Inclined indentation of smooth wedge in rock mass

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Abstract. The article focuses on the inclined rigid wedge indentation into a rigid-plastic half-plane of rocks with the Mohr–Coulomb-Mohr plasticity. The limiting loads on different sides of the wedge are determined versus the internal friction angle, cohesion and wedge angle. It is shown that when the force is applied along the symmetry axis of the wedge, the zone of plasticity is formed only on one wedge side. In order to form the plasticity zone on both sides of the wedge, it is necessary to apply the force asymmetrically relative to the wedge symmetry axis. An engineering solution for the asymmetrical case implementation is suggested.

The problem on destruction of a material by a wedge is the basic in mining [1–5]. It has been solved in elasticity [6–7] and in plasticity [8–10]. The most commonly used approach is rigid–plastic body when elastic strains are neglected and fully plastic strains are the framework. A wedge can be indented along a straight line and at an angle. The straight-line indentation is when the wedge faces have the same angles with the surface of a body the wedge penetrates. The inclined indentation when the inclines of the wedge faces relative to the body are different on all sides. Both direct and inclined inundation processes have been studied [2–3 and 8–11]. The objective was to determine forces that make the wedge penetrate in the body.

Unlike the earlier research, the present paper authors aim to determine loading conditions when a wedge moves in the preset direction. In this case, it is required to control forces and moments applied to the wedge. Let us discuss the case of plain strain deformation of rock mass under perfectly rigid interaction between a nondeformable wedge and a perfectly smooth surface. These requirements are traditional and can be withdrawn by adding the mathematical model with the contact friction and considering more complex deformation state. We take this simplest example to understand the essence of the problem.

So, we analyze the problem on the inclined indentation of a smooth wedge with a nose angle 2γ and the axis of symmetry inclined at the abscissa axis at a constant angle α (Figure 1).

It is assumed that the half-lane material is a medium under limit equilibrium by the Mohr–Coulomb condition:

$$\max_n \left\{ \tau_n + \operatorname{tg} \psi \cdot \sigma_n \right\} = K ,\tag{1}$$

where ψ —internal friction angle; K —cohesion coefficient; n —normal to an arbitrarily oriented plane in the coordinate xO_yz . In the principal axes of the stress tensor, (1) is rewritten as:

$$\frac{\sigma_1 - \sigma_3}{2\cos\psi} + \text{tg}\,\psi \cdot \frac{\sigma_1 + \sigma_3}{2} = K\,, \tag{2}$$

where σ_1, σ_2, σ_3—principal stresses of the tensor T_σ. The medium equilibrium equations are:

$$\partial\sigma_x/\partial x + \partial\tau_{xy}/\partial y = 0,\ \partial\tau_{xy}/\partial x + \partial\sigma_y/\partial y = 0\,. \tag{3}$$
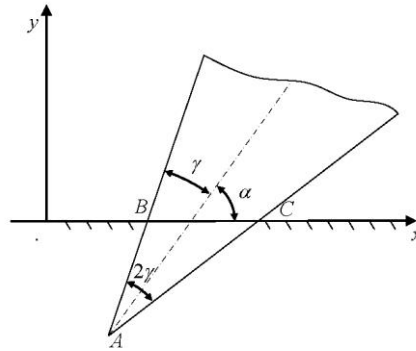

Figure 1. Schematic indentation of wedge with the nose angle 2γ in the rigid–plastic half-plane.

After substitutions $(\sigma_x - \sigma_y)/2 = T\cos 2\Theta$, $\tau_{xy} = T\sin 2\Theta$, $(\sigma_x + \sigma_y)/2 = \sigma$, where $T = (\sigma_1 - \sigma_3)/2$, $\sigma = (\sigma_1 + \sigma_3)/2$, $\text{tg}\,2\Theta = 2\tau_{xy}/(\sigma_x - \sigma_y)$, $\sigma_1 \ge \sigma_2 \ge \sigma_3$, Θ—angle between the first direction for T_σ and the x axis, the stresses $\sigma_x, \sigma_y, \tau_{xy}$ are related to two independent values $\sigma = \sigma(x, y)$, $\Theta = \Theta(x, y)$:

$$\begin{cases} \sigma_x = \sigma(1 - \sin\psi \cdot \cos 2\Theta) + K \cdot \cos\psi \cdot \cos 2\Theta, \\ \sigma_y = \sigma(1 + \sin\psi \cdot \cos 2\Theta) - K \cdot \cos\psi \cdot \cos 2\Theta, \\ \tau_{xy} = -\sigma \cdot \sin\psi \cdot \sin 2\Theta + K \cdot \cos\psi \cdot \sin 2\Theta. \end{cases} \tag{4}$$

