IOP Conf. Series: Earth and Environmental Science **153** (2018) 042001 doi:10.1088/1755-1315/153/4/042001

On the Diophantine equations $x^2 - 27y^2 = 1$ and $y^2 - 2^p z^2 = 25$

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Abstract. In recent years, scholars have paid much attention to the problem of solving the diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$.
At present, there are only a few conclusions on $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 1$, and the conclusions mainly concentrated in the number of solutions and the range of it, see Ref [1] and [3].
For odd numbers D_2, the integer solution of $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 4$, see Ref [4] - [6]. For even numbers D_2, the integer solution of (2), see Ref [7] - [11].
Up to now, there is no relevant result on the integer solution of $x^2 - 27y^2 = 1$ and $y^2 - 2^pz^2 = 49$, this paper mainly discusses the integer solution of it.

1. Introduction

In recent years, scholars have paid much attention to the problem of solving the diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$. When $m = 1, n = 1$, the equations turns into:

$$x^2 - D_1y^2 = 1\text{and } y^2 - D_2z^2 = 1 \tag{1}$$

At present, there are only a few conclusions on (1), and the conclusions mainly concentrated in the number of solutions and the range of it, see Ref [1] and [3].

When $m = 1, n = 4$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1\text{and } y^2 - D_2z^2 = 4 \tag{2}$$

For odd numbers D_2, the integer solution of (2), see Ref [4] - [6]. For even numbers D_2, the integer solution of (2), see Ref [7] - [11].

When $m = 1, n = 25$, the diophantine equations (1) turns into: $x^2 - D_1y^2 = 1$ and $y^2 - D_2z^2 = 25$.

In this case, $D_1 = 27$, D_2 can be expressed as 2^p. Up to now, there is no relevant result on the integer solution of $x^2 - 27y^2 = 1$ and $y^2 - 2^pz^2 = 49$, this paper mainly discusses the integer solution of it.

2. Critical lemma

Lemma 1[12] When $D = 1785, 4 \times 1785, 16 \times 1785$, In addition to 2 sets of integer solution $(x, y) = (13,4), (239,1352) (x, y) = (13,2), (239,676);\ (x, y) = (13,1), (239,338)$.the indeterminate equations has 1 set of positive integer solution (x_1, y_1), where ${x_1}^2 = x_0$ or ${x_1}^2 = 2{x_0}^2 - 1$, and$\varepsilon = x_0 + y_0\sqrt{D}$ is the basic solution of the Pell equations $x^2 - Dy^2 = 1$.

Lemma 2[12] Diophantine equations $x^4 - 27y^2 = 1$ only has common solution $(x, y) = (\pm 1,0)$.

Proof: For $D = 27$, from Lemma 1 we can get the equation $x^4 - 27y^2 = 1$ has 1 set of positive integer solution at most. $(x_0, y_0) = (26,3)$ is the basic solution of $x^2 - 27y^2 = 1$, then $x_0 = 26$ is a non square number, $2{x_0}^2 - 1 = 1351$ is a non square number, therefore Diophantine equations $x^4 - 27y^2 = 1$ only has common solution $(x, y) = (\pm 1,0)$.

Lemma 3[13] Let $(x_n, y_n), n \in Z$ be all of the solutions on $x^2 - 27y^2 = 1$, then x_n is a square number if and only if $n = 0$ for any x_n.

Proof: Replacing $x_n = a^2$ into the original equation, we can get $x^4 - 27y^2 = 1$. From Lemma 2 we can get Diophantine equations $x^4 - 27y^2 = 1$ only has common solution $(x, y) = (\pm 1,0)$, then $x_n = 1$, therefore $n = 0$.

3. Theorem and proof

By using elementary method such as congruence, the integer solution of the diophantine equations on $x^2 - 27y^2 = 1$ and $y^2 - 2^p z^2 = 25$ can be obtained.

3.1. Theorem

Let $p \in Z^+$, then the diophantine equations

$$x^2 - 27y^2 = 1 \text{ and } y^2 - 2^p z^2 = 25 \tag{3}$$

has one and only one common solution $(x, y, z) = (\pm 26, \pm 5,0)$.

