

Dynamic Analysis Model of Rubber Bearing Based on Geometric Nonlinearity

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Abstract. According to the rubber bearing, the mathematical model of rubber bearing based on geometric nonlinear is established by using Hamilton's principle. Based on the assumption of homogeneous column, the dynamics model is investigated considering the cross section rotated and the influence of the shear deformation and axial pressure. Finally, the geometric nonlinearity governing equation and boundary conditions of the rubber bearing is deduced.

1 Introduction

The key issues of isolation technology is to install flexible device on the base or upper part of building structure to extend the natural vibration period of isolation system, so as to avoid the relatively concentrated frequency of seismic energy and reduce the seismic response of superstructure. The laminated rubber bearing is the key component in the isolated building. The horizontal and vertical mechanical properties and stability of the laminated rubber bearing is researched [1-5]. But all the research based on the basic mechanics model of static and it is larger difference from the mechanical model in dynamic finite element analysis and experimental research. Therefore, it is necessary to study a dynamic model for the rubber bearing.

2 Governing Differential Equation

As shown in figure 2, considering that the total height of the isolation system is H with laminated rubber seismic isolation bearing on the top, the laminated structure consisting of steel plates and rubber sheets is simplified as equivalent, continuous and uniform columns. Meanwhile, considering the influence of bend deformation and shear deformation, the cross section is A_r ; the diameter is d ; equivalent density is ρ_r ; modified bend elastic modulus is E_r ; shear modulus is G_r ; moment of inertia of cross section is I_r . The lower part is reinforced concrete column, to which the seismic isolation bearing is connected at the height of h . The cross section is A_c ; equivalent density is ρ_c ; elastic modulus is E_c ; shear modulus is G_c and moment of inertia of cross section is I_c . The other side $y = 0$ is connected to the foundation.



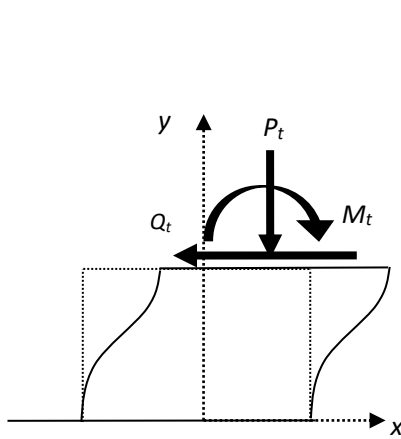


Figure 1 Deformation of rubber bearing

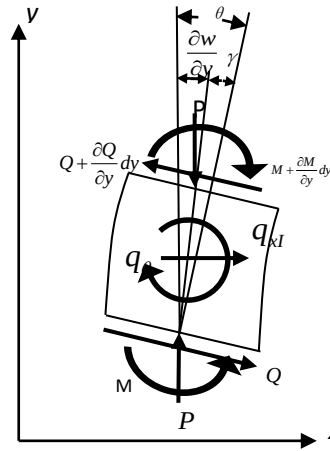


Figure 2 Differential element

2.1 Displacement Field

The centre line of homogeneous column [6-10] is y-axis and the symmetry axis of cross section is x-axis, according to first-class shear deformation beam theory, the displacement fields are listed as below:

$$\begin{cases} U_y = u(y, t) - x\varphi(y, t) \\ U_x = v(y, t) \end{cases} \quad (1)$$

In the equation, u and v refer to the vertical and horizontal displacement on the column centre line, respectively; φ refers to the normal angle of cross section after deformation.

2.2 Nonlinear Geometric Equation

Based on the displacement fields mentioned above, under the framework of limited deformation, the nonlinear geometric equation can be obtained as below:

$$\begin{cases} \varepsilon_y = \frac{\partial u}{\partial y} + \frac{1}{2} \left(\frac{\partial v}{\partial y} \right)^2 - x \frac{\partial \varphi}{\partial y} \\ \gamma_{yx} = \frac{\partial v}{\partial y} - \varphi \end{cases} \quad (2)$$

2.3 Constitutive Equation

It can be seen from the beam theory that as for homogeneous column, if linear and isotropic elastic materials are assumed, the constitutive equation is:

$$\sigma_y = E \varepsilon_y, \quad \tau_{yx} = G \gamma_{yx}, \quad G = \frac{E}{2(1 + \mu)} \quad (3)$$

In the equation, E refers to the modified bend elastic modulus of laminated rubber bearing; G refers to shear modulus; μ refers to Poisson ratio.