Substituting (4) in (3) results in two equilibrium equations:

$$d\sigma[\sin 2\Theta \cdot dy + (\sin\psi + \cos 2\Theta) \cdot dx] + 2 \cdot (\sigma\sin\psi - K\cos\psi) \cdot d\Theta \cdot dy = 0 \tag{5}$$

along two coordinate lines with the equations:

$$(dy/dx)_1 = \text{tg}(\Theta + \pi/4 + \psi/2),\ (dy/dx)_2 = \text{tg}(\Theta - \pi/4 - \psi/2). \tag{6}$$

It follows from (6) that that the characteristics (lines) are arranged symmetrically relative to the first principal direction.

In order to determine limiting loads on the right-hand and left-hand faces of the wedge (Figure 1), we take first the right-hand face AC depicted in Figure 2 with a segment CE showing the free boundary. At the boundary CE, the normal stress $\sigma_y = 0$ and the shear stress $\tau_{xy} = 0$. Consequently, the normal to the boundary CE and the tangent (axis X) to this boundary are the principal directions. Inasmuch as the segment CE undergoes compression under the wedge penetration, $\sigma_y = \sigma_1 = 0$, $\sigma_x = \sigma_3 < 0$, which means that the first principal; direction is perpendicular to the axis X. In other words, the angle $\Theta = \pi/2$ and the characteristics are the line with the equations:

$$dy = \text{tg}(3\pi/4 + \psi/2)dx,\ dy = \text{tg}(\pi/4 - \psi/2)dx\,. \tag{7}$$
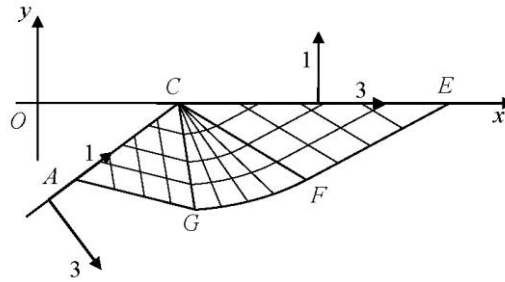
Figure 2. Determination of limit load on the right-hand face of the wedge indenter: 1, 3—directions of the principal stresses.

It is assumed that the angle Θ is constant in the triangle CEF, i.e. the lines (7) are the straight lines from CE to AC and are described by the equations:

$$y = \mathrm{tg}(\pi/4 - \psi/2) \cdot x + \mathrm{const}\,.$$

Inasmuch as (2) should be valid in the whole triangle CEF, under $\sigma_1 = 0$, $\sigma_x = \sigma_3$, we obtain $\sigma_3 = -2K \cdot \cos\psi/(1 - \sin\psi)$; i.e. $\sigma = (\sigma_1 + \sigma_3)/2$ in CEF is:

$$\sigma = -K \cdot \cos\psi/(1 - \sin\psi)\,. \tag{8}$$

In the same manner, we find stress state in the triangle ACG (Figure 2). In this case, the principal directions are the normal to AC and the tangent. The face is inclined to the axis X at an angle $\alpha - \gamma$. It is assumed that the pressure long the normal to AC (negative) is higher in absolute value than along AC, i.e. the first principal direction in T_σ is oriented along AC. In other words, Θ is equal to $\alpha - \gamma$. The pressure applied to the face AC is denoted by $p_n = \sigma_3$. Let us express p_n in terms σ. From the limit equilibrium condition (2), we have:

$$\sigma_3 = \sigma(1 + \sin\psi) - K\cos\psi\,. \tag{9}$$

This means that finding $p_n = \sigma_3$ requires calculating σ on AC. As previously, Θ is assumed constant in the triangle ACG, which implies uniform stress state.

