Numerical solving of highly viscous fluids filtration in porous media for nonlinear filtration laws with power growth

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Abstract. We study the steady-state filtration process of an incompressible high-viscosity fluid follows the nonlinear filtration law. The generalized statement of this problem is formulated in the form of an operator equation with a monotone operator in a Banach space. To solve this operator equation, we propose an iteration method that does not require the inversion of the original operator. Each step of the iterative process reduces to solving the boundary value problem for the Laplace equation. In the Matlab environment, a software complex was developed, with the help of which numerical calculations were performed for model filtration problems. The analysis of numerical results is carried out.

1. Introduction

The specialists are interested in rational development of oil fields, aimed at minimizing the number of wells, as their increase leads to increased environmental pollution. The specificity of a viscoplastic fluid consists in the fact that it remains immobile in a porous medium if the pressure gradient modulus does not exceed a certain limiting value (limiting gradient). During the filtration, areas of immobile oil (stagnant zones) may arise. Determining the boundaries of such zones is undoubtedly an urgent problem.

In this paper, we study the process of steady-state filtration of an incompressible high-viscosity fluid follows a nonlinear filtration law with a limiting gradient (see, e.g., [1–7]). We assume that the function corresponding to this law is continuous, non-decreasing and has power growth, not necessarily linear, on sets corresponding to the flow zone. The generalized statement of this problem is formulated in the form of an operator equation in a Banach space. Under the conditions formulated above on the function that defines the filtration law, the operator in this equation is monotone, continuous, and coercive [8]. These properties of the operator allow us to use the general results of the theory of monotone operators [8–10] to prove the solvability of the equation. To solve the operator equation, an iterative method is proposed that does not require the inversion of the original operator. Each step of the iterative process reduces to solving the boundary value problem for the Laplace equation. Under additional conditions on the function describing the filtration law, the convergence of the process is established. The proposed method was implemented numerically. The results of numerical experiments are carried out for model problems. These results confirmed the effectiveness of the proposed iteration method. The behavior of the boundaries of stagnant zones is studied depending on the exponent in the filtration law function.

We note that exact solutions are known for a number of filtration area and filtration laws (when the filtration law function has linear growth in the flow zone) (see, e.g., [1–3]). However, in the case of arbitrary filtration area (including three-dimensional ones) and an arbitrary degree of growth in the filtration law function, the solving of the seepage problems is possible only using approximate methods.

2. The problem statement

We consider the problem of determining the stationary pressure fields p and the filtration rate $w = (w_1, w_2)$ of a high-viscosity fluid in the domain Ω of the plane R^2 satisfying the continuity equation and the effective nonlinear filtration law with a limiting gradient and the corresponding boundary conditions

$$\text{div } w(x) = f_*(x), \, w(x) = \frac{g(|\nabla p(x)|)}{|\nabla p(x)|} \nabla p(x), \, x \in \Omega, \, p(x) = 0, x \in \Gamma_1, (w, \mathbf{n}) = 0, x \in \Gamma_2 \,, \tag{1}$$

where $\Gamma = \Gamma_1 \bigcup \Gamma_2$, Γ is the boundary of the domain Ω, $\mathbf{n}$ is the unit outer normal to Γ vector, f_* is the density of external sources, and g is the function that determines a filtration law. We assume that the function g satisfies the following conditions (see Figure 1):

$$g \text{ is continuous, does not decrease, } g(\xi) = 0 \text{ for } \xi \le \beta, \tag{2}$$

here $\beta \ge 0$ is a limiting gradient; there exist $c_0 > 0$, $c_1 > 0$, $c_2 > 0$, $p \ge 2$ such that

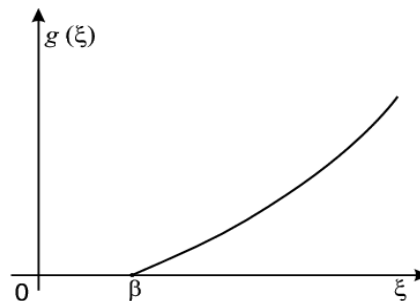$$c_0 \xi^{p-1} - c_1 \le g(\xi) \le c_2 \xi^{p-1} \text{ for } \xi \ge \beta. \tag{3}$$

