

# Infiniteness of stresses for regions with angular points— A class of ill-conditioned problems

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**Abstract.** The current theory of failure is connected with the stress intensity factors at angular points. All known solutions with the infinite stresses at some points are the elasticity problem solutions according to their authors. The analysis of such solutions shows that though these are the fundamental solutions of the elasticity theory, the use of them to model subproblems of the elastic theory is unpromising.

## 1. Introduction

In view of the dozens of monographs devoted to the theory of elasticity, it might be expected that the logic and style of the presentation are set. Nonetheless, the elasticity research is yet faced with the known difficulties defined by the diversity of the problems and the solution method, sophisticated mathematical apparatus and the method of presentation. As a rule, having derived basic relations, the researchers turn to examples of solving specific problems while the issues of the qualitative theory are left aside. The given research breaks the rule and addresses the qualitative principles in the first place. These principles rest upon the confession that deviation from from the theoretical framework of elasticity, no matter how small, is an evidence of improper formulation of the problem.

The author focuses on a plane elastic problem in polar coordinates for regions with angular points of the boundary. Here belong finite and infinite domains with fractures, piecewise-homogenous wedge structures, half-planes under point force or subjected to indentation with a rigid press-tool etc.

## 2. Point force application at the apex of a wedge

To start with, let us discuss a partial solution of a plane problem with the function of stresses given by:

$$\varphi = r\theta(A_1 \sin \theta + B_1 \cos \theta), \quad (1)$$

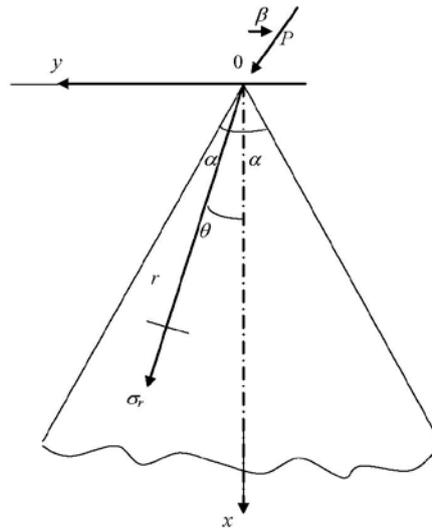
which conforms with the simple radial stress state (see the figure 1). According to [1], the stresses are normal and shear:

$$\sigma_r = \frac{2}{r}(A_1 \cos \theta - B_1 \sin \theta), \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0, \quad (2)$$

The stress distribution (2) is valid for an infinite wedge-shaped domain with the boundary conditions set as the zero normal stresses  $\sigma_\theta$  and shear stresses  $\tau_{r\theta}$  at the wedge boundary  $\theta = \pm\alpha$  (figure 1):

$$\sigma_\theta = 0, \quad \tau_{r\theta} = 0 \quad \text{at } \theta = \pm\alpha. \quad (3)$$





**Figure 1.** Pattern of calculation to fit the solution (1)

It is readily checked that the function (1) is the solution of a boundary value problem of elasticity. At the same time, this solution determines the stresses (2) that to zero as  $r \rightarrow \infty$  and infinitely increase when  $r \rightarrow 0$ , i.e. indefinite stresses. On the other hand, the boundary conditions (3) assume that a wedge with an apex angle  $2\alpha$  is free from external forces, i.e. unloaded. Then where the stresses (2) come from? The source of the stress state (2) in the body under analysis can be either in the coordinate origin or at infinity.

In order to answer the question, let us analyze the behavior of the solution (2) at infinity. We take a part of the wedge as a sector limited with the section  $r = R$ , and the action of the withdrawn part is replaced by the response  $\sigma_r(R)$ . Calculate the components of the resultant vector of forces which are external relative to the wedge sector under analysis and are applied to its cambered boundary:

— $x$ -component

$$I_x = \int_{-\alpha}^{\alpha} \sigma_r \cos \theta ds, \tag{4}$$

— $y$ -component

$$I_y = \int_{-\alpha}^{\alpha} \sigma_r \sin \theta ds. \tag{5}$$

With regard to (2) and  $ds = R d\theta$ , we find:

$$I_x = A_1(2\alpha + \sin 2\alpha), \quad I_y = -B_1(2\alpha - \sin 2\alpha). \tag{6}$$

Thus, when  $A_1$  and  $B_1$  are not concurrently zero:

—the analyzed wedge sector is not in the equilibrium and the resultant solution is impossible to interpret as the solution of a boundary value elastic problem;

—the values of  $I_x, I_y$  are independent of  $r$ , accordingly, at infinity, despite  $\lim_{r \rightarrow \infty} \sigma_r = 0$ , the resultant vector of external forces is non-zero.