We suppose that the fields ACG and CEF are connected by a centered field with the polar coordinate system with the pole C. Meanwhile:

$$x = \rho \cdot \cos\varphi,\ y = \rho \cdot \sin\varphi\,, \tag{10}$$

where the φ is related with Θ:

$$\varphi = \Theta - 3\pi/4 + \psi/2\,. \tag{11}$$

When $\Theta = \pi/2$ the angle φ equals $-\pi/4 + \psi/2$, when $\Theta = \gamma - \alpha$ $\varphi = \Theta - 3\pi/4 + \psi/2 + \gamma - \alpha$.

$$dx = d\rho \cdot \cos\varphi - \rho \cdot \sin\varphi \cdot d\varphi,\ dy = d\rho \cdot \sin\varphi + \rho \cdot \cos\varphi \cdot d\varphi\,, \tag{12}$$

where dx, $d\rho$, $d\varphi$, dy —differentials of functions. Placement of (12) in (7) brings the coordinate lines:

$$d\varphi = 0,\ \rho = \rho_0 \cdot e^{\mathrm{tg}\psi \cdot (\varphi - \varphi_0)}\,,$$

where $\varphi_0 = -\pi/4 + \psi/2$. At $\varphi = -3\pi/4 + \psi/2 + \gamma - \alpha$ we have $\rho = \rho_0 \cdot e^{-\mathrm{tg}\psi \cdot (\pi/2 - \gamma + \alpha)}$, which means that ρ for this value of φ is less than ρ_0 for $\varphi = \varphi_0 = -\pi/4 + \psi/2$.

Then, we analyze the relation in the characteristics (5). We have $d\varphi = d\Theta$, $dy/dx = \text{tg}(\Theta - \pi/4 - \psi/2) = \text{tg}(\varphi - \psi + \pi/2) = -\text{ctg}(\varphi - \psi)$. Substitution of this value in (5), after transformations, allows a relation:

$$\frac{d\sigma}{\sigma \cdot \sin\psi - K \cdot \cos\psi} + \frac{2d\varphi}{\cos\psi} = 0;$$

the latter is integrated and yields:

$$\sigma \cdot \sin\psi - K \cdot \cos\psi = (\sigma \cdot \sin\psi - K \cdot \cos\psi) \cdot e^{-2\text{tg}\psi \cdot (\varphi - \varphi_0)}.$$

The integration constant is found from the condition that at $\varphi = \varphi_0 = -\pi/4 + \psi/2$ the value of σ coincides with the value of σ from (8). In this case, at $\varphi = -3\pi/4 + \psi/2 + \gamma - \alpha$, we have:

$$\sigma \cdot \sin\psi - K \cdot \cos\psi = -\frac{K \cdot \cos\psi}{1 - \sin\psi} \cdot e^{2\text{tg}\psi \cdot (\pi/2 - \gamma + \alpha)}$$

or

$$\sigma = \frac{K \cdot \cos\psi}{\sin\psi} \cdot \left[1 - \frac{e^{2\text{tg}\psi \cdot (\pi/2 - \gamma + \alpha)}}{1 - \sin\psi} \right].$$

Using (9), find σ_3 at AC:

$$\sigma_3 = \frac{K \cdot \cos\psi}{\sin\psi} \cdot \left[1 - \frac{1 + \sin\psi}{1 - \sin\psi} \cdot e^{2\text{tg}\psi(\pi/2 - \gamma + \alpha)} \right]. \tag{13}$$

Multiplying (13) by the length of AC gives the force required to be applied to AC to obtain stress fields as in Figure 2.

For another thing, in the analysis of the stress field to the left of the line AB in Figure 1, it is assumed that it is inclined to the axis x at an angle $\alpha + \gamma$ less than $\pi/2$ (otherwise we have the situation des cribbed above). In so doing, we obtain two stress fields HDB and ADB (Figure 3).
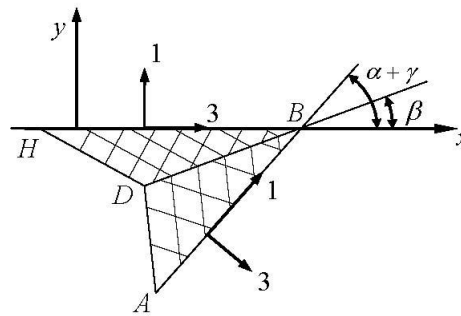

Figure 3. Stress fields HDB and ADB separated by a straight line with the unknown angular coefficient $\text{tg}\beta$.

