

Uncertainty qualification for the free vibration of a functionally graded material plate with uncertain mass density

Ta Duy Hien^{1*}, Bui Tien Thanh¹ and Nguyen Thi Quynh Giang²

¹ Department of Civil Engineering, University of Transport and Communications, Ha Noi, Vietnam

² University of Education, Vietnam National University Hanoi, Vietnam

* Corresponding author, Email: tdhien@utc.edu.vn

Abstract. This work studies the response variability in the eigenvalue of free vibration of functionally graded material (FGM) plates with uncertain mass density. The mass density of plates is assumed to be a Gaussian random field on the plane of the plate. The formulation of free vibration of FGM plates is derived using isogeometric analysis based on higher-order shear deformation theory. The stochastic isogeometric analysis is established based on standard isogeometric analysis and a perturbation approach. Also, the formulation for variation of the eigenvalue of free vibration of the plate is issued. The present approach is validated with Monte Carlo simulation (MCS) based on a spectral representation method. The numerical examples investigated show good agreement between the present approach and Monte Carlo simulation. Based on the numerical results, the effect of a parameter of uncertain mass density on variability in the natural frequencies is drawn with respect to FGM plates.

1. Introduction

The stochastic problem based on an analytical and numerical approach with uncertain system parameters (e.g. material property, model geometry and loads) has been studied for many years. For dealing with uncertain parameters, most stochastic problems fall into two main categories; non-statistical approach and Monte Carlo simulation. Normally, Monte Carlo simulation repeats the deterministic problem by generating random parameters and it can be used for complex problems such as those which are nonlinear; however, it is time consuming. The non-statistical approach can produce statistics of responses, such as the mean and standard deviation of output parameters. The serval numerical approach for solving stochastic problems has been used for many years; for example, the stochastic finite element method [1-4], the spectral stochastic finite element method [5, 6], and the spectral stochastic element-free Galerkin method [7]. A number of researchers have worked on the stochastic dynamic problem. Zhu et al. [8] used the local averages of a random vector field for developing the stochastic finite element method for the eigenvalues problem. Achchhe, Lal and Singh [9] investigated the free vibration analysis of laminated composite plates resting on elastic foundations with the random variable of system method. Thuan and Noh [10] used Monte Carlo simulation to investigate the variability of the natural frequency of functionally graded beams with random fields of material properties. Kang *et al.* [11] studied the random vibration of laminated FRP plates with material nonlinearity. Chang [12] analysed the stochastic dynamics of bridge-vehicle systems subjected to random material properties and loadings using the finite element method.



This paper presents a stochastic isogeometric analysis (SIGA) for free vibration of functionally graded plates that involves uncertain mass density. The random field was discretized into a set of random variables with statistical properties obtained from the statistical properties of the random field. In conjunction with the standard isogeometric analysis, the first-order perturbation approach was performed to predict the variability eigenvalue of free vibration of FGM plates. Numerical examples are presented to validate the accuracy of the proposed method with those using Monte Carlo simulation.

2. Formulation of free vibration of FGM plate using non-uniform rational basis spline (NURBS)

2.1. FGM plate model

We consider an FGM plate, for which the geometry of FGM plate and the coordinate system is shown in Fig.1:

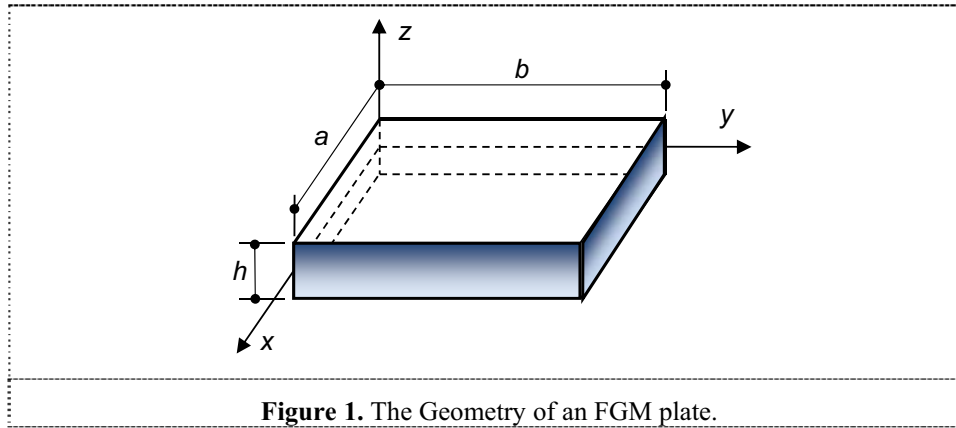


Figure 1. The Geometry of an FGM plate.