2.4 Control Equation

In order to deduce the motion differential equation, boundary conditions and initial condition that

displacement, u , v and φ , of geometric nonlinear elastic homogeneous column structure satisfy, Hamilton variation principle is applied. In satisfying geometric nonlinear equation, displacement and boundary conditions and enabling all the possible displacement of designated motion at the initial and termination moment, the actual displacement, u , v and φ , make the functional select stationary values.

$$\Pi = \int_0^T H dt = \int_0^T -(U - W - T) dt \quad (4)$$

In the equation, $H = -(U - W - T)$ refers to Hamilton function and T refers to kinetic energy of the structure; $U = U_1 + U_2$ refers to strain energy and U_1 and U_2 refer to the strain energy raised by normal strain and shear strain respectively; W refers to the work applied by horizontal external load and axial force.

$$T = \int_0^H \frac{1}{2} \rho_r \left\{ A_r \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right] + I_r \left(\frac{\partial \varphi}{\partial t} \right)^2 \right\} dy \quad (5)$$

In the equation, the second term is the kinetic energy of rotation, $I_r = \iint_A x^2 dA$, it is the moment of inertia.

The strain energy includes two part, that is tension-compression strain energy and the shear strain energy. The strain energy density is:

$$W_1 = \int_0^{\varepsilon_y} \sigma_y d\varepsilon_y = \int_0^{\varepsilon_y} E_r \varepsilon_y d\varepsilon_y = \frac{1}{2} E_r \varepsilon_y^2 \quad (6)$$

$$W_2 = \kappa_r \int_0^{\gamma_{yx}} \tau_{yx} d\gamma_{yx} = \kappa_r \int_0^{\gamma_{yx}} G_r \gamma_{yx} d\gamma_{yx} = \frac{1}{2} \kappa_r G_r \gamma_{yx}^2 \quad (7)$$

In the equation, κ_r is the shear correction factor, it represent to modify the assumption of the shear strain is constant in the cross section [11]. And it is related to the cross section shape and the poisson ratio,

$$\text{Rectangular section: } \kappa = \frac{10(1+\mu)}{12+11\mu};$$

$$\text{Circular section: } \kappa = \frac{6(1+\mu)}{7+6\mu}; \quad (8)$$

$$\text{circular ring section: } \kappa = \frac{6(1+\mu)(1+m^2)^2}{(7+6\mu)(1+m^2)^2 + (20+12\mu)m^2}, \quad m = r_{\text{inner}}/r_{\text{outer}};$$

So,

$$\begin{aligned} U_1 &= \iint_A \int_0^H W_1 dy dA = \iint_A \int_0^H \frac{1}{2} E_r \varepsilon_y^2 dy dA \\ &= \iint_A \int_0^H \frac{1}{2} E_r \left(\frac{\partial u}{\partial y} + \frac{1}{2} \left(\frac{\partial v}{\partial y} \right)^2 - x \frac{\partial \varphi}{\partial y} \right)^2 dy dA \\ &= \frac{1}{2} E_r A_r \int_0^H \left\{ \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial u}{\partial y} \left(\frac{\partial v}{\partial y} \right)^2 + \frac{1}{4} \left(\frac{\partial v}{\partial y} \right)^4 \right\} dy + \frac{1}{2} E_r I_r \int_0^H \left(\frac{\partial \varphi}{\partial y} \right)^2 dy \\ &= U_{11} + U_{12} \end{aligned} \quad (9)$$

And,

$$\begin{aligned}
U_2 &= \iint_A \int_0^H W_2 dy dA = \iint_A \int_0^H \frac{1}{2} \kappa_r G_r \gamma_{yx}^2 dy dA \\
&= \iint_A \int_0^H \frac{1}{2} \kappa_r G_r \left(\frac{\partial v}{\partial y} - \phi \right)^2 dy dA \\
&= \frac{1}{2} \kappa_r G_r A_r \int_0^H \left(\frac{\partial v}{\partial y} - \phi \right)^2 dy
\end{aligned} \tag{10}$$

The work of external force includes distribution force and the known force at the end, that is:

$$W = \int_0^H (pu + qv) dy + \frac{P}{2} \int_0^H \left(\frac{\partial v}{\partial y} \right)^2 dy + Pu(H) + Q_t v(H) + M_t \phi(H) \tag{11}$$

In the equation, P, Q_t and M_t are the force and moment at y=H respectively.