In other words, the analyzed wedge is subjected at infinity to the action of the resultant vector of forces (6). This resultant force has no counter-balance and the solution is, therefore, senseless. Given any equilibration of the vector of the external forces applied at infinity to the wedge  $0 \leq r \leq \infty, -\alpha \leq \theta \leq \alpha$ , the solution (2) has sense and the constants  $A_1$  and  $B_1$  are determined unambiguously only in this case. The only possible place for an equilibrating vector is the coordinate origin ( $r = 0$ ), i.e. the uncertainty point of the solution (2). The present research author believes this chain of reasoning is

more natural and correct than in [1–5]. In all of the known textbooks on the theory of elasticity, this chain of speculations is inverted. The primary element there is the point force applied at the wedge apex and the solution in form of (1) is selected for this force. The direction of the point force  $P$  is characterized by the angle  $\beta$  (refer to the figure 1). From the condition of equilibrium of the wedge, the constants  $A_1$  and  $B_1$  are found in the form of:

$$A_1 = -\frac{P \cos \beta}{2\alpha + \sin 2\alpha}, \quad B_1 = \frac{P \sin \beta}{2\alpha - \sin 2\alpha}. \quad (7)$$

The most interesting seems to be the partial solution of (4)–(7) when  $\alpha = \pi/2$  and  $\beta = 0$ . This so-called solution of the problem on a point force directed in perpendicular to the boundary of an elastic half-plane (Flamant solution):

$$\sigma_r = -\frac{2P \cos \theta}{\pi r}, \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0. \quad (8)$$

With the known stress distribution (8), it is possible to find the related displacements of points at the half-plane boundary:

$$u = -\frac{2P}{\pi E} \cos \theta \ln r - \frac{(1-\nu)P}{\pi E} \theta \sin \theta, \quad (9)$$

$$v = \frac{2P}{\pi E} \sin \theta \ln r - \frac{(1-\nu)P}{\pi E} \theta \cos \theta + \frac{(1+\nu)P}{\pi E} \sin \theta.$$

The elasticity problem solutions should ensure continuous extension to the boundary irrespective of our way from the body to a point of this boundary [5].

Regarding stresses, it can be emphasized that:

$$\lim_{r \rightarrow 0} \sigma_r = 0 \quad \text{at} \quad \theta = \pm \frac{\pi}{2},$$

$$\lim_{r \rightarrow 0} \sigma_r = -\infty \quad \text{at} \quad \theta = 0,$$

i.e. the limit of  $\sigma_r$  is different in different directions from  $\theta = \pm \frac{\pi}{2}$  to  $\theta = 0$  to  $r = 0$ .

The same is valid for the displacements:

$$\lim_{r \rightarrow 0} u = -\frac{(1-\nu)P}{2E} \quad \text{at} \quad \theta = \pm \frac{\pi}{2}, \quad \lim_{r \rightarrow 0} u = \infty \quad \text{at} \quad \theta = 0;$$

$$\lim_{r \rightarrow 0} v = \pm \infty \quad \text{at} \quad \theta = \pm \frac{\pi}{2}, \quad \lim_{r \rightarrow 0} v = 0 \quad \text{at} \quad \theta = 0.$$

Despite the solutions (8) and (9) are of no significant interest in applied problems such as the problem discussed above, theoretically, the solution for the point force is the fundamental solution in the theory of elasticity. Its fundamentality is connected with its ability to be integrated over an arbitrary boundary of a body and with obtainability of a finite solution everywhere at the boundary subject the resultant vector and resultant moment are zero, i.e. the problem is properly posed. In this sense, an example of an ill-posed problem, included in [1–5] too, is distributed load applied to a finite section of the boundary of a half-plane with the non-zero resultant vector, or indentation of a press-tool in the half-plane.

