$$B(r)\dot{t} = k_1, r^2\sin^2\theta\dot{\phi} = k_2.$$

Also,

$$\frac{\partial r^2\dot{\theta}}{\partial\phi} = \frac{\partial r^2\sin^2\theta\dot{\phi}}{\partial\theta} = 0$$

implies $\theta = k_3$. Taking $k_3 = \frac{\pi}{2}$, the Hamilton-Jacobi function takes the form $s(r,\theta,\phi,t) \equiv k_1t + k_2 + k_3\phi + h(r)$ with p_t, p_θ, p_ϕ being constants of the motion, and h a function such that $\frac{\partial h}{\partial r} = -A(r)$.

The three constants are well known and can be easily shown to be associated with geodesic motion. To fully obtain a geodesic, we would also require that $f_r = 0$, in addition to f_t, f_θ and f_ϕ already given above. Also by noting that any two constants can be related by a constant of proportionality ϵ, it follows that if $f_t = f_\phi = 0$ then any trajectory for such a motion must obey the equation $k_1 = \epsilon k_3$ or equivalently $\frac{B(r)}{r^2} = \epsilon\frac{d\phi}{dt}$.

Example 4: Similarly in the case of the Robertson-Walker metric

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right\} \tag{19}$$

there exists trajectories for which $\dot{t}, R^2(t)\frac{\dot{r}}{1-kr^2}, R^2(t)r^2\dot{\phi}$ and θ are constants of the motion, and in this case a generalized first law of Kepler would require that galaxies move on trajectories given by $\frac{\dot{r}}{r^2(1-kr^2)} = \epsilon\dot{\phi}$. This can be integrated out, using partial fractions, to give the family of curves

$$-\epsilon r\phi + k_5r + \frac{r\sqrt{k}}{2}\ln\left(\frac{1+\sqrt{k}r}{1-\sqrt{k}r}\right) = 1. \tag{20}$$

These curves are not geodesics, since only $R^2(t)r^2\dot{\phi} = constant$ and $\theta = constant$ satisfy the geodesic equation $f_t(s) = 0$.

2. Universal time

As previously noted, the requirement that $dW(\tau) \equiv \frac{ds}{d\tau}ds$ be a Hamilton-Jacobi function such that

$$p_j(\tau) = \frac{\partial W(\tau)}{\partial x^j} = g_{ij}\frac{dx^i}{d\tau}, \qquad \tau \text{ a parameter}$$

begs the question as to how we should determine the parameter τ. On the one hand it does not matter in that one can easily pass from one parameter τ to another by means of the transformation

$$p(\tau') = \frac{d\tau}{d\tau'}p(\tau). \tag{21}$$

Note the similarity between this and coordinate transformations of the form

$$p_i^* = p_j'^* \frac{\partial x'^j}{\partial x^i}.$$

This latter case applies when we always use the same proper time s/c. Indeed, if we were confined to a single curve then the proper time would be the most convenient. However, things are more complicated as soon as we pass over into higher dimensions and try to seek a consistent parametrization that applies to all curves. In this case, we will be guided by equation (21). Either way, as we shall see below, it will be necessary to define a universal time scale and basis within the space in order to have a common and consistent parametrization. We now explore this more in depth.

Consider for example the difficulty in trying to coherently compare the two straight lines,

$$ds^2 = c^2dt^2 \text{ and } ds^2 = c^2dt^2 - dx^2 = c^2dt^2(1 - \tanh^2\phi) = c^2dt^2\sec^2\theta$$

defined over the two dimensional Minkowski space $M(t,x)$. Using the proper time s/c as a parameter and $\phi = i\theta$ a constant, the first can be parameterized in terms of s by $t = s/c$ and defines the proper time for the curve, while the second line will have a parametrization given by $t = \sec(\theta)s/c$. Unless $\theta = 0$, a clear contradiction arises on equating s or t in each equation. To avoid this paradox there are two choices to make: either we rewrite the equations of the two curves as $s = tc$ and $s' = c\sec(\theta)t$ ($s \neq s'$) or as $s = tc$ and $s = c\sec(\theta)t'$ ($t \neq t'$). Both choices are valid and should never be confused.

At the core of the distinction is the difference between having two different lines or parameterizations with respect to the same reference frame versus the same line defined with respect to two different references frames. Equivalently it is the difference between parameterizing the two curves with respect to the same local time (common reference frame) or parameterizing with respect to the same proper time. The advantage of the latter case is that one can easily pass from one curve to another by means of a Lorentz transformation, if they are of the same type. A simple rotation of the axes will transform one curve into another. This is Weinberg's approach. It reflects the fact that one is examining the same phenomena from two different references frames ([7]).

However, in the event that two particles are moving along the same curve (geodesic) but with different speeds (not to mention accelerations) this second perspective is inadequate. Indeed, the requirement that $ds = ds'$ would be equivalent to saying that both particles have proper speed c by definition (albeit defined with respect to two different references frames) although they are clearly moving at different speeds from the perspective of the same reference frames. From this perspective, the equations $ds^2 = c^2dt^2$ and $ds'^2 = c^2\sec^2\theta dt^2$ can be viewed as two different parameterizations along the same curve reflecting the different speeds of the particles with respect to the laboratory frame. The problem becomes even more pronounced when we try to compare the velocities and accelerations between two particles moving on totally different curves. In such cases, unless the problem is formulated with respect to the same basis $\{cdt, dx, dy, dz\}$ it risks being ill-posed.