The variational expression of the kinetic energy, strain energy and the work of external force are:

$$\begin{aligned}
\delta T &= \rho_r \int_0^H \left\{ A_r \left[\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \delta u \right) + \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial t} \delta v \right) \right] + I_r \frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial t} \delta \phi \right) \right\} dy \\
&\quad - \rho_r \int_0^H \left[A_r \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v \right) + I_r \frac{\partial^2 \phi}{\partial t^2} \delta \phi \right] dy
\end{aligned} \tag{12}$$

$$\begin{aligned}
\delta U_{11} &= \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \delta u \right\} dy + \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] \delta v \right\} dy \\
&\quad - \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \right\} \delta u dy - \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] \right\} \delta v dy
\end{aligned} \tag{13}$$

$$\delta U_{12} = \frac{1}{2} E_r I_r \int_0^H \left[\frac{\partial}{\partial y} \left(2 \frac{\partial \phi}{\partial y} \delta \phi \right) \right] dy - \frac{1}{2} E_r I_r \int_0^H \left[\frac{\partial}{\partial y} \left(2 \frac{\partial \phi}{\partial y} \right) \right] \delta \phi dy \tag{14}$$

So,

$$\delta U_1 = \delta U_{11} + \delta U_{12} \tag{15}$$

$$\begin{aligned}
\delta U_2 &= \frac{1}{2} \kappa_r G_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \left(\frac{\partial v}{\partial y} - \phi \right) \delta v \right] \right\} dy - \frac{1}{2} \kappa_r G_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \left(\frac{\partial v}{\partial y} - \phi \right) \right] \right\} \delta v dy \\
&\quad - \frac{1}{2} \kappa_r G_r A_r \int_0^H \left[2 \left(\frac{\partial v}{\partial y} - \phi \right) \delta \phi \right] dy
\end{aligned} \tag{16}$$

$$\begin{aligned}
\delta W &= \int_0^H (p \delta u + q \delta v) dy + \frac{P}{2} \int_0^H 2 \left[\frac{\partial v}{\partial y} \delta \left(\frac{\partial v}{\partial y} \right) \right] dy \\
&\quad + P \delta u(H) + Q_t \delta v(H) + M_t \delta \phi(H) \\
&= \int_0^H \left(q - P \frac{\partial^2 v}{\partial y^2} \right) \delta v dy + P \int_0^H \left[\frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \delta v \right) \right] dy \\
&\quad + \int_0^H p \delta u dy + P \delta u(H) + Q_t \delta v(H) + M_t \delta \phi(H)
\end{aligned} \tag{17}$$

Eq.12~Eq.17 are substituted into equation $\delta \Pi = 0$. So,

$$\begin{aligned}
 \delta \Pi &= \int_0^T (\delta T + \delta W - \delta U) dt \\
 &= \rho_r \int_0^T \int_0^H \left\{ A_r \left[\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \delta u \right) + \frac{\partial}{\partial t} \left(\frac{\partial v}{\partial t} \delta v \right) \right] + I_r \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial t} \delta \varphi \right) \right\} dy dt \\
 &\quad - \rho_r \int_0^T \int_0^H \left[A_r \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v \right) + I_r \frac{\partial^2 \varphi}{\partial t^2} \delta \varphi \right] dy dt \\
 &\quad + \int_0^T \int_0^H \left(q - P \frac{\partial^2 v}{\partial y^2} \right) \delta v dy dt + \int_0^T P \int_0^H \left[\frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \delta v \right) \right] dy dt \\
 &\quad + \int_0^T \int_0^H p \delta u dy dt + \int_0^T [P \delta u(H) + Q_t \delta v(H) + M_t \delta \varphi(H)] dt \\
 &\quad - \int_0^T \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \delta u \right\} dy dt \\
 &\quad - \int_0^T \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] \delta v \right\} dy dt \\
 &\quad + \int_0^T \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \right\} \delta u dy dt + \int_0^T \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] \right\} \delta v dy dt \\
 &\quad - \int_0^T \frac{1}{2} E_r I_r \int_0^H \left[\frac{\partial}{\partial y} \left(2 \frac{\partial \varphi}{\partial y} \delta \varphi \right) \right] dy dt + \int_0^T \frac{1}{2} E_r I_r \int_0^H \left[\frac{\partial}{\partial y} \left(2 \frac{\partial \varphi}{\partial y} \right) \right] \delta \varphi dy dt \\
 &\quad - \int_0^T \frac{1}{2} \kappa_r G_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \left(\frac{\partial v}{\partial y} - \varphi \right) \delta v \right] \right\} dy dt \\
 &\quad + \int_0^T \frac{1}{2} \kappa_r G_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \left(\frac{\partial v}{\partial y} - \varphi \right) \right] \right\} \delta v dy dt + \int_0^T \frac{1}{2} \kappa_r G_r A_r \int_0^H \left[2 \left(\frac{\partial v}{\partial y} - \varphi \right) \delta \varphi \right] dy dt = 0
 \end{aligned} \tag{18}$$