Using the higher-order shear deformation plate theory proposed by Hien *et al.* Noh [13], the displacement components at an arbitrary point (x, y, z) in the plate are rearranged as follows:

$$\begin{aligned} U(x, y, z, t) &= u(x, y, z, t) - z \frac{\partial w_b}{\partial x} + z \left[-\frac{1}{5} + \frac{8}{5\pi} \cos\left(\frac{\pi z}{h}\right) \right] \frac{\partial w_s}{\partial x} \\ V(x, y, z, t) &= v(x, y, z, t) - z \frac{\partial w_b}{\partial y} + z \left[-\frac{1}{5} + \frac{8}{5\pi} \cos\left(\frac{\pi z}{h}\right) \right] \frac{\partial w_s}{\partial y} \\ W(x, y, z, t) &= w_b(x, y, t) + w_s(x, y, t) \end{aligned} \quad (1)$$

where (u, v, w_b, w_s) are the displacement components on the mid-plane, respectively.

The linear strains can be obtained by differentiating Eq. (1) as:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 + z\boldsymbol{\kappa}_b + \hat{q}\boldsymbol{\kappa}_s, \quad \boldsymbol{\gamma} = \hat{p}\boldsymbol{\varepsilon}_s \quad (2)$$

where:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad \boldsymbol{\varepsilon}_0 = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}, \quad \boldsymbol{\kappa}_b = \begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix},$$

$$\mathbf{\kappa}_s = \begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \boldsymbol{\gamma} = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \quad \boldsymbol{\epsilon}_s = \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix}$$

The mechanical properties of FGM such as Young's modulus E and mass density ρ are assumed as

$$E(z) = (E_c - E_m) \left(\frac{z}{h} + \frac{1}{2} \right)^n + E_m$$

$$\rho(z) = (\rho_c - \rho_m) \left(\frac{z}{h} + \frac{1}{2} \right)^n + \rho_m \quad (3)$$

where the subscripts m and c represent the metallic and ceramic constituents, respectively; and n is the power index of the volume fraction.

3. Stochastic isogeometric analysis for free vibration of FGM plates

3.1. A brief of isogeometric analysis

The isogeometric analysis was introduced in 2005 by Thomas Hoge *et al.* [14, 15]. In this analysis, the non-uniform rational B-spline (NURBS) functions are used to construct a finite approximation.

The B-spline basis functions are defined recursively as follows [14-17]:

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

and

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (5)$$

The Non-Uniform Rational B-Splines (NURBS) basis functions in two dimension, and can be constructed by taking the tensor product of two one-dimensional B-spline basis functions represented as [14, 16]:

$$R_{i,j}^{p,q}(\xi, \eta) = \frac{N_{i,p}(\xi) N_{j,q}(\eta) w_{i,j}}{\sum_{i=1}^m \sum_{j=1}^n \{N_{i,p}(\xi) N_{j,q}(\eta) w_{i,j}\}} \quad (6)$$

where $N_{i,p}(\xi)$, $N_{j,q}(\eta)$ are the B-spline basis functions of order p in the ξ , and order q in the η direction, respectively.

3.2. Stochastic isogeometric analysis for free vibration of FGM plates

Using the NURBS basis functions above, the displacement field of the plate is approximated as:

$$\begin{Bmatrix} \hat{u}_{0A} \\ \hat{v}_{0A} \\ \hat{w}_{0A}^b \\ \hat{w}_{0A}^s \end{Bmatrix} = \sum_{A=1}^{m \times n} R_A(\xi, \eta) \begin{Bmatrix} u_{0A} \\ v_{0A} \\ w_{0A}^b \\ w_{0A}^s \end{Bmatrix} \quad (7)$$

Substituting Eq. (7) into Eq.(2) the in-plane and shear strains can be rewritten as:

$$\begin{bmatrix} \boldsymbol{\epsilon}_0^T & \boldsymbol{\kappa}_b^T & \boldsymbol{\kappa}_s^T & \boldsymbol{\epsilon}_s^T \end{bmatrix}^T = \sum_{A=1}^{m \times n} \begin{bmatrix} (\mathbf{B}_A^0)^T & (\mathbf{B}_A^{b1})^T & (\mathbf{B}_A^{b2})^T & (\mathbf{B}_A^s)^T \end{bmatrix}^T \mathbf{d}_A \quad (8)$$

where $\mathbf{d}_A$ is displacement parameter at the control point A :

$$\mathbf{d}_A = \begin{Bmatrix} u_{0A} \\ v_{0A} \\ w_{0A}^b \\ w_{0A}^s \end{Bmatrix}$$

