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# One-dimensional vertical model of permafrost dynamics

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**Abstract**. Simple description of the temperature field in soils during their freezing or thawing is considered with the help of solutions of Stefan problems. A mathematical model is based on the equations of thermal conductivity for the frozen and thawed layers. In the vertical structure of the permafrost zone, three layers are distinguished: thawed soil, frozen soil, snow. A simplified numerical algorithm for solving one-dimensional (in the vertical direction) thermal conductivity problems with moving phase transition boundaries with the formation of new and cancellation of existing layers is proposed.

## 1. Introduction

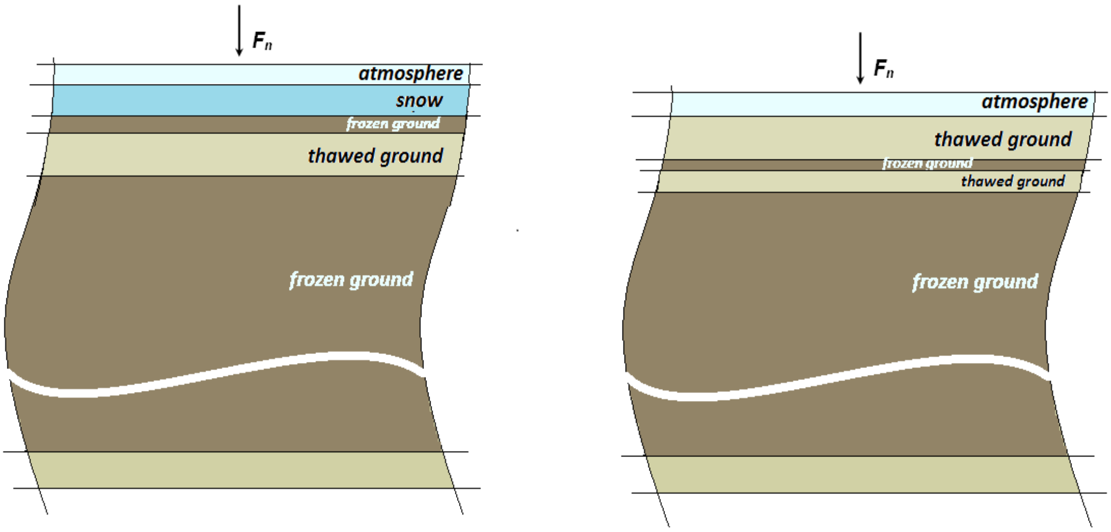
In connection with the change in global temperature the response of permafrost to climate change is interested. Both the thickness of a layer of thawed soil and the duration of its existence are increased by the climate warming. There is extensive literature on the mathematical modeling of permafrost (see [1-7]). In this paper the small scale numerical model is proposed for description vertical temperature distribution in thawed and frozen layers taking into account the formation of new and revocation of the existing layers (proposed in [8] for swamp-lake landscapes). Since the vertical temperature gradients are usually larger than the horizontal one all physical process are assumed one-dimensional in the vertical direction in the description of the heat transfer. We distinguish the following layers in the vertical direction: the snow, thawed soil, frozen soil. The theoretical description of the temperature field in soils during their freezing and thawing is carried out using solutions of Stefan problem. A mathematical model based on the equations of thermal conductivity for frozen and thawed areas. At the borders of phase transition (freezing-thawing) the conditions of equality of temperatures to the phase transition temperature and the Stefan condition are used. The formulated mathematical model of vertical temperature distributions in thawed and frozen layers takes into account the formation of new and cancellation of existing layers.

There are various options for the location of frozen and thawed layers. When switching from one variant to another the layers are added or deleted. Five options are considered (table 1). The conditions of switching from one option to another are determined. Figure 1 illustrates a vertical structure for variants 3 and 4.