3.2. Proof of main theorem

3.2.1. Primary analysis.

Because $(x_1, y_1) = (26,5)$ is the basic solution of the Pell equation $x^2 - 27y^2 = 1$, then all solution of the Pell equation $x^2 - 27y^2 = 1$ can be expressed as:

$$x_n + y_n\sqrt{D} = (26 + 5\sqrt{27})^n, n \in Z^+.$$

It is easily shown that

(I) ${y_m}^2 - 25 = y_{m+1}y_{m-1}$;

(II) $y_{2m} = 2x_m y_m$;

(III) $\gcd(x_{2m+1}, y_{2m}) = \gcd(x_{2m+1}, y_{2m+2}) = 26$,
$\gcd(x_{2m}, y_{2m+1}) = \gcd(x_{2m+1}, y_{2m+2}) = 1$.

(IV) $\gcd(x_m, y_m) = 1, \gcd(x_{m+1}, y_{m+1}) = 1, \gcd(x_m, x_{m+1}) = 1, \gcd(y_m, y_{m+1}) = 5$;

(V) $x_{2m} \equiv 1(mod 2)$; $y_{2m+1} \equiv 1(mod 2)$.

Suppose that $(x, y, z) = (x_m, y_m, z), m \in Z$ is the positive integer solution of the diophantine equation (3), from(I), we can get: $2^p z^2 = {y_m}^2 - 25 = y_{m+1}y_{m-1}$, it is

$$2^p z^2 = y_{m+1}y_{m-1} \tag{4}$$

As a result the equation (4) will be:

Case 1 p is an positive odd number.

Case 2 p is an positive even number.

3.2.2. Discusion on Case 1

Let $p = 2l - 1, l \in Z$,(4) is equivalent to:

$$2^{2l-1}z^2 = y_{m+1}y_{m-1} \tag{5}$$

1. m is an odd number.

Let $m = 2k - 1, k \in Z$,(5) is equivalent to:

$$2^{2l-1}z^2 = y_{2(k-1)}y_{2k} \tag{6}$$

From (II), (6) is equivalent to:

$$2^{2l-1}z^2 = 4x_{k-1}y_{k-1}x_k y_k \tag{7}$$

1.1 k is an odd number.

From (III) and (IV), we can get $\gcd(x_{k-1}, y_{k-1}) = \gcd(x_k, y_k) = 1$, $\gcd(x_k, x_{k-1}) = 1$, $\gcd(x_k, y_{k-1}) = 26, \gcd(y_k, y_{k-1}) = 5$, it means $\gcd\left(\frac{y_k}{5}, \frac{y_{k-1}}{5}\right) = 1$. $\gcd\left(\frac{x_k}{26}, \frac{y_{k-1}}{26}\right) = 1$. x_{k-1}, $\frac{y_{k-1}}{52}$, $\frac{x_k}{26}$, $\frac{y_k}{5}$ are pairwise coprime.

When $k \neq 1$, from (V), we can get $x_{k-1} \equiv 1(mod2)$, from Lemma 3, we can get x_{k-1} is a square number if and only if $k = 1$. When $k = 1$, $x_{k-1} = x_0 = 1$, $\frac{x_k}{26} = \frac{x_1}{26} = 1$, $\frac{y_k}{5} = \frac{y_1}{5} = 1$, and $\frac{y_{k-1}}{52} \neq 1$ for any $k \in Z$. So $x_{k-1}, \frac{y_{k-1}}{52}, \frac{x_k}{26}, \frac{y_k}{5}$ can not be 2 times of any square number when $k \neq 1$. It means $4x_{k-1}y_{k-1}x_k y_k = 20^2 \times \frac{x_k}{26} \times \frac{y_k}{5} \times x_{k-1} \times \frac{y_{k-1}}{52}$ can not be 2 times of any square number. Therefore, (5) has no integer solution, the diophantine equation (1) has no integer solution.

When $k = 1$, (7) is equivalent to: $2^{2l-1}z^2 = 4x_0 y_0 x_1 y_1 = 4 \times 26 \times 5 \times 1 \times 0 = 0$, then $z = 0$, Therefore, the diophantine equation (3) has common solution $(x, y, z) = (\pm 26, \pm 5, 0)$.