The stress state is assumed constant (uniform) in the triangle HDB:

$$\tau_{xy} = 0, \ \sigma_y = \sigma_1 = 0, \ \sigma_x = \sigma_3, \ (\sigma_1 - \sigma_3)/2\cos\psi + \text{tg}\,\psi \cdot (\sigma_1 + \sigma_3)/2 = K, \tag{14}$$

$$\sigma_x = \sigma_3 = -2K\cos\psi/(1 - \sin\psi).$$

The same computations are made for the triangle ADB. In this case,

$$\Theta = \alpha + \gamma \,, \ \sigma_x = \sigma + T \cos 2\gamma \,, \ \sigma_y = \sigma - T \cos 2\gamma \,, \ \tau_{xy} = T \sin 2\gamma \,, \tag{15}$$

where $T = (\sigma_1 - \sigma_3)/2, \quad \sigma = (\sigma_1 + \sigma_3)/2.$ By the Mohr–Coulomb condition, $T = K \cos \psi - \sigma \sin \psi$. Therefore, instead of (15), we have:

$$\begin{cases} \tau_{xy} = K \cos \psi \sin 2\gamma - \sigma \sin \psi \sin 2\gamma \,, \\ \sigma_x = K \cos \psi \cos 2\gamma + \sigma (1 - \sin \psi \cos 2\gamma), \\ \sigma_y = -K \cos \psi \cos 2\gamma + \sigma (1 + \sin \psi \cos 2\gamma). \end{cases} \tag{16}$$

Figure 3 shows the straight line with the angular coefficient tgβ separating the fields (14) and (14). To connect the fields, we have the condition of continuity of the Cauchy vector when intersecting this line (third Newton law). Introduce the vector of the normal n to DB: $n = (\sin \beta - \cos \beta)$ and obtain:

$$\begin{cases} \sigma_x^+ \sin \beta - \tau_{xy}^+ \cos \beta = \sigma_x^- \sin \beta - \tau_{xy}^- \cos \beta \,, \\ \tau_{xy}^+ \sin \beta - \sigma_y^+ \cos \beta = \tau_{xy}^- \sin \beta - \sigma_y^- \cos \beta \,. \end{cases} \tag{17}$$

If the stress fields to the left and right of DB in Figure 3 are given the plus and minus signs, respectively, we have:

$$\begin{cases} \left(\sigma_x^+ - \sigma_x^-\right) \sin \beta + \tau_{xy}^- \cos \beta = 0, \\ -\tau_{xy}^- \sin \beta + \left(\sigma_y^- - \sigma_y^+\right) \cos \beta = 0. \end{cases} \tag{18}$$

The system (18) is a uniform system of two equations to determine two unknown values: $\sin \beta$ and $\cos \beta$. In order that solution exists, it necessary that the determinant of the system (18) is zero. With the zero determinant, the stress σ in ADB in Figure 3 is given by:

$$\frac{\sigma}{K} = -\frac{\cos^2 \psi + \sin \psi 2 \cos^2 \gamma}{\cos \psi (1 - \sin \psi)} \pm \frac{2 \cos \gamma \sqrt{1 - \sin^2 \gamma \sin^2 \psi}}{\cos \psi (1 - \sin \psi)} \,, \tag{19}$$

and the angle β is found from (18), e.g.:

$$\mathrm{tg}\beta = \left(\sigma_y^- - \sigma_y^+\right) / \tau_{xy}^- \,. \tag{20}$$

The calculation of σ_3 at AB use the Mohr–Coulomb condition:

$$\sigma_3 = \sigma (1 + \sin \psi) - K \cos \psi \,. \tag{21}$$

Comments. It is apparent that the limit loads on the right-hand size (13) and left-hand side (21) of the wedge are different. Furthermore, the forces are different (as the stresses are multiplied by different areas). This fact means that under the force applied to the wedge along the wedge axis of symmetry, the material where the limit force is lower (on the left-hand side of the wedge in Figure 1) will 'yield' first. Naturally, to reach the plastic deformation of the material on the left and right of the wedge, it is required to apply the force not along the axis of symmetry of the wedge but at a certain angle connected with the formulas (13), (21) and the lengths of the faces AB and AC of the wedge in Figure 1. One of the possible diagrams of nonsymmetrical application of force along the wedge axis is shown in Figure 4.

Conclusion
    1. Limit loads on a wedge indented in rock mass at an angle have been determined.

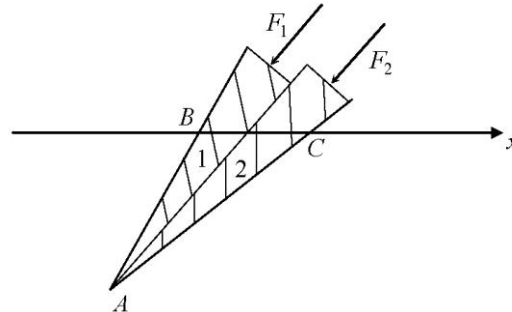2. The authors have suggested the scheme of nonsymmetrical loading of a wedge during its inclined indentation.

Figure 4. Different forces F_1 and F_2 applied to 'constituents' 1 and 2 of a wedge.

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