In terms of the general theory, consider two curves defined with respect to a basis $\{x^i\}$ and a parameter τ such that

$$\frac{ds}{d\tau}ds = p_i(s)dx^i, \qquad \frac{ds'}{d\tau}ds' = p'_i(s')dx^i$$

on a pseudo-Riemannian manifold. Since both s and s' are considered to be Hamilton-Jacobi functions the existence of p_i and p'_i are guaranteed by equation (6). Let us further assume that the first curve is both a geodesic and a unit speed curve when parameterized with respect to s, while the second curve parameterized with respect to s is an arbitrary non-unit speed curve. In other words, motion along the non-unit curve is defined in terms of a tetrad $\{x^i\}$ which can be Fermi transported along the unit geodesic. Transposed to the laboratory rest frame these can be expressed by the equations

$$ds^2 = c^2 d\tau^2, \qquad ds'^2 = p'_\tau d\tau^2,$$

with $p'_t = c\frac{ds'}{ds}$. By construction these curves cannot be transformed one into another by means of a Lorentz transformation. Their corresponding proper times are different. However, it is possible to define one proper time as a non-trivial function of the other. To do this in an effective and consistent way it is necessary to establish a universal measure of time, a standard time to which all others can be compared. In theory, by the Principle of Equivalence it is sufficient to parameterize any curve in terms of a local unit speed geodesic proper time and use this as a universal time parameter. In practice, the only way one can know if this has been successful is by comparing the time calibrations of the standard laboratory clock with a universally pre-established time standard. It can be identified with the worldtime defined by Horwitz et al. in [2].

3. Mass and momentum

Thus far the role of mass in our equations has been ignored and the above discussion related to different parameterizations of a curve suggests that the concept of rest mass will also be affected by such changes. Returning to equation (1), we see that for the curve $x^i = x^i(s)$ defined with respect to a local tetrad, the function $W(s) = m_o s$ is a Hamilton-Jacobi function with $\frac{\partial s}{\partial x^i} = \frac{dx^i}{ds}$, where s/c is the proper time along the curve. It follows that $p_i = \frac{\partial W}{\partial x^i} = m_o c\frac{dx}{ds}$. However, if the parameter is changed to s' then

$$p^i = m_o c\frac{dx^i}{ds} = m(s')c\frac{dx^i}{ds'}, \qquad \text{with} \qquad m(s') \equiv m_o\frac{ds'}{ds} \tag{22}$$

which means the momentum remains invariant under a change of parameter.

In particular, in the case of motion defined with respect to two different parameters s and s' respectively, the four momentum $m_o c$ in the rest frame associated with s will will transform into

$$m_o c = m_o c\frac{ds}{ds'}\frac{ds'}{ds} = m'_o c\frac{ds}{ds'} \tag{23}$$

in the frame associated with s'. Equivalently, the rest mass defined with respect to two different parameterizations of a curve may be different. For example, if $s' = t$ the local time then the change in mass will correspond to the usual mass change under Lorentz transformations. If $s' = 2.2s$ then a mass measurement in kilos in the s frame would correspond to the same measurement in pounds in the s' frame. It also follows from equation (23) that if s'/c is the worldtime parameter defined in [3] then

$$ds^2 = \frac{m^2}{M^2} ds'^2, \qquad m_o = m, \; M = m(s')$$

corresponds to equation (14) in [3] and $m^2 = M^2$ defines the so called "mass shell" condition. Indeed, the mass will be the same under any change of basis that does not affect the curve parameter. If $ds = ds'$ then $m(s') \equiv c\frac{dx^i}{ds} = m_o$. In other words, the rest mass along any geodesic measured with respect to a standard clock and length will always be the same. Real differences will only occur in a gravitational field.

On a final note, if we return to equation (22), then the conservation of momentum requires

$$\frac{ds}{m(s)} = \frac{ds'}{m(s')},$$

which implies that $\frac{ds^2}{m^2(s)}$ is invariant both under general coordinate transformations given by equation (7) and under changes of parameter given by equation (21) and it unifies the two different perspectives as outlined in the previous section. For example, it allows one to pass from the worldtime parameter to a proper time parameter in one step.

4. Conclusion

In the spirit of Born's observation, there is something special about Hamilton-Jacobi functions. Not only can they be used to derive Hamilton's equations but they allow us to identify equations and constants of motion as well as a new relativistic invariant $\frac{ds}{m(s)}$ associated with the motion, with the rest mass m_o being meaningfully defined only if a "worldtime" parameter is introduced. They also determine the trajectories in general for natural motions.

In that regard, it should be recalled that if $s(\tau)$ is a Hamilton-Jacobi function then so also are smooth functions $W(s)$ and more sophisticated motions will require their use. Indeed, in the context of the overall field of mechanics the Hamiltonian-Jacobi functions with gradient ∇W serve as gauge terms for the more general motion which can be written (see equation (2)) as

$$\frac{\tilde{ds}}{d\tau}.\tilde{\partial}_s\psi = \frac{dW}{d\tau} + \vec{A}.\frac{ds}{d\tau} + \vec{A} \wedge \frac{\vec{ds}}{d\tau}. \tag{24}$$

But this is a discussion for another day.

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