Using the virtual work principle, the weak form of free vibration of FGM plate can be obtained as follows:

$$\int_{\Omega} \delta \boldsymbol{\varepsilon}_b^T \hat{\mathbf{D}} \boldsymbol{\varepsilon}_b d\Omega + \int_{\Omega} \delta \boldsymbol{\gamma}^T \mathbf{A}^s \boldsymbol{\gamma} d\Omega = \int_{\Omega} \delta \hat{\mathbf{u}}^T \mathbf{m} \ddot{\mathbf{u}} d\Omega \quad (9)$$

The formulation of free vibration of the FGM plate is expressed as follows:

$$\mathbf{K} \mathbf{d} + \mathbf{M} \ddot{\mathbf{d}} = \mathbf{0} \quad (10)$$

where $\mathbf{M}$, $\mathbf{K}$ represent mass matrix and stiffness matrix, respectively.

The random field in mass density is assumed as homogeneous Gaussian in the plane of the plate:

$$\rho(x, y, z) = \left\{ (\rho_c - \rho_m) \left(\frac{z}{h} + \frac{1}{2} \right)^p + \rho_m \right\} (1 + r(x, y)) \quad (11)$$

The random fields are discretized into a set of N_r random variables with pre-assigned statistical properties as follows:

$$\mathbf{r} = \{r_1, r_2, \dots, r_{N_r}\} \quad (12)$$

Substituting random variables in Eq.(12) into the governing equation of free vibration, we obtain the discretization of the stochastic isogeometric analysis:

$$[\mathbf{K} - \omega^2 \mathbf{M}(\mathbf{r})] \mathbf{X}(\mathbf{r}) = \{\mathbf{0}\} \quad (13)$$

where ω and $\mathbf{X}$ are natural frequencies and displacement vector (eigenvectors).

The Taylor series expansion at $\mathbf{r}=\mathbf{0}$ gives:

$$\begin{aligned} \mathbf{M}(\mathbf{r}) &= \mathbf{M}_0 + \sum_{i=1}^{N_r} \mathbf{M}_{,i}^r r_i + \frac{1}{2} \sum_{i,j=1}^{N_r} \mathbf{M}_{,ij}^r r_i r_j + \dots \\ \mathbf{X}(\mathbf{r}) &= \mathbf{X}_0 + \sum_{i=1}^{N_r} \mathbf{X}_{,i}^r r_i + \frac{1}{2} \sum_{i,j=1}^{N_r} \mathbf{X}_{,ij}^r r_i r_j + \dots \\ \lambda(\mathbf{r}) &= \lambda_0 + \sum_{i=1}^{N_r} \lambda_{,i}^r r_i + \frac{1}{2} \sum_{i,j=1}^{N_r} \lambda_{,ij}^r r_i r_j + \dots \end{aligned} \quad (14)$$

where $\lambda = \omega^2$ is an eigenvalue.

Substituting series in Eq.(14) into Eq.(13), solving Eq.(13) using the perturbation technique, we obtain the eigenvalues and their derivatives. The mean and variance of the eigenvalues are given as:

$$\begin{aligned} \mu_{\lambda} &= \lambda_0^k \\ \text{Var}_{\lambda} &= \sum_{i=1}^{NR} \sum_{j=1}^{NR} \lambda_{,i}^{r-k} \lambda_{,j}^{r-k} \mathbf{R}_{ij}^r \end{aligned} \quad (15)$$

The response variability COV of eigenvalue λ is computed as follows:

$$COV = \frac{\sqrt{\text{Var}_{\lambda}}}{|\mu_{\lambda}|} \quad (16)$$

For validating the present approach, the Monte Carlo simulation is performed using the spectral representation method as proposed by Shinozuka *et al.* [18, 19].

