Invariant submodels of model of nonlinear diffusion in inhomogeneous medium with nonstationary absorption or source

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Abstract. We carried out a complete group classification of the nonlinear differential equation of the model, describing the nonlinear diffusion process (or the process of heat propagation) in an inhomogeneous medium with nonstationary absorption or source. We found all models whose equations admit a nontrivial continuous Lie group of the transformations. We researched invariant submodels of the obtained models. Some invariant solutions, describing these submodels were found either explicitly, or their search was reduced to solving of the nonlinear integral equations. In particular, we obtained an invariant solution that describes the nonlinear diffusion process (or heat distribution process) with two fixed "black holes", and invariant solution that describes the nonlinear diffusion process (or heat distribution process) with a fixed "black hole", and with a moving "black hole". The solutions with "black holes" can be used to describe the process of destruction of glaciers under the influence of the external heat source, for example a solar energy. Using other invariant solution, we studied a diffusion process (or heat distribution process) for which at the initial moment of the time at a fixed point a concentration (a temperature) and its gradient are specified. The solution of the boundary value problem describing this process reduces to the solution of nonlinear integral equations. We have established the existence and uniqueness of solutions of this boundary value problem under certain conditions. The mechanical significance of the obtained solutions is as follows: 1) they describe specific nonlinear diffusion processes (or heat distribution processes), 2) they can be used as tests in numerical calculations in studies of nonlinear diffusion (or heat distribution) in an inhomogeneous medium with nonstationary absorption or source, 3) they allow to assess the degree of adequacy of a given mathematical model to real physical processes, after carrying out experiments corresponding to these decisions, and estimating the arising deviations.

1. Introduction

Symmetry properties of the model of nonlinear diffusion (or heat distribution) in an inhomogeneous medium without absorption or a source and also with stationary absorption or a source were studied in many papers (see, for example, bibliographies given in [1]). Symmetry properties of the model of nonlinear diffusion (or heat distribution) in an inhomogeneous medium with nonstationary absorption or a source of a special type were studied in [2]. A model of nonlinear diffusion (or heat distribution) in an inhomogeneous medium with nonstationary absorption or a source of general form was investigated in [3]. In this report we will use the materials of this article.

A model describing the nonlinear process of diffusion (or heat distribution) in an inhomogeneous medium with nonstationary absorption or a source is determined by the equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(x^{\alpha} u^{\beta} \frac{\partial u}{\partial x}\right) - \varphi(t)u, \tag{1}$$

where $u = u(t,\ x)$ is the concentration of the substance (or temperature) at the point $x \in (-\infty,\ \infty)$ at time t; α and β are parameters of the diffusion coefficient $D = x^{\alpha}u^{\beta}$, which characterizes the diffusion rate (α and β are arbitrary real constants); $\varphi(t)$ is non-stationary absorption or source coefficient.

We assume further that $\varphi(t)$, α and β satisfy the condition

$$\alpha\beta\varphi'(t) \neq 0. \tag{2}$$

This condition means that the diffusion is nonlinear; the medium is inhomogeneous; for $\varphi(t) > 0$ there is an absorption, and for $\varphi(t) < 0$ there is an external source.

The model (1) under condition (2) will be studied by methods of group analysis [4, 5].

2. Group classification

We fulfilled group classification of equation (1). We solved the problem of the group classification of this equation using the algorithm proposed in [4–6]. This algorithm has been successfully used in [1–3, 7–10] for group classification of the various equations of mechanics and mathematical physics. The results of the group classification are as follows [3]:

The kernel of the main groups of the equation (4) consists only of the identity transformation of the space $R^3(t,\ x,\ u)$.

For each pair of non-zero model parameters $(\alpha,\ \beta)$ and

$$\varphi(t) = -\frac{1}{\beta}\left(\ln\left(\psi'(t)\right)\right)',\ \ \psi'(t) > 0,\ \ \left(\ln\left(\psi'(t)\right)\right)'' \neq 0 \tag{3}$$

the main group of the equation (1) is generated by the operators

$$X_1 = x\partial_x + \frac{2-\alpha}{\beta}u\partial_u,\quad X_2 = \beta\frac{\psi}{\psi'}\partial_t - \left(\frac{\psi}{\psi'}\right)'u\partial_u,\quad X_3 = \frac{\beta}{\psi'}\partial_t - \left(\frac{1}{\psi'}\right)'u\partial_u. \tag{4}$$

Note. *The function $\psi(t)$ is expressed via the function $\varphi(t)$, by the formula*

$$\psi(t) = k_1\int \exp\left(\beta\int\varphi(t)dt\right)dt + k_2,$$

where k_1 and k_2 are real constants.

