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# The Application of Inverse Transformation in the Isoparametric Singular Element

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**Abstract**. The isoparametric element is a sort of common element which is widely used in the finite element complex calculations, and the inverse transformation of which is also used in the field of engineering, and it has the advantages such as the high iteration speed and the high computational accuracy. But when the unit form is complicated or the nodes coincide, the Jacobian Matrix will be close to 0 and singular phenomenon will exist, and also in this case the finite element calculations may be adversely affected. This article verifies the singular phenomenon of the 5-nodes and 6-nodes degraded unit when the nodes coincide, and discusses the calculation results and its accuracy of the method of inverse transformation when it comes to singular model. According to the programming calculation, the experiment result shows that the method can meet the requirements in the field of actual project, and it has a good adaptability in the calculation of engineering.

## 1. Introduction

Isoparametric element [1] is a finite element calculation unit that is designed for the transformation of the geometric shape of the unit and the parameter function in the element with the same number of node parameters and the same shape functions. Isoparametric element is widely applied in the finite element analysis of complexities with its good adaptability and coordination. Inverse transformation [2] is the coordinate transformation from the holistic coordinate system to the local coordinate system, which is common and well known as the way of reducing local coordinate from the holistic coordinate. In geotechnical engineering, on one hand, the calculation range is often larger and the finite element mesh is coarser because of the considerations of topographic features and the influences of faults and joints. However, on the other hand, the finite element mesh needs dividing more closely due to the sizes of the concerned architecture (such as slope support, underground cavern, etc.) are relatively smaller. It is usually difficult to take account of the above two points in calculations, so calculating and interpolating in large model is often used to acquire the data of small model. The method is also common in the interpolation of crustal stress and seepage field [3].

At the same time, in the process of using isoparametric element to model, the complex simulation and irregular shapes are often encountered. For example, when the seepage flow model is built, drainage and water stop will be set up. This joint unit will also be encountered in the simulation of fault and joint fracture. After the fault joint passes through the rock mass, the ordinary element is cut into degradation unit [4].Singular phenomena will also occur in calculation. Such singular elements are usually in the shape of long and narrow, and the distances between joints are close. The Jacobi matrix may tend to be 0[5].

Based on the inverse transformation of isoparametric elements, the accuracy of the algorithm is specifically discussed on the singular element in this article. Through discussion, it is found that this method has good adaptability to singular elements. The speed of calculation is fast and the accuracy is high as well, which can meet the engineering requirements.

## 2. Singularity caused by the Degradation of Isoparametric Elements

For the 4-Node isoparametric unit, a triangle unit can be obtained by merging two adjacent nodes. For example, if the 1-Node and the 2-Node in Figure 1 are combined, a constant strain triangular unit [6] will be obtained.

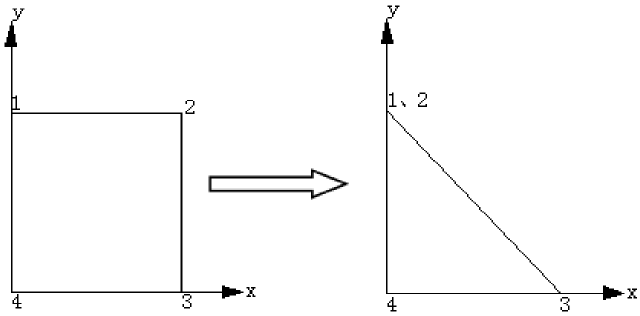


Figure 1. The 4-Node unit degenerates into the 3-Node unit

By the coordinate transformation of the isoparametric unit, so

$$\begin{cases} x = \frac{1}{4}(1+\xi)(1+\eta)x_1 + \frac{1}{4}(1-\xi)(1+\eta)x_2 + \frac{1}{4}(1-\xi)(1-\eta)x_3 + \frac{1}{4}(1+\xi)(1-\eta)x_4 \\ y = \frac{1}{4}(1+\xi)(1+\eta)y_1 + \frac{1}{4}(1-\xi)(1+\eta)y_2 + \frac{1}{4}(1-\xi)(1-\eta)y_3 + \frac{1}{4}(1+\xi)(1-\eta)y_4 \end{cases} \quad (1)$$