**Table 1.** Variants of the location for frozen and thawed soil.

| Variant | Snow | Frozen soil | Thawed | Frozen | Thawed | Frozen | Thawed |
|---|---|---|---|---|---|---|---|

| nubmer | | | soil | soil | soil | soil | soil |
|---|---|---|---|---|---|---|---|
| 1 | + | | | | | + | + |
| 2 | | | | | + | + | + |
| 3 | + | | | + | + | + | + |
| 4 | | | + | + | + | + | + |
| 5 | + | + | + | + | + | + | + |




a) b)

**Figure 1.** Variants 3 (a) and 4 (b).

## 2. Mathematical model of dynamics of freezing and thawing of permafrost

The vertical temperature distribution in every layer are determined by solving the heat equation, satisfying the appropriate boundary conditions:

$$\frac{\partial T_i}{\partial t} = K_i \frac{\partial^2 T_i}{\partial z^2}. \tag{1}$$

Here $T_i(t,z)$ is a temperature of $i$-th layer $h_{i-1} \le z \le h_i$ ), $K_i$ is a thermal diffusivity, $t$ is time, $z$ is a vertical coordinate (downward).

Boundary conditions.

The condition on the boundary of the atmosphere–soil in summer ($z$=0):

$$\left(K\frac{\partial T}{\partial z}\right) = -\frac{F_n}{c_w \rho_w}, \tag{2}$$

the condition on the boundary of the atmosphere-snow ($z$=0): the snow temperature is set

$$T = T_{sn}. \tag{3}$$

The condition on the boundary of a snow and frozen ground ($z$=0):

$$T_i = T_{i+1}, \qquad \left(\lambda\frac{\partial T}{\partial z}\right)_i = \left(\lambda\frac{\partial T}{\partial z}\right)_{i+1}. \tag{4}$$

On moving boundaries of phase transition ( $z = h_i$ )

$$T_i = T_{i+1} = T_{ph}\,,$$

$\rho_i L_i \dfrac{d h_i}{dt} = \left(\lambda \dfrac{\partial T}{\partial z}\right)_i - \left(\lambda \dfrac{\partial T}{\partial z}\right)_{i+1}$ , if the $i$-th layer is frost and the ($i$+1)-th is thawed one, (5)

$\rho_{i+1} L_{i+1} \dfrac{d h_i}{dt} = \left(\lambda \dfrac{\partial T}{\partial z}\right)_{i+1} - \left(\lambda \dfrac{\partial T}{\partial z}\right)_i$ , if the $i$-th layer is thawed and the ($i$+1)-th is frost one. ($5^1$)

Here $\rho_w$ is water density, $c_w$ is the specific heat capacity of water, $F_n$ is the total heat flow on the boundary of the atmosphere-ground, $T_s$ is temperature on soil surface, $z = h_i$ is coordinate of the boundary between $i$-th and ($i$+1)-th layers, $\lambda$ is coefficient of heat conductivity, $T_{ph}$ is phase transition temperature, $\rho_i$ is density of $i$-th layer, $L_i = L_w \cdot W_{Vi}$ is volumetric latent heat of melting environment in $i$ – th layer, $W_{Vi} = \Omega_{wi} \cdot (\Omega_i)^{-1}$ is soil humidity in $i$-th layer, $\Omega_{wi}$ is the volume of water in the soil, $\Omega_i$ is the volume of the soi, $W_{Vi} = 10^{-3} W_i$, $W_i$ (mm/m) – is soil humidity.

The initial conditions:

$$T_i(0,z) = T_i^0, \qquad \delta_i = \delta_i^0.$$

*Parameterization of heat flow through the underlying surface.*

Heat flows are important parameters that affect soil temperature. The total heat flow $F_n$ is defining by known formulas [9,10]:

$$F_n = (1-\alpha)F_I - (F_{ef} + F_t + F_{ev}),$$

where $F_{ef}$ is effective long-wave radiation, $F_{ev}$ is heat transfer by evaporation, $F_t$ is convective heat exchange, $F_I$ is the incoming short wave radiation.