Let $\psi(t)$ be determined by the formula (3). Then:

for

$$\alpha = 1,\quad \beta = -1 \tag{5}$$

the main group of the equation (1) is generated by the operators (4) and operator

$$X_4 = x(\ln x - 2)\partial_x - u\ln x\partial_u$$

for

$$\alpha = \frac{3\beta+4}{2\beta+3},\quad \beta(\beta+1)\left(\beta+\frac{3}{2}\right)\left(\beta+\frac{4}{3}\right) \neq 0 \tag{6}$$

the main group of the equation (1) is generated by the operators (4) and operator

$$X_5 = x^{-\frac{\beta+1}{2\beta+3}}\left((2\beta+3)x\partial_x - u\partial_u\right).$$

Thus, the set of the models (1), having essentially different group properties consists of models (3) - (6). We give some invariant solutions for these models.

3. Invariant solutions

Exact solutions for each mathematical model are important. They allow us to assess the degree of the adequacy of the mathematical model of real physical processes, after carrying out experiments appropriate to these solutions, and an evaluation of the arising deviations. Exact solutions can be used to describe of some physical processes. Exact solutions are good tests to check the approximate numerical solutions.

For $(\beta+1)(2-\alpha+\beta) \neq 0$ invariant $\left\langle \frac{\beta(\beta+1)}{2-\alpha+\beta}X_1 + X_2,\ X_3 \right\rangle$ -solution is determined by the formula

$$u = c_0 x^{\frac{1-\alpha}{\beta+1}}\left(\psi'(t)\right)^{\frac{1}{\beta}} \quad \left(c_0 = \text{const}\right). \tag{7}$$

When $\alpha = 1$, this solution describes the nonlinear diffusion process (or the heat propagation process) with a concentration (or a temperature) which is independent of the inhomogeneity of the medium at any given time.

For $(\alpha-2)(2-\alpha+\beta) \neq 0$ invariant $\langle X_1,\ X_2\rangle$–solution is determined by the formula

$$u = \left(\frac{\beta}{(\alpha-2)(2-\alpha+\beta)}\right)^{\frac{1}{\beta}} x^{\frac{2-\alpha}{\beta}}\left(\left(\ln\psi(t)\right)'\right)^{\frac{1}{\beta}}. \tag{8}$$

Let $\left(\frac{\beta}{(\alpha-2)(2-\alpha+\beta)}\right)^{\frac{1}{\beta}} > 0$. When $\frac{2-\alpha}{\beta} > 0$, this solution describes the nonlinear diffusion process (or the heat propagation process) with a concentration (or a temperature) $u \to \infty$ when $x \to \infty$. When $\frac{2-\alpha}{\beta} < 0$, this solution describes the nonlinear diffusion process (the heat propagation process) with a concentration (or a temperature) $u \to 0$ when $x \to \infty$.

Model (5) has the following invariant $\langle X_2,\ X_4\rangle$–solution which is determined by the formula

$$u = \frac{2\psi(t)}{x(\ln x - 2)^2\psi'(t)}.$$

Let $\psi(t) > 0$. This solution describes the nonlinear diffusion process (or the heat distribution process) in which a concentration (or a temperature) infinitely increases when $x \to +0$ or $x \to \mathrm{e}^2$. The points $x = 0$ and $x = \mathrm{e}^2$ are similar to the fixed "black holes", well known in astrophysics.

Model (5) has the following invariant $\langle X_1 + X_3,\ X_4 - X_2\rangle$–solution which is determined by the formula

$$u = -\frac{1}{x\left(\ln x - 2 + \psi(t)\right)\psi'(t)}.$$

This solution describes the nonlinear diffusion process (or the heat distribution process) in which a concentration (or a temperature) infinitely increases when $x \to +0$. The point $x = 0$ is similar to the fixed "black hole", well known in an astrophysics. In the vicinity of the trajectory $x = \mathrm{e}^{2-\psi(t)}$ the concentration (or the temperature) also infinitely increases. This trajectory describes the evolution of the "black hole".