### 3. Solutions for V-shaped domains

Let in a finite V-shaped domain with an apex angle  $2\theta > \pi$  similarly to (1):

$$\varphi(r, \theta) = r^{1+\varphi} F(\theta), \quad (10)$$

where

$$F(\theta) = A\cos(\varphi + 1)\theta + B\cos(\varphi - 1)\theta + C\sin(\varphi + 1)\theta + D\sin(\varphi - 1)\theta,$$

$A, B, C, D$ —arbitrary constants;  $\varphi$ —constant less than unit. For the stress state symmetrical relative to the axis  $\theta = 0$  of the wedge, it can be set that  $C = D = 0$ . Given asymmetry,  $A = B = 0$ . Finally, after analysis of the stress state, we have:

$$\begin{aligned}\sigma_r &= -A\varphi r^{\varphi-1} \left[ (1 + \varphi)\cos(1 + \varphi)\theta + (3 - \varphi)\frac{\cos(\varphi + 1)\alpha}{\cos(\varphi - 1)\alpha}\cos(\varphi - 1)\theta \right], \\ \sigma_\theta &= -A\varphi(\varphi + 1)r^{\varphi-1} \left[ \cos(\varphi + 1)\theta - \frac{\cos(\varphi + 1)\alpha}{\cos(\varphi - 1)\alpha}\cos(\varphi - 1)\theta \right], \\ \sigma_{r\theta} &= -A\varphi r^{\varphi-1} \left[ (\varphi + 1)\sin(\varphi + 1)\theta + (1 - \varphi)\frac{\cos(\varphi + 1)\alpha}{\cos(\varphi - 1)\alpha}\sin(\varphi - 1)\theta \right], \\ u &= \frac{A}{2G} r^\varphi \left[ -(\varphi + 1)\cos(\varphi + 1)\theta + \left(\varphi + 1 - \frac{\varphi}{1+\varphi}\right)\frac{\cos(\varphi + 1)\alpha}{\cos(\varphi - 1)\alpha}\cos(\varphi - 1)\theta \right], \\ v &= \frac{A}{2G} r^\varphi \left[ (\varphi + 1)\sin(\varphi + 1)\theta - \left(\varphi - 1 + \frac{\varphi}{1+\varphi}\right)\frac{\cos(\varphi + 1)\alpha}{\cos(\varphi - 1)\alpha}\sin(\varphi - 1)\theta \right].\end{aligned}\quad (11)$$

By way of example, consider a wedge  $2\alpha = 270^\circ$ . For such angle,  $\varphi = 0.545$ , i.e. the singularity of the stresses, as per [10], is  $1 - \varphi = 0.455$ . The classical solution (11) determines the pre-deformation angle  $2\alpha = 270^\circ$  as the post-deformation angle of  $180^\circ$ , which means the assumption of smallness of deformation is no more valid. On the other hand, the opposite sign stress state, which induces displacement of the boundaries  $\theta = \pm 135^\circ$  and results in material-material penetration, contradicts the physical sense of the problem.

In this manner, the solutions (11) in the interpretation by the authors [4–10] are senseless. Indeed, on the one part, the V-shaped domain is free from any force:

$$\sigma_r = 0, \quad \tau_{r\theta} = 0 \quad \text{at} \quad \theta = \pm\alpha \quad (12)$$

but is deformed and one of the backbone hypothesis of the linear elasticity theory is violated; on the other part, the solution is found accurate to a constant.

Aiming to find a real ground of the solution (11), we take a sector with a radius  $r = R$  out of a V-shaped domain and replace the action of the withdrawn portion by the responses in accordance with (11). In this case, it is assumed that  $2\alpha = 270^\circ$ ,  $\varphi = 0.545$ . At the radial boundaries of the selected sector, the forces are absent as per [5–10], i.e. the condition (12) is fulfilled. It is readily checked that the sector is out of equilibrium when subjected to the forces  $\sigma_r$  and  $\tau_{r\theta}$  applied at the cambered boundary, which means that the problem formulation is ill-posed in [5–10]. Actually, the resultant vector of external forces applied to the selected sector is:

$$Q_x = 2.888AR^\varphi.$$

In order to eliminate this deficiency, an additional condition should be set. Such condition may be a force applied at the apex of the V-shaped domain, or any other permissible influence. In the meanwhile, the statement in [5–10] suggests no additional forces applied at the boundary or inside the domain. Let us loop the setting of the boundary conditions with the assumption that the apex is rigidly fixed ('nailed') at a point at the coordinate origin. In this case, the selected sector is balanced by the unknown response of the nailing. Mathematically, this condition means that  $u(0,0) = v(0,0) = 0$ . Now, the constant  $A$  can be determined in terms of the nailing response  $-Q$ . For such kind of fixation at a point, the classical elasticity theory has no definition of a response as against the case when the singularity of stresses is equal to unit (so called point force).