Figure 1. Nonlinear power filtration law

We define the operator $G: R^2 \to R^2$ by the formula $G(y) = g(|y|)y/|y|$, $y \ne 0$, $G(0) = 0$. Let's $V = \{\eta \in W_p^{(1)}(\Omega) : \eta(x) = 0, x \in \Gamma_1\}$ is the Sobolev space, f_* generate a linear continuous functional on V. By virtue of conditions (2), (3), the form $a(u, \eta) = \int_\Omega (G(\nabla u)(\nabla u, \nabla \eta) dx$ generates the operator $A: V \to V^* = W_{p_*}^{(-1)}(\Omega)$, $p^* = p/(p-1)$, by the formula $\langle Au, \eta \rangle = a(u, \eta)$ (see [11]), where $\langle \cdot, \cdot \rangle$ is the duality relation between V and V^*, V^* is the space conjugate to V (see [9]). By the solution of the filtration problem (1) we mean a function $u \in V$ satisfying equation

$$Au = f \,, \tag{5}$$

where the element $f \in V^*$ is determined by the formula $\langle f, \eta \rangle = \int_\Omega f_* \eta \, dx$. The operator A is continuous, monotone, and coercive; therefore, equation (1) has at least one solution (see, e.g., [8–10]). We note that in [12–16] the filtration problems with the multivalued filtration law in the case $p = 2$ are studied. The filtration problem was formulated in the form of a variational inequality.

3. Iterative method and numerical experiments.

To solve the operator equation (5), we use a two-layer iterative process of the form [17–20]

$$-\Delta(u^{k+1} - u^k) = \tau_k (Au^k - f), \quad k = 0, 1, 2, \dots, \tag{6}$$

where $\tau_k > 0$ are the iteration parameters, u^0 is given element. If the function g, in addition to (2), (3) satisfies also condition (see. [11, 17]) $(g(\xi) - g(\zeta))/(\xi - \zeta) \le c_3(1 + \xi + \zeta)^{p-2}$, $c_3 > 0$ and $\tau_k = \min\{1, 1/(\alpha + \mu_k)\}$, $\mu_k = \mu(\|u^k\| + \|Au^k - f\|_{V^*}^{-1})$, , where $\mu(\xi) = c_4(1 + 2\xi)^{p-2}$; $c_4 > 0$; α is an arbitrary positive number, then the iterative sequence $\{u^k\}$ constructed according to (6) is bounded; any weakly limit point of the sequence $\{u^k\}$ is a solution of problem (5) [18]. A study of the convergence of the method is based on the proof of the quasi-potentiality and bounded Lipschitz continuity [9, 15] of the

operator A: for all $u, \eta \in V$, the inequality $\|Au - A\eta\|_{V^*} \leq \rho(R)\Phi(\|u - \eta\|)$ holds, where $R = \max\{\|u\|, \|\eta\|\}$, ρ is the non-decreasing on $[0, +\infty)$ function; Φ is the increasing on $[0, +\infty)$ function, $\Phi(0) = 0$, $\Phi(\xi) \to +\infty$ for $\xi \to +\infty$. In this case, in the two-layer iterative method considered in [6, 16], as a pre-conditioner, generally speaking, is the nonlinear duality operator [8], generated by the function Φ. For the filtration problem studied in the present paper, the operator A generated by the form (4) satisfies the condition of bounded Lipschitz continuity with functions $\rho = \mu$ and $\Phi(\xi) = \xi$, and the Laplace operator acts as a pre-conditioner, and the iterative process is written in the form (6).