Model (6) has the following invariant $\langle X_3 + X_5 \rangle$–solution which is determined by the formula

$$u = x^{-\frac{1}{2\beta+3}} \left(-\frac{\beta^2 (2\beta+3)^2}{(\beta+1)^3} \left(x^{\frac{\beta+1}{2\beta+3}} - \frac{\beta+1}{\beta} \psi(t) \right) \psi'(t) \right)^{\frac{1}{\beta}}.$$

This solution describes the nonlinear diffusion process (or the heat propagation process) with a concentration (or a temperature) $u = 0$ along a trajectory $\psi(t) = \dfrac{\beta}{\beta+1} x^{\frac{\beta+1}{2\beta+3}}$.

Model (6) has the following invariant $\left\langle \dfrac{\beta(2\beta+3)}{\beta+1} X_1 + X_2,\ X_3 + X_5 \right\rangle$–solution which is determined by the formula

$$u = x^{-\frac{1}{2\beta+3}} \left(\psi'(t)\right)^{\frac{1}{\beta}} V(\xi), \quad \xi = x^{\frac{\beta+1}{2\beta+3}} - \frac{\beta+1}{\beta} \psi(t),$$

where the function $V(\xi)$ is implicitly determined by a quadrature

$$\beta \int \left(c_1 - (2\beta+3)^2 V^{\frac{1}{\beta+1}} \right)^{-\frac{1}{2}} \mathrm{d}V = \xi + c_2 \quad (c_1,\ c_2 = \mathrm{const}). \tag{9}$$

For some values of parameter β the integral in the left-hand side of (9) is expressed in terms of elementary functions. This solution describes the nonlinear diffusion process (or the heat propagation process), for which the value $x^{\frac{1}{2\beta+3}} \left(\psi'(t)\right)^{-\frac{1}{\beta}} u(t,\ x)$ is constant along each trajectory $x = \left(\dfrac{\beta+1}{\beta} \psi(t) + c_3 \right)^{\frac{2\beta+3}{\beta+1}}$ $(c_3 = \mathrm{const})$.

Invariant $\langle \gamma X_1 + X_2 \quad (\gamma = const) \rangle$–solution has the form

$$u = \left(\psi(t)\right)^{\frac{\gamma(2-\alpha)}{\beta^2}} \left(\left(\ln \psi(t) \right)' \right)^{\frac{1}{\beta}} U(\xi), \quad \xi = x \left(\psi(t)\right)^{-\frac{\gamma}{\beta}}. \tag{10}$$

Substitution (8) into (1) gives a factor-equation

$$\xi^{\alpha} U^{\beta} U'' + \beta \xi^{\alpha} U^{\beta-1} U'^2 + \left(\alpha \xi^{\alpha-1} U^{\beta} + \frac{\gamma}{\beta} \xi \right) U' + \frac{\gamma(\alpha-2)+\beta}{\beta^2} U = 0. \tag{11}$$

The equation (11) is reduced to the following integral equations:

if $(\alpha-1)(\beta+1) \neq 0,\ U = V^{\frac{1}{\beta+1}}(\xi)$, then

$$V(\xi)=c_4+c_5\xi^{1-\alpha}+$$
$$+\frac{(\beta+1)}{\beta^2(1-\alpha)}\int_{\xi_0}^{\xi}\Big(\big(\gamma(\alpha-2)(\beta+1)+\beta\big)\tau^{1-\alpha}-\big(\gamma(\alpha-2-\beta)+\beta\big)\xi^{1-\alpha}\Big)V^{\frac{1}{\beta+1}}(\tau)\,\mathrm{d}\tau \qquad (12)$$

if $\alpha=1,\ \beta\neq-1,\ U=V^{\frac{1}{\beta+1}}(\xi)$, then

$$V(\xi)=c_6+c_7\ln\xi+\frac{(\beta+1)}{\beta^2}\int_{\xi_0}^{\xi}\left(\big(\beta-\gamma(\beta+1)\big)\ln\frac{\tau}{\xi}-\beta\gamma\right)V^{\frac{1}{\beta+1}}(\tau)\,\mathrm{d}\tau \qquad (13)$$

if $\alpha\neq1,\ \beta=-1,\ U=\mathrm{e}^{V(\xi)}$, then

$$V(\xi)=c_8+c_9\xi^{1-\alpha}+\frac{1}{(1-\alpha)}\int_{\xi_0}^{\xi}\Big(\big(\gamma(1-\alpha)+1\big)\xi^{1-\alpha}-\tau^{1-\alpha}\Big)\mathrm{e}^{V(\tau)}\,\mathrm{d}\tau \qquad (14)$$

if $\alpha=1,\ \beta=-1,\ U=\mathrm{e}^{V(\xi)}$, then

$$V(\xi)=c_{10}+c_{11}\ln\xi+\int_{\xi_0}^{\xi}\left(\ln\frac{\xi}{\tau}+\gamma\right)\mathrm{e}^{V(\tau)}\,\mathrm{d}\tau, \qquad (15)$$

where c_k $(k=4,\ 5,\ \ldots 11)$, $t_0>0,\ x_0>0,\ \psi_0>0$ are arbitrary real constants, $\xi_0=x_0\psi_0^{-\frac{\gamma}{\beta}}$, $\psi_0=\psi(t_0)$.