For the V-shaped domains described by the stress function (1), the physical sense is supported by introduction of a hypothetical 'point force' (or determination of the constant  $A_1$ ). In case of (1) the science has no even a hypothetical structure to explain the very existence of the 'solution'. Thus, the constant  $A$  in (10) is indefinite.

The presented analysis of the solutions for the domains with an infinitely remote point and infinite stresses displays weakness of complementary explanations aimed to justify the results which drop out the framework of the theory of elasticity in [5–10]. It seems to be worthy of adding that when points with infinite solutions are withdrawn, the withdrawn portion should be replaced by responses based on an infinite solution. This results in a partial stress distribution over rounded boundary, which never exists in the applied problems, and the size of the withdrawn portion is uncertain.

#### 4. Connection between the solutions (1) and (10)

For the V-shaped domains, irrespective of the boundary conditions, the singularity of stresses (strains) is determined from the equations [10]:

$$\sin 2\varphi\alpha = \pm \varphi \sin 2\alpha,$$

where “ $\pm$ ”—respectively, in the case of symmetric ( $\beta = 0$ ) and asymmetry ( $\beta = 90^\circ$ ) of the problem. In particular, it is simple to obtain the singularity for a plane with a half-infinite cut (V-shaped domain with an angle  $\alpha = \pi$ ) using the mapping of the half-plane with the so-called 'point force' solution on a wedge with an apex angle  $2\pi$ . The mapping of a half-plane on a wedge with an apex  $2\alpha = 2\pi$  is given by the relations:

$$r = \sqrt{r_1}, \theta = 0.5\theta_1, \quad (13)$$

where  $r_1, \theta_1$ —polar coordinate system connected with the wedge  $2\alpha = 2\pi$ . Given the mapping (13), the boundary condition and the solution for the half-plane convert, respectively, to the boundary conditions and the solution for a plane with a half-infinite cut. The boundary condition in the half-plane is set as the force  $P$ , accordingly, the boundary condition in the V-shaped domain is the imaging force  $P$ .

Consequently, assuming that the force  $P$  has a physical sense, its imaging becomes physically real. As has been demonstrated above, this interpretation represents the response that balances forces at infinity for a wedge. The common interpretation of the solutions at a crack tip (a V-shaped domain with an apex angle  $2\alpha = 360^\circ$ ), stating that the tip is free from forces, is incorrect. In point of fact, the tip of a cut is subjected to a concentrated impulse—the imaging of a 'point force' applied to a half-plane.

This investigation offers a most complete explanation of the sense, inter-relation and place of indefinite stresses in the theory of elasticity.

#### 5. Mathematical cuts

Let us fix on a problem on a plane with a weakening in the form of a finite straight-line cut. Irrespective of the approached to solving the problem, the results coincide; we focus on the solution using the conformal mapping, e.g. [5]. The solution in [5] is incorrect as the conformality of mapping is violated at the angular points, which automatically brings infinite stresses and is inadmissible. This could be the end but the authors [1–10] permit using this inelastic solution at the distance from the cut tips. In this solution, it is essential that the tips of a cut (for any cracks) are 'nailed' at the points where  $u = v = 0$ . It is a subconscious comprehension that in the elastic solution, when the cut widens, its tips displace toward the cut center. On the other hand, when the tips are 'nailed', the responses arise at the nailing points (and such notions is absent in the theory of elasticity), and such responses can be infinite. Deformation of a cut with the 'nailed' tips is a complex structure and cannot be modeled by a plane with a mathematical cut that opens under normal forces. The theory of failure based on the stress intensity factor is no more satisfactory. Stress intensity factor is an absolute number divided by zero,

which is wrong. The discussed classical solutions are mathematically ill-conditioned and physically inadmissible

## 6. Conclusion

The known solutions with the infinite stresses are impossible to assume elastic as they violate the provisions of the elastic model of a medium.

The singularity of stresses in all known solutions is connected with the conformal mapping of the analyzed domains and is the result of the rigid fixation of the domains at the angular points.

In case of mathematical cuts, large deformation or penetration of a material into a material in the vicinity of the tips of a cut prove ill-posedness of the solutions which are mathematically and physically senseless.

It is incorrect to map domains with angular points on domains with smooth boundaries inasmuch as the conformality of the mapping is violated.

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