Short-wave radiation is calculated by the formula [11]

$$F_I = 0.94 \cdot Q(h_c)(1 - 0.65 n^2),$$

where

$$Q(h_c) = \left(0.66 + 0.34 \frac{\gamma - 0.9 + 0.4 \sin h_c}{0.1 + 0.4 \sin h_c}\right) \frac{\kappa_n \sin^2 h_c}{\rho^2 (\sin h_c + 0.107)},$$

$$h_c = \arcsin\left(\sin\varphi_k \sin\gamma_1 + \cos\varphi_k \cos\gamma_1 \cos\left((t - t_n)\frac{\pi}{12}\right)\right),$$

$$\gamma_1 = 0.4 + 23.4 \cdot \cos\left(\frac{2\pi}{365}(d+192)\right) - 0.4 \cdot \cos\left(\frac{2\pi}{365}(d-192)\right),$$

$\kappa_n$=1.11÷1.23 depending on the humidity content of the atmosphere, $n$ is total cloud score in fractions of a unit, $h_c$ is height of the sun in degrees, $\rho$ is air density, $\gamma$ = 0.94, $\varphi_k$ is latitude in degrees, $t$ = 0,1,…,23 is the local astronomical time; $t_n$=12 is the noon local time; $\gamma_1$ is the sun declination, $d$ is the ordinal day number since the beginning of the year.

The semi-empirical formula was used to calculate the effective radiation of the $F_{ef}$ [9,10]:

$$F_{ef} = F_{ef}^0 (1 - 0.79 n) + 3.8 \sigma (T_a + 273.15)^3 (T_s - T_a),$$

$$F_{ef}^0 = 0.95 \sigma (T_s + 273.15)^4 (0.39 - 0.058\sqrt{e}).$$

Here $T_a$ is the surface air temperature (°*C*), $T_s$ is the temperature of the underlying surface (°*C*), $n$ is total cloud score (fractions of a unit), $\sigma = 5.67\cdot 10^{-8}$ is the Stefan-Boltzmann constant (W/(m$^2$ degree$^4$ )), $e$ is the water vapor pressure in the atmosphere (in milibar).

Turbulent exchange between the underlying surface and the atmosphere is determined by the formula [9,10]

$$F_t = \rho_a c_a D(T_s - T_a),$$

where $\rho_a$ is the air density, $c_a$ is the specific heat of air, $D$ =0.0063 is the external diffusion coefficient (m/s).

Evaporation $E$ (mm/m) is calculated by the formula [9,10]

$$E = \begin{cases} E_0 & for\ W > W_0, \\ E_0 \dfrac{W}{W_0} & for\ W \le W_0, \end{cases}$$

here $W$ (mm/m) is the ground (soil) moisture, the critical soil moisture content $W_0$ is assumed equal to 200 mm/m. The evaporability of $E_0$ is determined by the [9,10]

$$E_0 = \rho_a D(e_0 - e),$$

$e_0$ - the saturating water-vapour pressure near the earth's surface is found by the Magnus formula

$$e_0 = 6.11\cdot \exp\left(\frac{17.57\cdot T_s}{241.9 + T_s}\right).$$

To determine the moisture content of the top layer of soil, a method based on the solution of the moisture balance equation is used [9,10]:

$$\delta_T \frac{dW}{dt} = r - f - E \quad , \tag{6}$$

$$f = \begin{cases} r\dfrac{W}{W_0}\sqrt{m_0{}^2\left[1-\left(1-\dfrac{E_0}{r}\right)^2\right]+\left(1-\dfrac{E_0}{r}\right)^2} & \text{for} \quad r > E_0, \\ m_0 r \dfrac{W}{W_0} & \text{for} \quad r \le E_0, \end{cases}$$

where $r$ is is rainfall (mm/h), $f$ is the water runoff (mm/h), $\delta_T$ is the thickness of the upper melt layer of soil (m), $m_0 = 0.2$.