The function $V(\xi)=c_0^{\beta+1}\xi^{1-\alpha}$ is a particular solution of the nonlinear integral equation (12) for

$$(2-\alpha+\beta)(3+2\beta-2\alpha-\alpha\beta)\neq0,\ c_4=\frac{c_0(\beta+1)\big(\gamma(\alpha-2)(\beta+1)+\beta\big)}{\beta^2(1-\alpha)(2-\alpha+k)}\xi_0^{2-\alpha+k},$$

$$c_5=c_0^{\beta+1}-\frac{c_0(\beta+1)^2\big(\gamma(\alpha-2-\beta)+\beta\big)}{\beta^2(1-\alpha)(2-\alpha+\beta)},\ k=\frac{1-\alpha}{\beta+1},\ \gamma=\frac{\beta(\beta+1)}{2-\alpha+\beta}.$$

Substitution of this function into (10) gives a solution (7).

The function $V(\xi)=\left(\frac{-\beta\xi}{\beta+1}\right)^{\frac{\beta+1}{\beta}}$ is a particular solution of the nonlinear integral equation (13) for

$$c_6=\left(\frac{-\beta}{\beta+1}\right)^{\frac{1}{\beta}}\xi_0^{\frac{\beta+1}{\beta}}\left(\frac{\big(\beta-\gamma(\beta+1)\big)}{\beta}\left(\ln\xi_0-\frac{\beta+1}{\beta}\right)-\gamma\right),\ c_7=\left(\frac{-\beta}{\beta+1}\right)^{\frac{1}{\beta}}\frac{\big(\gamma(\beta+1)-\beta\big)}{\beta}\xi_0^{\frac{\beta+1}{\beta}}.$$

Substitution of this function into (10) gives a solution (8) for $\alpha=1$.

The function $V(\xi)=\big((\alpha-1)(\alpha-2)\xi^{\alpha-2}\big)^{\beta+1}$ is a particular solution of the nonlinear integral equation (14) for

$$c_8=\ln\big((\alpha-1)(\alpha-2)\big)+\frac{(\alpha-1)(\alpha-2)\big(\gamma(1-\alpha)+1\big)}{(1-\alpha)^2}-\frac{(\alpha-1)(\alpha-2)}{(1-\alpha)}\ln\xi_0,$$

$$c_9=-\frac{(\alpha-1)(\alpha-2)\big(\gamma(1-\alpha)+1\big)}{(1-\alpha)^2}\xi_0^{\alpha-1}.$$

Substitution of this function into (10) gives a solution (8) for $\beta=-1$.

The function $V(\xi) = c_{12}^{\beta+1}\xi^{-\frac{(\gamma+1)(\beta+1)}{\gamma}}$ $(\gamma \neq 0,\ c_{12} = \text{const} > 0)$ is a particular solution of the nonlinear integral equation (15) for

$$c_{10} = \ln c_{12} - \gamma c_{12}\xi_0^{-\frac{1}{\gamma}}(\gamma - \ln\xi_0 + 1),\ c_{11} = -\gamma c_{12}\xi_0^{-\frac{1}{\gamma}} - \frac{\gamma+1}{\gamma}.$$

Substitution of this function into (10) gives a solution $u = \frac{c_{12}}{\psi'(t)} x^{-\frac{\gamma+1}{\gamma}}$ of the equation (1).