Thus, the motion control equation of rubber bearing based on the geometric nonlinear is:

$$\rho_r A_r \frac{\partial^2 u}{\partial t^2} - E_r A_r \frac{\partial^2 u}{\partial y^2} - E_r A_r \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} - p = 0 \tag{19}$$

$$\begin{aligned}
 &\rho_r A_r \frac{\partial^2 v}{\partial t^2} + P \frac{\partial^2 v}{\partial y^2} - E_r A_r \frac{\partial^2 u}{\partial y^2} \frac{\partial v}{\partial y} - E_r A_r \frac{\partial u}{\partial y} \frac{\partial^2 v}{\partial y^2} - \frac{3}{2} E_r A_r \left(\frac{\partial v}{\partial y} \right)^2 \frac{\partial^2 v}{\partial y^2} \\
 &- \kappa_r G_r A_r \left(\frac{\partial^2 v}{\partial y^2} - \frac{\partial \varphi}{\partial y} \right) - q = 0
 \end{aligned} \tag{20}$$

$$\rho_r I_r \frac{\partial^2 \varphi}{\partial t^2} - E_r I_r \frac{\partial^2 \varphi}{\partial y^2} - \kappa_r G_r A_r \left(\frac{\partial v}{\partial y} - \varphi \right) = 0 \tag{21}$$

It can be seen that these are coupled nonlinear partial differential equations with u , v and φ .

2.5 Boundary Conditions

Considering homogeneous column, the boundary virtual work equation is:

$$\begin{aligned}
& \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \delta u \right\} dy + \frac{1}{2} E_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] \delta v \right\} dy \\
& + \frac{1}{2} E_r I_r \int_0^H \left[\frac{\partial}{\partial y} \left(2 \frac{\partial \varphi}{\partial y} \delta \varphi \right) \right] dy + \frac{1}{2} \kappa_r G_r A_r \int_0^H \left\{ \frac{\partial}{\partial y} \left[2 \left(\frac{\partial v}{\partial y} - \varphi \right) \delta v \right] \right\} dy \\
& - P \int_0^H \left[\frac{\partial}{\partial y} \left(\frac{\partial v}{\partial y} \delta v \right) \right] dy - P \delta u(H) - Q_t \delta v(H) - M_t \delta \varphi(H) = 0
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
& \frac{1}{2} E_r A_r \left\{ \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] \delta u \right\} \Big|_0^H - P \delta u(H) \\
& + \left\{ \left[\frac{1}{2} E_r A_r \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] + \frac{1}{2} \kappa_r G_r A_r \left[2 \left(\frac{\partial v}{\partial y} - \varphi \right) \right] - P \frac{\partial v}{\partial y} \right] \delta v \right\} \Big|_0^H - Q_t \delta v(H) \\
& + \frac{1}{2} E_r I_r \left(2 \frac{\partial \varphi}{\partial y} \delta \varphi \right) \Big|_0^H - M_t \delta \varphi(H) = 0
\end{aligned} \tag{23}$$

Boundary conditions at $y=H$ is:

$$\begin{cases} \frac{1}{2} E_r A_r \left[2 \frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^2 \right] = P \\ \frac{1}{2} E_r A_r \left[2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y} \right)^3 \right] + \kappa_r G_r A_r \left[\left(\frac{\partial v}{\partial y} - \varphi \right) \right] - P \frac{\partial v}{\partial y} = Q_t \\ E_r I_r \frac{\partial \varphi}{\partial y} = M_t \end{cases} \tag{24}$$

2.6 Initial Conditions

If homogeneous column remains in natural status when $t < 0$, the following initial conditions can be satisfied when $t \geq 0$:

$$\begin{aligned}
u|_{t=0} &= 0, \quad \dot{u}|_{t=0} = 0; \\
v|_{t=0} &= 0, \quad \dot{v}|_{t=0} = 0; \\
\varphi|_{t=0} &= 0, \quad \dot{\varphi}|_{t=0} = 0;
\end{aligned} \tag{25}$$

It can be seen that, although the assumed material is linear, the end conditions of the force are nonlinear due to the influence of finite deformation. Therefore, the solution of the problem is difficult, and the numerical solution can only be solved by numerical method.

3 Conclusion

The seismic isolation technique has been widely applied in our country since 1990s. In the existing developed structure-control technology, basic seismic isolation is considered as a kind of structurally passive seismic control method, which has a simple concept, stable performance and cost effective. In

the paper, the mathematical model of rubber bearing based on geometric nonlinear is established by using Hamilton's principle. Based on the assumption of homogeneous column, the dynamics model is investigated considering the cross section rotated and the influence of the shear deformation and axial pressure. Finally, the geometric nonlinearity governing equation and boundary conditions of the rubber bearing is deduced. And it is obviously that the solution of the problem is difficult, and the numerical solution can only be solved by numerical method.

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