Heat expenditure on evaporation $F_u$ is calculated by the ratio

$$F_{ev} = L_{ev} E,$$

where $L_{ev} = 600$ (kcal / kg) is latent heat of evaporation.

The snow cover is an important factor of interaction in a system of the atmosphere and underlying surface. One of the main parameters of the snow cover, which determines the heat-shielding properties, is the coefficient of the snow thermal conductivity ( $\lambda_{sn}$ ).The dependence of the coefficient of snow thermal conductivity on its density ($\rho_{sn}$) can be estimated by the Abels formula $\lambda_{sn} = 0.0068\rho_{sn}^2$ (cal (cm s degree)$^{-1}$).

The influence of snow on soil freezing was estimated by using the quasi-stationary approximation for the two layers of snow with different densities and thermal conductivity. On the border atmosphere-snow set snow temperature equals to air temperature; at the boundary of snow layers the conditions of continuity of temperature and heat flow are fulfilled. It is assumed that the temperature in the frozen soil layer ($T_2^{fr}$) is known ($z$=$z$*). To calculate the temperature of the soil on the border of snow – soil, the formula is obtained:

$$T_1^{fr} = \frac{T_a + \mu \cdot T_2^{fr}}{1+\mu}, \qquad \mu = \frac{\delta_{sn1}}{\lambda_{sn1}} + \frac{\delta_{sn2}}{\lambda_{sn2}} \cdot \frac{\lambda_{fr}}{z*}.$$

where $T_a$ is the air temperature, $\lambda_{sn1}, \lambda_{sn2}$ - snow thermal conductivity coefficients in layers 1 and 2, respectively; $\delta_{sn1}, \delta_{sn2}$ are the the thickness of the layers 1 and 2, $\lambda_{fr}$ is the thermal conductivity coefficient of frozen soil.

Snow melting begins when the temperature of the snow rises to zero in the spring. The expenses of heat for melting snow with thickness $\delta_{sn}$ is $F_{sn} = 80\rho_{sn}\delta_{sn}$ (cal/cm$^2$). Heat exchange at the snow – atmosphere boundary is $F_{a-sn} = \alpha(T_{sn} - T_a)$, heat transfer on the border of a snow – frozen ground is $F_{sn-fr} = 0.5\lambda_{fr} \cdot (T_2^{fr} - T_{sn})(\delta_{fr})^{-1}$, $\alpha = 6.44 \cdot (W_2 + 0.3)^{0.5}$ in cal/(cm$^2$s), $W_2$ is the wind speed. The time spent on the snow melting is determined by the ratio of $\tau_{sn} = F_{sn} \cdot (F_{a-sn} - F_{sn-fr})^{-1}$.

## 3. Numerical algorithm

It is assumed that the transition of moisture in the soil from the frozen state to the melt and vice versa does not change the overall moisture content. The numerical solution of the equation (6) has the form

$$\hat{W} = W^n + \Delta t \frac{(r - f - E)^n}{\delta_T},$$

$$W^{n+1} = \frac{\hat{W}\,\delta_T^n + W_{fr}(\delta_T^{n+1} - \delta_T^n)}{\delta_T^{n+1}} \quad \text{for} \quad \delta_T^{n+1} > \delta_T^n,$$

$$W_{fr}^{n+1} = \frac{W_{fr}^n\,\delta_{fr}^n + W_T^n(\delta_{fr}^n - \delta_{fr}^{n+1})}{\delta_{fr}^{n+1}} \quad \text{for} \quad \delta_T^{n+1} \le \delta_T^n, \qquad W^{n+1} = \hat{W}.$$

Where $\delta_{fr}$ is the thickness of adjacent layer of frozen soil, $W_{fr}$ is moisture reserves of frozen soil. Consider an arbitrary $i$-th layer: $h_{i-1} \le z_i \le h_i$, $\delta_i = h_i - h_{i-1}$, where $\delta_i$ is the thickness of $i$-th layer. We introduce new independent variables $(t,\xi)$:

$$t = t, \qquad \xi_i = \frac{z_i - h_{i-1}}{\delta_i}, \qquad 0 \le \xi_i \le 1.$$