Make $x_1 = x_2, y_1 = y_2$, so

$$\begin{cases} x = \frac{1}{2}(1+\eta)x_2 + \frac{1}{4}(1-\xi)(1-\eta)x_3 + \frac{1}{4}(1+\xi)(1-\eta)x_4 \\ y = \frac{1}{2}(1+\eta)y_2 + \frac{1}{4}(1-\xi)(1-\eta)y_3 + \frac{1}{4}(1+\xi)(1-\eta)y_4 \end{cases} \quad (2)$$

Bring the specific node coordinates shown in the above figure into the above two formulas, so

$$\begin{cases} x = \frac{1}{2}(1+\xi)(1-\eta) \\ y = 1+\eta \end{cases} \quad (3)$$

Thus,

$$\frac{\partial x}{\partial \xi} = \frac{1}{2}(1-\eta), \quad \frac{\partial y}{\partial \xi} = 0, \quad \frac{\partial x}{\partial \eta} = -\frac{1}{2}(1+\xi), \quad \frac{\partial y}{\partial \eta} = 1 \quad (4)$$

Therefore, the Jacobian matrix can be obtained,

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-\eta & 0 \\ -(1+\xi) & 2 \end{bmatrix} \rightarrow [J]^{-1} = \begin{bmatrix} \frac{2}{1-\eta} & 0 \\ \frac{1+\xi}{1-\eta} & 1 \end{bmatrix} \quad (5)$$

From the above formula, when $\eta$ approaches 1, the Jacobian Matrix J approaches 0 and the Jacobian Matrix J cannot be inverted.

Similarly, when the four points on the same plane of the spatially eight-node spatial isoparametric unit coincide, the Jacobian Matrix of the 8-Node spatial isoparametric element can be obtained.

$$[J] = \frac{1}{4}\begin{bmatrix} (1-\eta)(1-\xi) & 0 & 0 \\ -(1+\xi)(1-\zeta) & 2(1-\zeta) & 0 \\ -(1+\xi)(1-\eta) & -2(1+\eta) & 4 \end{bmatrix} \quad (6)$$

From the above formula, when $\eta$ approaches 1 or $\zeta$ approaches 1, the Jacobian Matrix J approaches 0, then the Jacobian Matrix J cannot be inverted.

## 3. Inverse Transform Algorithm

When the overall coordinates of the node to be determined are known, the inverse transform algorithm can be used to obtain the local coordinates. The method owns a wide applicability, with a high calculation precision. The specific algorithm is as follows:

Take spatial isoparametric unit as an example, the expression of the coordinate transformation M:L→G of the unit is as followed:

$$\begin{cases} x = \sum_{i=1}^{n} N_i(\xi,\eta,\zeta)x_i \\ y = \sum_{i=1}^{n} N_i(\xi,\eta,\zeta)y_i \\ y = \sum_{i=1}^{n} N_i(\xi,\eta,\zeta)y_i \end{cases} \quad (7)$$

Where Ni is a shape function, xi, yi, zi are the node coordinates of the unit, and n is the number of nodes of the unit. Let x, y, z in equation (3) be expanded with Taylor series at (0, 0, 0), only leaving linear terms, then

$$\begin{cases} y \approx y_0 + \left.\frac{\partial y}{\partial \xi}\right|_{(\xi_0,\eta_0,\zeta_0)}(\xi-\xi_0) + \left.\frac{\partial y}{\partial \eta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\eta-\eta_0) + \left.\frac{\partial y}{\partial \zeta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\zeta-\zeta_0) \\ y \approx y_0 + \left.\frac{\partial y}{\partial \xi}\right|_{(\xi_0,\eta_0,\zeta_0)}(\xi-\xi_0) + \left.\frac{\partial y}{\partial \eta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\eta-\eta_0) + \left.\frac{\partial y}{\partial \zeta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\zeta-\zeta_0) \\ z \approx z_0 + \left.\frac{\partial z}{\partial \xi}\right|_{(\xi_0,\eta_0,\zeta_0)}(\xi-\xi_0) + \left.\frac{\partial z}{\partial \eta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\eta-\eta_0) + \left.\frac{\partial z}{\partial \zeta}\right|_{(\xi_0,\eta_0,\zeta_0)}(\zeta-\zeta_0) \end{cases} \quad (8)$$

In the formula, (x0, y0, z0) is the overall coordinate corresponding to the local coordinates (0, 0, 0). It can be expressed as a matrix.

$$\tilde{X} = \tilde{X}_0 + \tilde{J}_0(\tilde{\xi} - \tilde{\xi}_0) \quad (9)$$

Where $J_0$ is the Jacobi matrix at local coordinates ( 0, 0, 0). Equation (5) is a linear expression that can be solved directly.

