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Mesh optimization based on a process of geometrical nonlinear analysis

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Abstract. The tangent stiffness method produces the strict equilibrium solutions using the geometrical stiffness and element force equations. The method based on the principle of stationary total potential energy can be applied to mesh optimization problems by using the potential functions of element measurement as objective functions. In this study, three different types of functions are examined with numerical examples. One is proportionate to the total area of curved surface consist of triangle elements, and it gives minimal and isotonic surfaces. Another is expressed as power functions of length of 1D elements, and it makes the total length between nodes minimalize. The last one is to minimalize the square of difference between the side lengths of triangle elements, and it may arrange the shapes of the elements equilateral triangles.

1. Introduction

It is very important for non-linear structural analyses to use suitable and available mesh configuration according to the purpose of the analyses. Plenty of investigations for “mesh optimization problem” have been proposed since the computational technology such as FEM began to develop to be able to adopt to complex non-linear phenomenon. General concept of mesh optimization is to minimize the objective function concern with topology of corresponding structure, for example, length of elements can be a typical target [1]. Moreover, the mesh optimization problem has analogic background as the geometrically nonlinear problem, so there exist many proposals [2][3] to use the common scheme with geometrically nonlinear analyses.

Authors have studied about geometrical nonlinear analyses using the tangent stiffness method in which the geometrical stiffness caused by elements’ rigid body displacement are perfectly separated from the element stiffness caused by elements’ own deformation. Therefore, the method produces the equilibrium solutions that strictly adjust to the element behavior defined in the element force equation with rapid and sure convergent process[4]. Furthermore, the element force equation can be derived by differential calculus of the strain energy of the elements. Thus, if we can use some mechanical values which are defined as potential of element measurement (after this, call it “the measure potential”) instead of strain energy, the object of the analysis shall be minimalizing of the functional. For example, when the measure potential is proportionate to the total area of triangle elements, the method will produce an isotonic and minimal surface, and the authors have applied it to the form finding problem for pneumatic structures [5]. Furthermore, when the measure potential is power function of the length of line elements, each node will be located to have shortest distance between two panel nodes next to each other, and this definition of potential is available for form finding problem of tensegrity structures [6].

In this study, three different types of measure potential for mesh optimization are shown. The first one and second one are proportionate to the total area and to the power of length, respectively. The last one is to adjust triangular elements to equilateral shape, using the measure potential expressed by the square total of difference of side length. The positioning of the elements by this measure potential is effective for extremely large deformational analyses of membrane structures even when the wrinkle occurs in some elements [7].

Two numerical examples are examined in this paper. One is to adjust the side length of triangular elements on the initial curved surface for membrane or shell structures determined by the form finding process using isotonic strain elements. The analysis is proceeded in 2 dimensional coordinate along the tangent plane of object surface, and the solution that reduced the difference between side length could be obtain with stable convergent process. The other is to examine adaptability when an extremely large deformational analysis is proceeding simultaneously with the mesh optimization analysis. This example discusses about comparison between two measure potentials of "the square total of difference of side length " and "the power function of element length", and it became evident that the both measure potentials give rational mesh division by renewing the tangent plane coordinate step by step, even when the displacements of the surface are so large.

2. Tangent stiffness method for geometrical nonlinear analysis

The measure potential P is defined as a functional of element measurement $\mathbf{s}$, and the element edge forces are expressed as following.

$$\mathbf{S} = \frac{\partial P}{\partial \mathbf{s}} \tag{1}$$

Furthermore, the equilibrium equation between the nodal forces $\mathbf{U}$ in the common coordinate and the element edge forces $\mathbf{S}$ in the local element coordinate can be expressed as following.

$$\mathbf{U} = \mathbf{J}\mathbf{S} \tag{2}$$

Here, $\mathbf{J}$ is the conversion matrix between two coordinates. In case of geometrically nonlinear analyses, the tangent stiffness equation can be derived by differential calculus of Eq(2).

$$\boldsymbol{\delta}\mathbf{U} = \mathbf{J}\boldsymbol{\delta}\mathbf{S} + \boldsymbol{\delta}\mathbf{J}\mathbf{S} = (\mathbf{K}_0 + \mathbf{K}_G)\boldsymbol{\delta}\mathbf{u} \tag{3}$$

In which, $\mathbf{K}_0$ is the element stiffness and $\mathbf{K}_G$ is the tangent geometrical stiffness. $\boldsymbol{\delta}\mathbf{u}$ is nodal displacement vector in general coordinate.