A software package in the MatLab environment is developed. Numerical experiments for model filtration problems are carried out. The dependence of the boundaries of stagnant zones, i.e., the sets in the filtration area, on which the modulus of the pressure gradient is less than the limiting one and the flow is absent, on the degree of p in the filtration law was investigated.
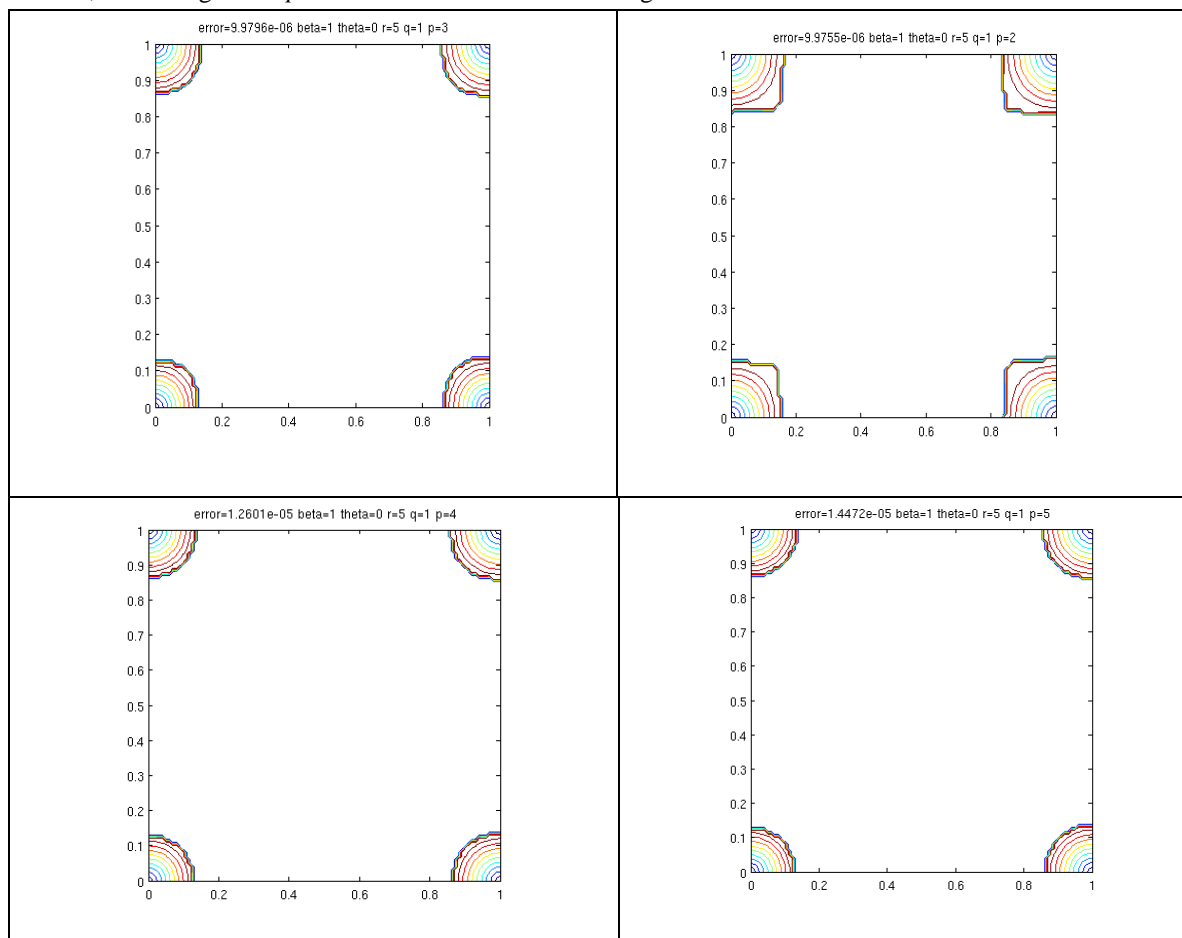

Figure 2. The boundaries of the stagnant zones at a values of p: a) 2; b) 3; c) 4; d) 5.

The results of numerical experiments, i.e., the boundaries of stagnant zones, are shown in Fig. 2. The filtration area was a unit square, in the center of which there is a well with a production rate $q = 1$, we put $\beta = 1$, $g(\xi) = (\xi - \beta)^{p-1}$ for $\xi \geq \beta$. It should be noted that for $p = 2$ (the case of a Hilbert space), the results (see Fig. 2a) correspond to the results obtained by the splitting method [21–25]. As can be seen from Fig. 2, with increasing p, the stagnant zones decrease, which is explained by the increase in the flow intensity in the flow zones. The results obtained correspond to the expected flow pattern.

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