$$\tilde{\xi} = \tilde{\xi}_0 + \tilde{J}_0^{-1}(\tilde{X} - \tilde{X}_0) \quad (10)$$

$J_0^{-1}$ is the Jacobi inverse matrix at ~0. From the equation (6), an iterative formula according to the local coordinates of any point in the unit can be established.

$$\tilde{\xi}_{k+1} = \tilde{\xi}_k + \tilde{J}_k^{-1}(\tilde{X}_P - \tilde{X}_k) \quad k = 0, 1, 2\ldots\ldots \quad (11)$$

$\tilde{\xi}_k$ is the $K_{th}$ approximation representing $\tilde{\xi}$.

When $\| \tilde{X}_P - \tilde{X}_k \| \le \varepsilon$ ,then $\tilde{X}_P = \tilde{X}_k$ .

## 4. Discussion on the Precision of Inverse Transform Algorithm

The iterative format of the inverse transform algorithm is

$$\tilde{\xi}_{k+1} = \tilde{\xi}_k + \tilde{J}_k^{-1}(\tilde{x}_p - \tilde{x}_k)$$

From the iterative format, Jacobian matrix needs to be inverted every time it is iterated. When the calculated unit is a singular element and the point distance to be determined is close, the Jacobian Matrix tends to 0. The accuracy problems of the inverse transform algorithm in the degenerate unit mode will be discussed below.

As shown in Figure 2 below, when Node 1, 2, 3, and 4 coincide [7], the point P to be determined is close to the coincidence point.

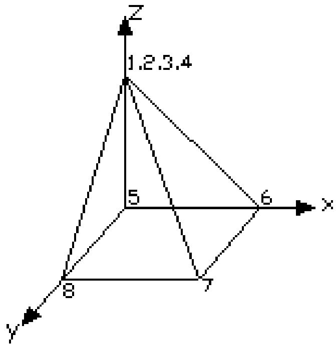


Figure 2. The 8-node unit degenerates into the 5-node singular unit

Assign the following three-dimensional coordinate values to eight nodes, where the coordinates of Node 1, 2, 3, and 4 are the same.

$(x_1,y_1,z_1)=(0,0,10)$ $(x_2,y_2,z_2)=(0,0,10)$ $(x_3,y_3,z_3)=(0,0,10)$ $(x_4,y_4,z_4)=(0,0,10)$

$(x_5,y_5,z_5)=(0,0,0)$ $(x_6,y_6,z_6)=(10,0,0)$ $(x_7,y_7,z_7)=(10,10,0)$ $(x_8,y_8,z_8)=(0,10,0)$

Meanwhile, the point P to be determined is very close to Point 1, and four sets of coordinate can be taken separately to test accuracy:

（0,0,9.9） （0,0 ,9.999） （0,0,9.99999） （0.0001,0.0001,9.9999）

Using the FORTRAN language [8], according to the inverse transformation of the isoparametric element, the program is programmed. From the overall coordinates to determine the local coordinates, the calculation results are as follows:

Table 1. Calculation results of inverse transformation of the 5-node degenerate unit

| CT | OCP(x, y, z) | LCP(ξ, η, ζ) | JMM[J] | IT | ES |
|---|---|---|---|---|---|
| 1 | （0,0,9.9） | (-1,-1,-0.98) | -1.25000500E-02 | 4 | (-1,-1,-0.9799999) |
| 2 | （0,0 ,9.999） | (-1,-1,-0.9997999) | -1.25116010E-06 | 4 | (-1,-1,-0.9997999) |
| 3 | （0,0,9.99999） | (-1,-1,-0.9999981) | -1.13686840E-10 | 4 | (-1,-1,-0.9999993 ) |
| 4 | （E-04, E-04,9.9999） | (-0.801,-0.801,-0.99979) | -1.25116010E-06 | 4 | (-0.801,-0.801,-0.9997999) |

Remark：CT- calculation time；OCP- Overall coordinate points；LCP- Local coordinate points；JMM- Jacques Matrix model；IT- iterative times；ES- Exact solution

From the calculation results in the above table, the 5-Node degenerate unit has higher calculation accuracy and can meet the engineering requirements. For further demonstration of the general applicability of the inverse transform method, the calculation accuracy in the 6-Node degenerate unit mode is analyzed below.

As shown in Table 3, when the Nodes 1 and 2 coincide at one point and the Nodes 3 and 4 coincide at one point, the point P to be determined is close to the coincidence Points 1 and 2.