In case of the large displacement problems with the elements having real material, the measure potential P is equal to the strain energy and Eq. (1) gives the element deformation including the material behaviour in the local element coordinate. Therefore, $\mathbf{K}_G$ in Eq.(3) can evaluate the geometrical non-linearity caused by rigid body displacement of elements strictly.

On the other hand, in case of the mesh optimization problems as the pre-process of main analyses, the measure potential P can be define depending on the purpose of the main analyses. Therefore, nodal forces may be derived by direct differential calculus as:

$$\mathbf{U} = \frac{\partial P}{\partial \mathbf{u}} \tag{4}$$

As well as the tangent stiffness equation are also expressed as:

$$\mathbf{\delta U} = \frac{\partial^2 P}{\partial \mathbf{u} \partial \mathbf{u}^{\mathrm{T}}} \mathbf{\delta u} \tag{5}$$

3. Potential of element measurement for mesh optimization

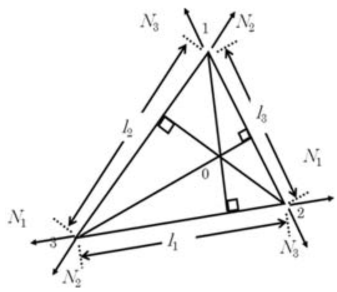


Figure 1. A triangular element and its measurement.

In this study, three different types of measure potentials for mesh optimization are defined and used. One has the measure potential proportionate to the area of triangular elements. Figure 1 shows its measurement. The potential of an element numbered "*i*" is expressed as:

$$P_i = CA_i \tag{6}$$

where, C is the constant corresponding to the product of the tensile stress and the thickness of the soap film, and A_i is the area of element numbered i. When the Eq.(6) is substituted to Eq.(1)-(3), the form of the tangent geometrical stiffness $\mathbf{K_G}$ in Eq.(3) becomes same as one of the triangular truss block. This potential will give the isotonic carved surfaces.

Second one is line elements whose measure potential expressed as the power functions of elements' own length:

$$P_i = Cl_i^n \tag{7}$$

where, C and n is the constant which can designated freely, and l is the element length. This potential is available for the form-finding of tensegrity structures.

Third one is the potential by square total of side length difference of triangular elements.

$$P_i = C\left\{ \left(l_3 - l_2\right)^2 + \left(l_1 - l_3\right)^2 + \left(l_2 - l_1\right)^2 \right\}_i \tag{8}$$

Therefore, this potential can be expected for each element to be closed to the shape of equilateral triangle. Even if any potential of Eq.(6) to Eq.(8) was used, the tangent stiffness equation can be expressed by superposition of Eq.(9).

$$\delta\begin{bmatrix} \vdots \\ \dfrac{\partial P_i}{\partial \mathbf{u}_{\mathbf{i1}}^{\mathbf{i}}} \\ \vdots \\ \dfrac{\partial P_i}{\partial \mathbf{u}_{\mathbf{i2}}} \\ \vdots \\ \dfrac{\partial P_i}{\partial \mathbf{u}_{\mathbf{i3}}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & & \vdots & & \vdots & \\ \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i1}}{}^2} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i1}}\partial \mathbf{u}_{\mathbf{i2}}{}^T} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i3}}\partial \mathbf{u}_{\mathbf{i1}}{}^T} & \cdots \\ & \vdots & \ddots & & & & \\ \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i2}}\partial \mathbf{u}_{\mathbf{i1}}{}^T} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i2}}{}^2} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i2}}\partial \mathbf{u}_{\mathbf{i3}}{}^T} & \cdots \\ & \vdots & & \vdots & \ddots & \vdots & \\ \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i1}}\partial \mathbf{u}_{\mathbf{i3}}{}^T} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i3}}\partial \mathbf{u}_{\mathbf{i2}}{}^T} & \cdots & \sum_{i=1}^{mi}\dfrac{\partial^2 P_i}{\partial \mathbf{u}_{\mathbf{i3}}{}^2} & \cdots \\ & \vdots & & \vdots & & \vdots & \ddots \end{bmatrix} \delta\begin{bmatrix} \vdots \\ \mathbf{u_1} \\ \vdots \\ \mathbf{u_2} \\ \vdots \\ \mathbf{u_3} \\ \vdots \end{bmatrix} \tag{9}$$

After convergent of the unbalanced forces by application of Newton-Raphson method, each node moved to rational position on a designated surface.

4. Numerical examples

4.1. Minimization of element side length difference

Figure 2 shows a shape of initial plane and mesh division. And Figure 3 is a shape of carved surface found by an isotonic stain analysis. Only the self-weight is loaded to each node under the boundary condition to be fixed the nodes around four corners. This algorithm is to renew the non-stressed shape of each triangular element in every iteration step. Therefore, the determined surface has equal strain all around on it.