4. Examples

A simply supported FGM rectangular plate with side-to-thickness ratio $a/h=40$, dimension of rectangular $a=0.4\text{m}$, $b=0.3\text{m}$, and a power index $p=2$ is considered. The elastic moduli and mean of mass density are chosen to be the same as [20] $E_m = 70\text{ GPa}$, $\rho_m=2707\text{kg/m}^3$, $E_c = 380\text{GPa}$, $\rho_c=3800\text{kg/m}^3$, with a Poisson's ratio of 0.3. The auto-correlation function for the stochastic fields function $r(x,y)$ of mass density is assumed as form:

$$R(\xi) = \sigma^2 \exp \left[- \left(\frac{\xi_x}{d_1} \right)^2 - \left(\frac{\xi_y}{d_2} \right)^2 \right] \quad (17)$$

where ξ_x, ξ_y are components of the separate vector ξ in the plane Cartesian coordinate and d_1, d_2 are correlation distances along the corresponding directions.

The coefficient of variance of random field is assumed $\sigma = 0.15$. We consider the correlation distance $d = d_1 = d_2$.

Fig.2. shows the first mode shape and second mode shape of a simply supported rectangular FGM based on deterministic analysis.

Fig.3 illustrates the response COV of eigenvalue at the first mode and the second mode using both approaches: the stochastic isogeometric analysis method and Monte Carlo simulation. It is clear that the response COV increases when correlation distance d increases. When correlation distance d is quite large, the response COV of an eigenvalue is nearly equal to the coefficient of variance of the random field σ . Also, Fig.3 illustrates that there is good agreement between the results of the present approach and the Monte Carlo simulation.

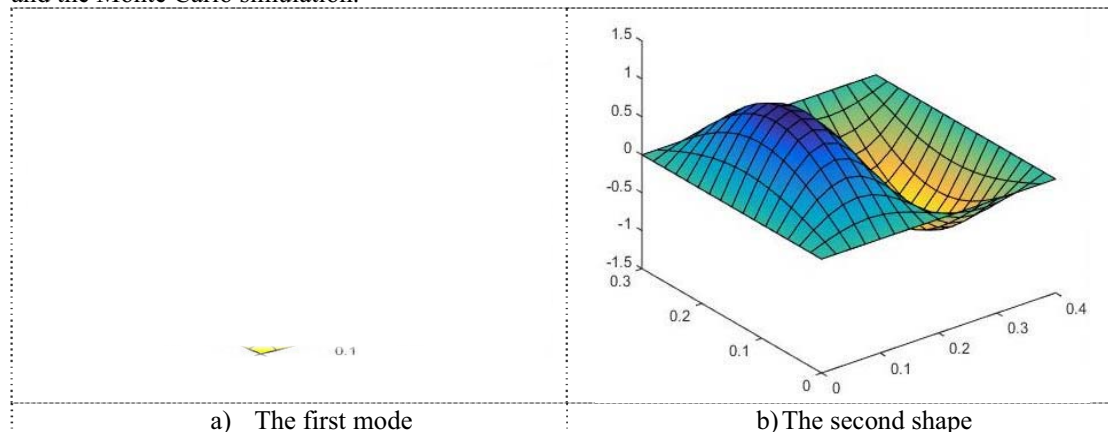


Figure 2. The mode shape of a simply supported FGM plate.

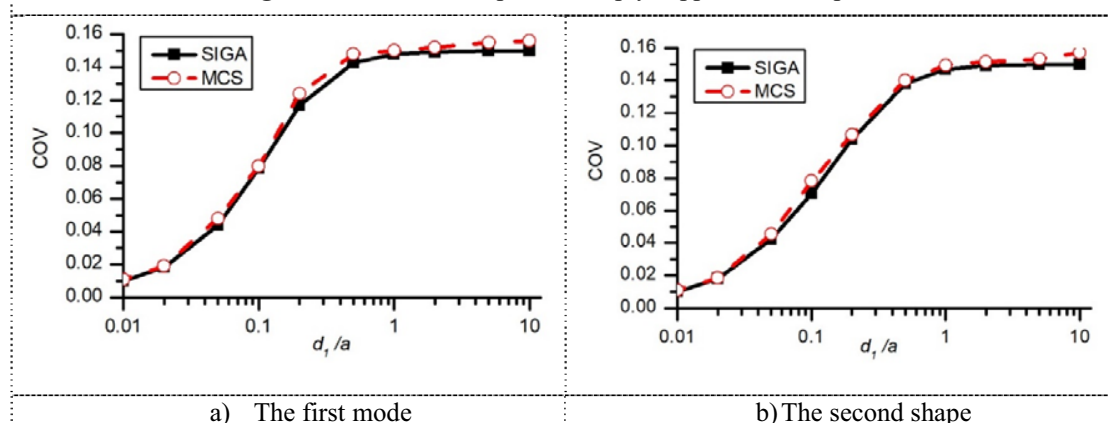


Figure 3. Response COV of the eigenvalue λ & correlation distance d .