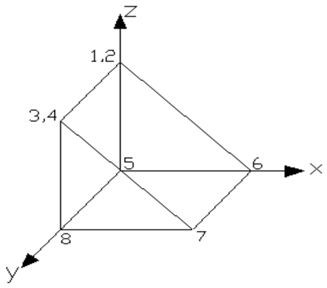


Figure 3. The 8-node unit degenerates into the 6-node singular unit

The following three-dimensional coordinate values are assigned to eight nodes, wherein the coordinates of the first and second nodes are the same, and the coordinates of the three and four nodes are the same.

$(x_1,y_1,z_1)=(0,0,10)$ $(x_2,y_2,z_2)=(0,0,10)$ $(x_3,y_3,z_3)=(0,10,10)$ $(x_4,y_4,z_4)=(0,10,10)$
$(x_5,y_5,z_5)=(0,0,0)$ $(x_6,y_6,z_6)=(10,0,0)$ $(x_7,y_7,z_7)=(10,10,0)$ $(x_8,y_8,z_8)=(0,10,0)$

Meanwhile the Point P to be determined is very close to Point 1, and four sets of coordinate can be taken separately to test accuracy:

(0,0,9.9) (0,0 ,9.999) (0,0,9.99999) (0.0001,0.0001,9.9999)

The same as the solution method of the 5-Node degenerate unit, the result of the inverse transformation of the 6-Node degenerate unit is as followed:

Table 2. Calculation results of inverse transformation of Node 6 degradation unit

| CT | OCP(x, y, z) | LCP(ξ, η, ζ) | JMM[J] | IT | ES |
|---|---|---|---|---|---|
| 1 | (0,0,9.9) | (-1,-1,-0.98) | -1.250003 | 4 | (-1,-1,-0.9799999) |
| 2 | (0,0 ,9.999) | (-1,-1,-0.9997999) | -1.25058E-02 | 4 | (-1,-1,-0.9997999) |
| 3 | (0,0,9.99999) | (-1,-1,-0.9999981) | -1.1920929E-4 | 3 | (-1,-1,-0.9999982) |
| 4 | (E-04,E-04,9.9999) | (-0.8003,-0.9999,-0.99998) | -1.2516975E-03 | 4 | (-0.8004,-0.999,-0.99998) |

Remark：CT- calculation time；OCP- Overall coordinate points；LCP- Local coordinate points；JMM- Jacques Matrix model；IT- iterative times；ES- Exact solution

From the calculation results, it can be found:

（1）The calculation accuracy is high, the result can be accurate to 7 digits after the decimal point, and the speed of iteration is fast. This calculation only needs to iterate 3 to 4 times to meet the accuracy requirement.

（2）In the calculation of the 5-Node degraded unit, the Jacobian Matrix can reach the rank of 10-10, which has met the engineering requirements, not requiring higher calculation accuracy.

（3）Comparing the program calculation results with the exact solution, the error is very small and can be ignored.

（4）Compared to the 5-Node degradation mode, in the 6-Node degradation mode, the modulus of the Jacobian Matrix is relatively large, far from E-06, and the singular phenomenon is not obvious, which can meet the engineering calculation accuracy requirements.

（5）Although the inverse transform algorithm cannot directly calculate the local coordinates close to the coincident singular points, according to the calculation data, the program can achieve the actual engineering requirements. For local coordinates that are very close to the singular point, the difference can be directly performed without program calculation.

（6）Having finished the transformation from overall coordinates to local coordinates, the correlation between the geostress field or the seepage field of the corresponding model unit can be obtained by the unit's searching and interpolation [9].

## 5. Conclusion

When the unit shape is complex or the nodes are coincident, the modulus of the Jacobian Matrix will be close to 0 and singular phenomenon will exist, and in the cause the finite element calculation will be affected. Two degenerate units, the 5-nodes and the 6-nodes, are used in this article. In the singular unit mode, the inverse transformation of the isoparametric element is used, achieving the transformation from overall coordinates to local coordinates. In these two degradation modes, the modulus of the Jacobian Matrix J is very small near the singular point with an obvious singularity. However, adopting the inverse transform algorithm in this article, the two degenerate units generally get the final calculation results after 3 to 4 iterations. The iterative speed is fast, the calculation accuracy is high, and the result can also meet the engineering requirements. Therefore, the method has good adaptability in various singular isoparametric units and can be widely applied in the interpolation calculation of various units.

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