The presence of the arbitrary constants in the integral equations (12) - (15), allows one to use the obtained solutions for solving of boundary value problems. In particular, these solutions describe a nonlinear diffusion process (or a heat propagation process) for which a concentration (or a temperature) and its gradient are specified at an initial time $t = t_0 \geq 0$ at a fixed point $x = x_0 > 0$:

$$u(t_0, x_0) = u_0 > 0,\ \frac{\partial u}{\partial x}(t_0, x_0) = u_1. \tag{16}$$

In this case the constants c_k $(k=4,\ 5,\ldots,11)$ are determined by the formulas $(\psi_1 = \psi'(t_0))$:

$$\begin{aligned}
c_4 &= \left(u_0\psi_0^{\frac{\gamma(\alpha-2)+\beta}{\beta^2}}\psi_1^{-\frac{1}{\beta}}\right)^{\beta+1} - c_5\left(x_0\psi_0^{-\frac{\gamma}{\beta}}\right)^{1-\alpha},\\
c_5 &= \frac{\beta+1}{\beta(1-\alpha)}\psi_0^{\frac{\beta-\gamma(\beta+2-\alpha)}{\beta^2}}\psi_1^{-\frac{\beta+1}{\beta}}\left(\gamma x_0 u_0\psi_1 + \beta x_0^{\alpha}u_0^{\beta}u_1\psi_0\right),\\
c_6 &= \left(u_0\psi_0^{\frac{\beta-\gamma}{\beta^2}}\psi_1^{-\frac{1}{\beta}}\right)^{\beta+1} - c_7\left(\ln x_0 - \frac{\gamma}{\beta}\ln\psi_0\right),\\
c_7 &= \frac{\beta+1}{\beta}\psi_0^{\frac{\beta-\gamma(\beta+1)}{\beta^2}}\psi_1^{-\frac{\beta+1}{\beta}}x_0\left(\gamma u_0\psi_1 + \beta u_0^{\beta}u_1\psi_0\right),\\
c_8 &= \ln u_0 + (\gamma(\alpha-2)-1)\ln\psi_0 + \ln\psi_1 - c_9\left(x_0\psi_0^{\gamma}\right)^{1-\alpha},\\
c_9 &= \frac{1}{(1-\alpha)}\psi_0^{\gamma(\alpha-1)}\left(-\gamma x_0 u_0\psi_0^{-1}\psi_1 + x_0^{\alpha}u_0^{-1}u_1\right),\\
c_{10} &= \ln u_0 - (\gamma+1)\ln\psi_0 + \ln\psi_1 - c_{11}\left(\ln x_0 + \gamma\ln\psi_0\right),\\
c_{11} &= x_0\left(-\gamma u_0\psi_0^{-1}\psi_1 + u_0^{-1}u_1\right).
\end{aligned} \tag{17}$$

The unique solution of the equation (1), satisfying to the conditions (16), for which the value $(\psi(t))^{\frac{\gamma(\alpha-2)+\beta}{\beta^2}}\left((\ln\psi(t))'\right)^{-\frac{1}{\beta}}u(t,x)$ is constant along an each trajectory $x = c(\psi(t))^{\frac{\gamma}{\beta}}$ $(c = \text{const})$ exists at the neighborhood of the point $(t_0,\ x_0)$. This solution is given by the formulas (10), (12) - (15), (17).

4. Conclusion and discussion

We performed the full group classification of the nonlinear differential equation of the model, describing the nonlinear diffusion process (or the heat propagation process) in an inhomogeneous medium with non-stationary absorption or source. We found all values of the diffusion and absorption or source coefficients, for which the equation admits a continuous Lie group of the transformations, acting on the set of the solutions of this equation. We researched invariant submodels of the obtained models. Some invariant solutions, describing these submodels are found either explicitly, or their search is reduced to solving of the nonlinear integral equations. For example, we obtained the invariant solution describing the nonlinear diffusion process (or the heat propagation process) with two fixed "black holes", and the invariant solution describing the nonlinear diffusion process (or the heat distribution process) with the fixed "black hole" and the moving "black hole". The solutions with "black holes" can be used to describe the process of destruction of glaciers under the influence of the external heat source, for example a solar energy. Namely, along each ray in the glacier, under the action of the external heat source, the temperature at two points is much higher than the temperature at other points of the ray. Thus, within the glacier, the most strongly heated zones are formed in which melting of ice occurs. Over time, these zones are combined, which leads to the formation of cracks and the destruction of the glacier. In particular, this way it is possible to explain the occurrence of icebergs off the coast of Antarctic continent. With a help of the other invariant solution we have studied a diffusion process (or heat distribution process) for which at the initial moment of the time at a fixed point a concentration (a temperature) and its gradient are specified. Solving of boundary value problem describing this process is reduced to the solving of nonlinear integral equations. We are established the existence and uniqueness of solutions of these boundary value problem under some conditions.

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