Since

$$\frac{\partial}{\partial z_i} = \frac{1}{\delta_i}\frac{\partial}{\partial \xi_i}, \qquad \frac{\partial^2}{\partial z_i^2} = \frac{1}{\delta_i^2}\frac{\partial^2}{\partial \xi_i^2}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + w_i \frac{\partial}{\partial \xi_i},$$

$$w_i = \frac{\partial \xi_i}{\partial t} = \frac{1}{\delta_i}\left[(\xi_i - 1)\frac{dh_{i-1}}{dt} - \xi_i \frac{dh_i}{dt}\right],$$

then equation (1) and boundary conditions (2) – (5) written as

$$\frac{\partial T_i}{\partial t} + w_i \frac{\partial T_i}{\partial \xi_i} = \frac{K_i}{\delta_i^2}\frac{\partial^2 T_i}{\partial \xi_i^2}, \tag{7}$$

$$\left(\frac{K_i}{\delta_i}\frac{\partial T_i}{\partial \xi_i}\right)_{\xi_i=0} = -\frac{F_n}{c_w \rho_w}, \qquad T_i\big|_{\xi_i=0} = T_{ice}, \qquad T_i\big|_{\xi_i=1} = T_{i+1}\big|_{\xi_{i+1}=0}, \tag{8}$$

$$\left.\left(\frac{\lambda_i}{\delta_i}\frac{\partial T_i}{\partial \xi_i}\right)\right|_{\xi_i=1} = \left.\left(\frac{\lambda_{i+1}}{\delta_{i+1}}\frac{\partial T_{i+1}}{\partial \xi_{i+1}}\right)\right|_{\xi_{i+1}=0}, \qquad T_i\big|_{\xi_i=1} = T_{i+1}\big|_{\xi_{i+1}=0} = T_{ph}, \tag{9}$$

$$\rho_i\, L_i \frac{d\,h_i}{d\,t} = \left( \frac{\lambda_i}{\delta_i} \frac{\partial T_i}{\partial \xi_i} \right)\Bigg|_{\xi_i=1} - \left( \frac{\lambda_{i+1}}{\delta_{i+1}} \frac{\partial T_{i+1}}{\partial \xi_{i+1}} \right)\Bigg|_{\xi_{i+1}=0} . \quad (10)$$

Let us consider the solution of the formulated problems on coarse (three-point) grid for each of layers: $\xi_{i1}=0, \quad \xi_{i2}=0.5, \quad \xi_{i3}=1.$

We approximate the equation (7) using the implicit difference scheme and and directional differences for the convective terms. Grid equations corresponding to the differential equation (7) for $i$–th layer are of the form:

$$\frac{T_{i,2}-T_{i,2}^n}{\Delta t} + w^- T_{i,3} + 2w^0 T_{i,2} - w^+ T_{i,1} = \frac{K_i (T_{i,1} - 2T_{i,2} + T_{i,3})}{(0.5\delta_i^n)^2}, \quad (11)$$

$$w_{i,2} = -\left( \frac{dh_i}{dt} + \frac{dh_{i-1}}{dt} \right) / (2\delta_i^n), \qquad w^0 = \left| w_{i,2} \right|, \qquad w^- = w_{i,2} - \left| w_{i,2} \right|, \qquad w^+ = w_{i,2} + \left| w_{i,2} \right|,$$

here $\Delta t$ is time step, $t_{n+1} = t_n + \Delta t$ , $T_{i,k}^{n+1} = T_i(t_{n+1}, \xi_{ik})$ .