1.2 k is an even number.

From (III), we can get $\gcd(x_{k-1}, y_k) = 26$, then $\gcd\left(\frac{x_{k-1}}{26}, \frac{y_k}{26}\right) = 1$. From (IV), we can get $\gcd(x_k, y_k) = 1$, and $\gcd(x_k, x_{k-1}) = 1, \gcd(y_k, y_{k-1}) = 5$. Then $\gcd\left(\frac{y_k}{5}, \frac{y_{k-1}}{5}\right) = 1$, so when k is an even number, x_k, $\frac{y_k}{52}$, $\frac{x_{k-1}}{26}$, $\frac{y_{k-1}}{5}$ are pairwise coprime.

When $k = 0, x_{k-1} = x_0 = 1$. When $k = 2, \frac{x_{k-1}}{26} = \frac{x_1}{26} = 1$, and $\frac{y_{k-1}}{5} = \frac{y_1}{5} = 1$. $\frac{y_k}{52} \neq 1$ for any $k \in Z^+$. So when even number $k \neq 0$, x_k, $\frac{y_k}{52}$, $\frac{x_{k-1}}{26}$, $\frac{y_{k-1}}{5}$ can not equal to 1.

From Lemma 3, we can get x_k is a square number if and only if $k = 0$. From (V), we can get $x_k \equiv 1(mod2)$, So x_k, $\frac{y_k}{52}$, $\frac{x_{k-1}}{26}$, $\frac{y_{k-1}}{5}$can not be 2 times of any square number when $k \neq 0$. It means $4x_{k-1}y_{k-1}x_k y_k = 20^2 \times \frac{y_k}{52} \times \frac{x_{k-1}}{26} \times x_k \times \frac{y_{k-1}}{5}$ can not be 2 times of any square number. Therefore, (5) has no integer solution, the diophantine equation (1) has no integer solution.

When $k = 0$, (5) is equivalent to: $2^{2l-1}z^2 = 4x_0 y_0 x_{-1} y_{-1} = 0$, then $z = 0$, Therefore, the diophantine equation (1) has common solution $(x, y, z) = (\pm 26, \pm 5, 0)$.

2. m is an even number.

Let $m = 2k, k \in Z^+$,(2) is equivalent to:

$$2^{2l-1}z^2 = y_{2k-1}y_{2k+1} \tag{8}$$

From (V), we can get $y_{2k-1} \equiv y_{2k+1} \equiv 1(mod2)$, the power of 2 on the right of (8) should be 0, it is even-power. At the same time, the power of 2 on the left of (8) should be odd-power. Which is contradict with each other. Therefore, (8) has no integer solution when m is an even number, and the Diophantine equation (1) has no integer solution.

3.2.3. Discusion on Case 2

Let $p = 2k, k \in Z^+$, then $D = 2^{2k}$, from $y^2 - 2^p z^2 = 25$, we can get $y^2 - 2^{2k}z^2 = 25$, it is equivalent to:

$$(y + 2^k z)(y - 2^k z) = 25 \tag{9}$$

Solve (9), we can get $y_1 = \pm 5, z_1 = 0, y_2 = \pm 13, z_2 = \pm 3, k = 2$. When $y_2 = \pm 13$, from $x^2 - 27y^2 = 1$ we can get $x^2 = 4564$, Obviously it has no integer solution. Therefore, the Diophantine equation (3) has one and only one common solution $(x, y, z) = (\pm 26, \pm 5, 0)$.

To sum up, the theorem is proved.

4. Conclusion

The integer solution of diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$ is a matter of great concern.

By using elementary number theory methods,we solved the common solution and nontrivial solution on the diophantine equation when $m = 1$,$n = 25$, $D_1 = 27$, D_2 can be expressed as 2^t, it is the diophantine equations $x^2 - 27y^2 = 1$ and $y^2 - 2^pz^2 = 25$ has one and only one common solution $(x, y, z) = (\pm26, \pm5, 0)$.

Acknowledgment
Supported by Scientific Research Project fund of Education Department of Yunnan Province: 2018JS608.

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