Mesh optimization by the measure potential by square total of side length difference (we call it "Minimization of element side length difference") is adopted to the found surface to be used as a primary solution for a large deformational analysis of the membrane structure. To make use of the performance of the constant strain membrane elements, conversion of its element edge forces to the direction along each sides is most rational, as N_i(i=1,2,3) shown in Figure 1. However, flatter shape of triangular element, in some cases(for example, [7]), causes the compressional side direction forces even under the condition of tensional principal strains.

"Minimization of element side length difference" is proceeded in 2-D coordinate along the tangent plane of the surface, and Figure 4 shows movement of each node and deformation of each element. Because total area of the surface does not change between before and after of the analysis, it seems that each node moved on the surface itself.

$$P_{ai} = \left.\frac{\left(l_3 - l_2\right)^2 + \left(l_1 - l_3\right)^2 + \left(l_2 - l_1\right)^2}{l_1^2 + l_2^2 + l_3^2}\right|_i \tag{10}$$

$$P_a = \sum_{i=1}^{m} P_{ai} \quad m : \text{Total number of elements} \tag{11}$$

Figure 5 is contour that indicates the deference of index by Eq.(10) between elements.
This index can express the difference between side lengths no concern with size of triangles. According to Figure 5, P_{ai} is reduced in almost of elements, and total amount of P_a in Eq.(11) is also reduced from 32.72 to 30.60.

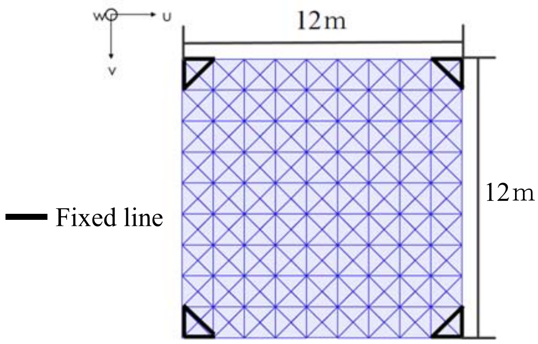


Figure 2. Shape of initial plane and mesh division.

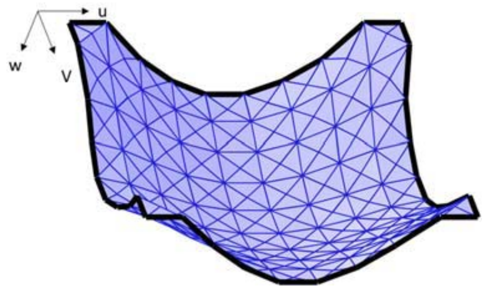
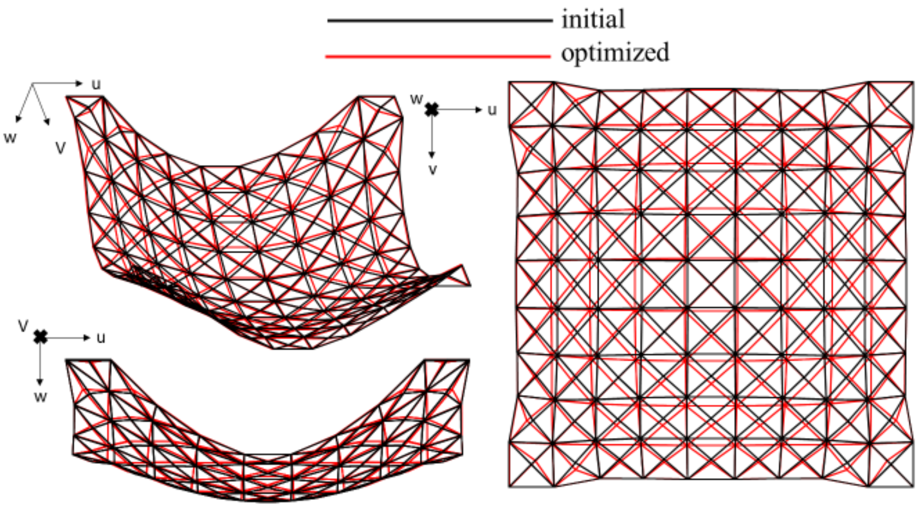


Figure 3. Determined shape by isotonic strain analysis.



Total area (initial) = 158.80m^2 Total area (optimized) = 158.83m^2

Figure 4. Movement of each node and deformation of each element by the mesh optimization.