Boundary conditions (8) – (10) for the difference grid are represented as:
on the border of the atmosphere – soil

$$\left[ 1 + \frac{8K_i \Delta t}{(\delta_i^n)^2} \right] T_{i,1} - \frac{8K_i \Delta t}{(\delta_i^n)^2} T_{i,2} = T_{i,1}^n + \frac{4\Delta t F_n}{c_w \rho_w} \quad , \quad (12)$$

on the border of the atmosphere - snow+soil

$$T_{i,1} = T_{sn} , \quad (13)$$

at the borders of phase transition $z = h_i$

$$T_{i,3} = T_{i+1,1} = T_{ph} \quad (14)$$

$$\rho_i\, L_i \frac{h_i^{n+1} - h_i^n}{\Delta t} = \lambda_i \frac{T_{i,3} - T_{i,2}}{0.5\delta_i^n} - \lambda_{i+1} \frac{T_{i+1,2} - T_{i+1,1}}{0.5\delta_{i+1}^n} . \quad (15)$$

If $\delta_i$ becomes less than a predetermined small value $\varepsilon$ ($\delta_i < \varepsilon$), the corresponding layer will be cancelled. If due to melting or freezing a new $k$–th layer is formed, the initial thickness $\delta_k^0 = \varepsilon$ and temperatures are set for this layer.

The following algorithm is used for the obtained tasks. If all parameters (temperature distributions in layers under study and the positions of phase transition) are known on $n$-th time step, then a finding the unknown parameters at time $t_{n+1}$ is performed in two stages:
The first step is to define the temperature distribution in the selected layers (taking into account the relations (11) – (14) by solution of systems of linear algebraic equations of small dimension.
In the second phase it is clarified the position of the phase boundary by the numerical solution of equation (15).

## 4. Calculations examples

Calculations were performed for a permafrost soil using weather data of 2010-2012 years for the weather station Dudinka ($69^0$24' North latitude). 2010 year could be considered as "cold" (about 240 days with mean daily negative temperature, the average temperature for the period from 1 October to 30 April was -20.72°C); 2011 – "warm" (190 days with mean daily negative temperature and the average temperature for October to April was -15.22°C). The depth of seasonal thawing of permafrost soils, the temperature of the melt and frozen layers, the time intervals of the existence for layer of thawed soil were determined. Figure 2 shows the calculated boundaries of thawing-freezing of the soil.

**Table 2**. The thickness of the layer of thawing frozen soil (cm) and the time of its existence

| year | thickness | time existence |
|---|---|---|
| 2010 г. | 99.87 | 4.06.10 – 7.12.10 |
| 2011 г. | 138.4 | 22.04.11 – 15.02.11 |
| 2012 г. | 136.5 | 21.04.12 – |

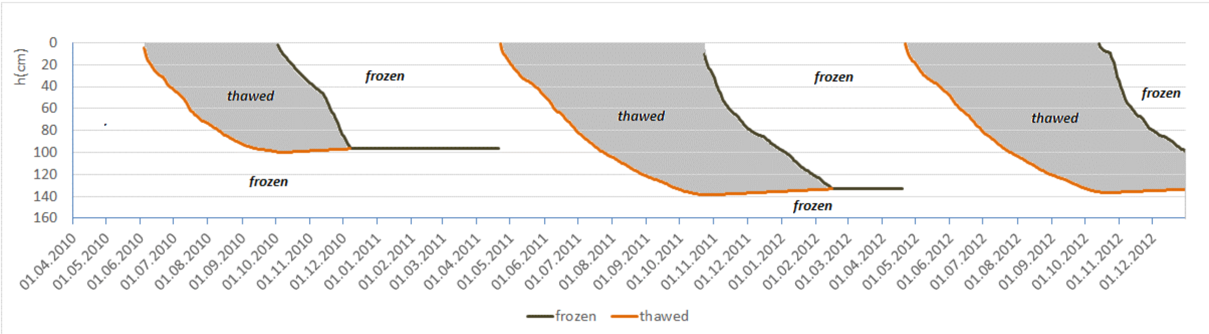



**Figure 2**. Dynamics of soil thawing-freezing.

Thus, a relatively simple one-dimensional model can be used to assess the current state and forecast of permafrost in the land regions.

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