5. Conclusions

The statistics of the eigenvalue of free vibration of FGM plates such as the mean, standard deviation, and response variability of eigenvalue have been obtained by using the perturbation technique in conjugation with standard isogeometric analysis. The response COV of the eigenvalue obtained by the present approach are in good agreement with the Monte Carlo simulation approach. From the results of numerical examples, we observed that the response COV is in the range of 0.9%–15% relative to the coefficient of variation of the random field, which is assumed to be 0.15. Also, the numerical examples clearly show that the effect of uncertain mass density on standard deviation and response COV are significant.

References

- [1] Liu WK, Belytschko T, Mani A 1986 Random field finite elements *International Journal for Numerical Methods in Engineering* 23(10):1831-1845.
- [2] Takada T 1990 Weighted integral method in stochastic finite element analysis *Probabilistic Engineering Mechanics* 5(3):146-156.
- [3] Michael Kleiber, Hien TD 1992 *The stochastic finite element method: basic perturbation technique and computer implementation* Wiley.
- [4] Noh H-C. 2011 Stochastic finite element analysis of composite plates considering spatial randomness of material properties and their correlations *Steel and Composite Structures* 11(2):115-130.
- [5] Ghanem RG, Spanos PD 1991 Spectral stochastic finite-element formulation for reliability-analysis *Journal of Engineering Mechanics* 117(10):2351-2372.
- [6] Chen NZ, Guedes Soares C 2008 Spectral stochastic finite element analysis for laminated composite plates *Computer Methods in Applied Mechanics and Engineering* 197(51–52) : 4830-4839.
- [7] Kim HM, Inoue J 2004 A spectral stochastic element free Galerkin method for the problems with random material parameter *International Journal for Numerical Methods in Engineering* 61(11) : 1957-1975.
- [8] Zhu WQ, Wu WQ 1991 A stochastic finite element method for real eigenvalue problems *Probabilistic Engineering Mechanics* 6(3–4) : 228-232.
- [9] Lal A, Singh BN 2009 Stochastic nonlinear free vibration of laminated composite plates resting on elastic foundation in thermal environments *Computational Mechanics* 44(1):15-29.
- [10] Van TN, Noh HC 2017 Investigation into the effect of random material properties on the variability of natural frequency of functionally graded beam *KSCE Journal of Civil Engineering* 21(4):1264-1272.
- [11] Kang J, Harichandran R 1999 Random Vibration of Laminated FRP Plates with Material Nonlinearity Using High-Order Shear Theory *Journal of Engineering Mechanics* 125(9):1081-1088.
- [12] Chang TP 2014 Stochastic dynamic finite element analysis of bridge–vehicle system subjected to random material properties and loadings *Applied Mathematics and Computation* 242(0):20-35.
- [13] Hien TD, Noh HC 2015 Analytical solution for the dynamic response of functionally graded rectangular plates resting on elastic foundation using a refined plate theory *Applied Mathematical Modelling* 39(20):6243-6257.
- [14] Hughes TJR, Cottrell JA, Bazilevs Y 2005 Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement *Computer Methods in Applied Mechanics and Engineering* 194(39–41):4135-4195.
- [15] Austin CJ, Hughes TJR, Bazilevs Y 2009 *Isogeometric Analysis: Toward Integration of CAD and FEA* Wiley.
- [16] Nguyen VP, Bordas S, Rabczuk T 2012 *Isogeometric analysis: an overview and computer implementation aspects*. arXiv:12052129 [csNA].
- [17] Piegl L, Tiller W 1995 *B-Spline Basis Functions* The NURBS Book. Berlin, Heidelberg: Springer

- Berlin Heidelberg p. 47-79.
- [18] Shinozuka M, Deodatis G 1991 Simulation of Stochastic Processes by Spectral Representation *Applied Mechanics Reviews* 44(4):191-204.
 - [19] Shinozuka M, Deodatis G 1996 Simulation of Multi-Dimensional Gaussian Stochastic Fields by Spectral Representation *Applied Mechanics Reviews* 49(1):29-53.
 - [20] Hasani Baferani A, Saidi AR, Ehteshami H 2011 Accurate solution for free vibration analysis of functionally graded thick rectangular plates resting on elastic foundation *Composite Structures* 93(7):1842-1853.

Acknowledgments

The authors would like to acknowledge the support from the University of Transport and Communications under the basic research fund and the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 107.01-2017.314.