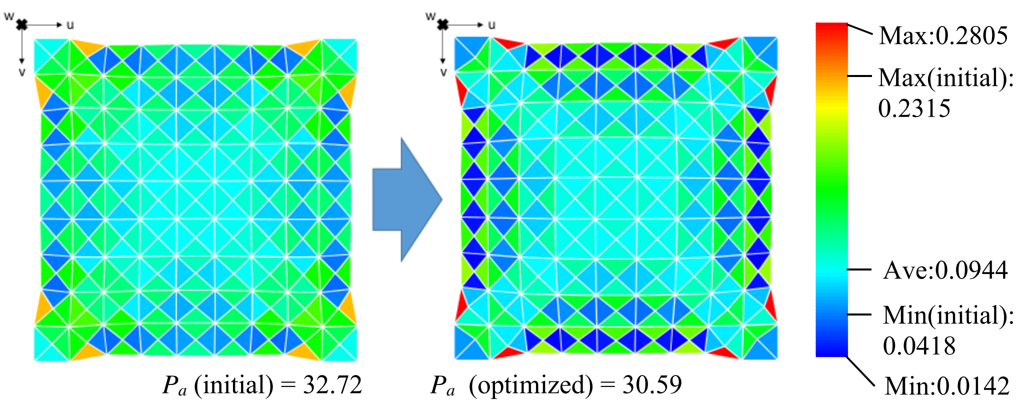


Figure 5. Comparison of contour by difference of side length.

4.2. Mesh optimization proceeding simultaneously with extremely large displacement analysis

Figure 6 shows the initial plane of square with 12m of side length. Form finding analysis of isotonic surface is proceeded with the potential of Eq.(6) under the loading of constant air pressure to normal direction of the surface. By application of the simultaneous control [8], which is an incremental technique different from the arc length method, compulsory displacements are given on a control point (shown in Figure 6) to normal direction of the surface. A path can be searched until the solutions with very large involved volume by displacing the control point compulsorily, as shown in Figure 7.

In case of isotonic surface analyses, each element has no stiffness to tangent direction but to normal direction of the surface. Therefore, each node should be fixed to tangent direction, or some rational mesh optimization methods are required. In this study, the measure potential of Eq.(7) and Eq.(8) are applied to tangent direction and are compared their performance. Namely, two cases of following are examined.

Case A: Soap film elements derived by the measure potential of Eq.(6) are applied to normal direction, and line elements derived by Eq.(7) is applied to tangent direction, when C=1 and n=1. In this study, we call this mesh optimization "Minimize of total element length".

Case B: Soap film elements derived by the measure potential of Eq.(6) are applied to normal direction, and "Minimization of element side length difference" by Eq.(8) is applied to tangent direction, when C=1.

In both cases, the stiffness equations to normal direction and to tangent direction are solved simultaneously and independently, and the coordinate which consist of 1st axis as normal direction and 2nd and 3rd axes as tangent direction is renewed in each iterative step of Newton-Raphson method. Therefore, the large displacement analysis and the mesh optimization can be proceeded simultaneously.

Figure 8 and Figure 9 are the found surfaces by Case A and B respectively, when adopting the rougher mesh division that a side is divided in 6. Furthermore, Figure 10 and Figure 11 are the surfaces when adopting the denser mesh division that a side is divided in 12. 10m of total compulsory displacement is given to the control point in each case.

According to Table 1 showing the comparison, Eq.(7) gives shorter total length of line between 2 nodes connected by each elements than Eq.(8). On the other hand, Eq.(8) makes smaller side length difference of triangular elements than Eq.(7). Therefore, it can be concluded that the each measure potential acts rationally, even if under the condition of extremely large displacement by using same algorithm as the tangent stiffness method.

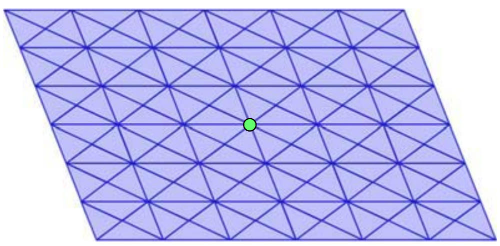


Figure 6. Shape of initial plane and mesh division.

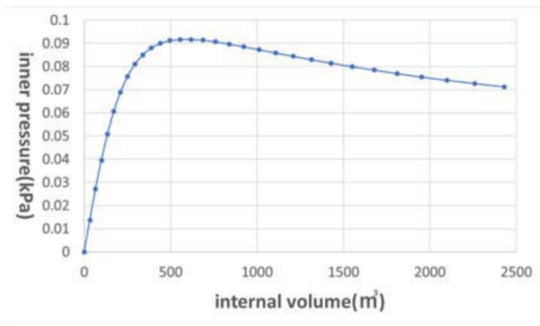


Figure 7. Inner pressure-volume path.

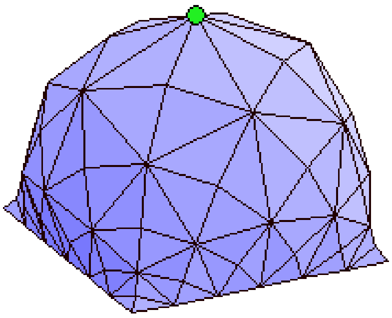


Figure 8. Determined form by measure potentials of Eq.(7) with rougher division

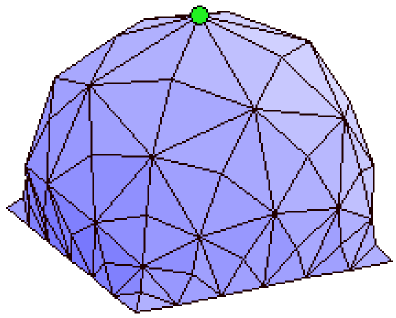


Figure 9. Determined form by measure potentials of Eq.(8) with rougher division

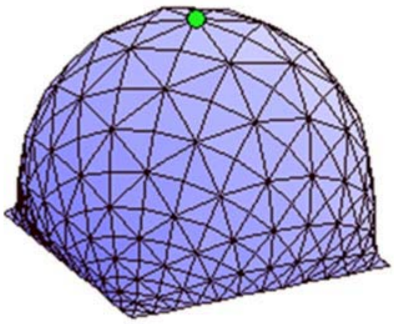


Figure 10. Determined form by measure potentials of Eq.(7) with denser division

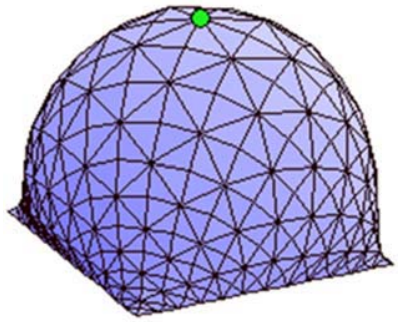


Figure 11. Determined form by measure potentials of Eq.(8) with denser division

Table 1. Comparison of solution between Case A and B.

Division number of a side	6		12	
Adopted potential to tangent direction	Case A by Eq.(7)	Case B by Eq.(8)	Case A by Eq.(7)	Case B by Eq.(8)
Inner pressure[kPa]	-0.08602	-0.08593	-0.08303	-0.08297
Pa in Eq.(10)	18.74196	16.33165	75.74562	67.02342
Total length of line between 2 nodes [m]	1166.43514	1184.67583	2358.368	2398.83

5. Conclusions

In this paper, we proposed three element potential functions for mesh optimization of curved surface. The first one is to be proportionate to the area of the triangle element, the second one is proportionate to the element side length, and the last one is to be proportionate to the square sum of the side length difference. The advantage of "Minimization of element side length difference" was discussed when it is applied to a surface found by constant strain analysis. Furthermore, in case of isotonic curved surface analysis until the decompression-expansion process with very large displacement, we examined the performance of both of "Minimization of total element length" and "Minimization of element side length difference", when the sequential updating of the local coordinates to be composed of one normal axis and two tangent axes at every node on the surface is adopted. The findings on the features of each method are summarized below.

(1) Optimized nodal positions by application of "Minimization of element side length difference" to an equilibrium solution determined by the constant strain analysis can be assumed to be on the curved surface because of no changing of the total surface area, even if the local coordinate system having two axes in the tangent direction is used without updating. For this example model, rationality of "Minimization of element side length difference" became evident, because it can be judged that all elements are approaching to the shape of equilateral triangle by checking the changing of the value of the measure potential.
(2) Geometrically nonlinear analysis with extremely large deformation and the mesh optimization analysis using the measure potential can be parallel procedure. On the isotonic form-finding analysis, comparing "Minimization of element side length difference" with "Minimization of total element length", it was found that both measure potentials are enough available for mesh optimization regardless of the coarseness and minuteness of the mesh.

Consequently, the application of the measure potential to the algorithm of the geometrically nonlinear analysis by the tangent stiffness method contributes the mesh optimization by choosing appropriate potential function. It is expected that some more different types of measure potentials for different objectives will be developed in near future.

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