Inclusion of inhomogeneous deformation and strength characteristics in the problem on zonal disintegration of rocks

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Abstract. The authors determine stress and deformation in a heterogeneous rock mass at the preset displacement and Cauchy stress vector at the boundary of an underground excavation. The influence of coordinates on Young's modulus, shear modulus and ultimate strength is shown. It is found that regions of tension and compression alternate at the excavation boundary—i.e. zonal rock disintegration phenomenon is observed.

1. Introduction
In solid mechanics and, specifically, in rock mechanics, preferable formulation of boundary value problems are: setting the Cauchy stress vectors at the known boundaries of a body, setting displacement vectors at the boundaries and setting the Cauchy stress vectors at some boundaries and displacement vector at the other boundaries (combined problem) [1–3]. In elasticity, the theorems of existence and uniqueness of solution are proved [4–5]. This study addresses such formulation of a boundary value problem when the Cauchy stress vector and the displacement vector are set at the same boundary. The problem includes the simplest boundary in the form of a circumference and the preset boundary conditions are independent of coordinates.

2. The problem of the theory of elasticity for rock mass surrounding a tunnel
Let there be a tunnel with a radius a (Figure 1) with the preset boundary conditions given by:

$$u_r = -u_0, \quad u_\theta = 0,$$
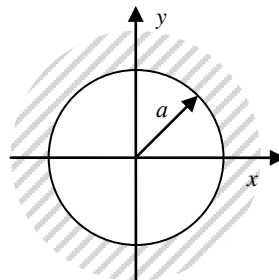$$\sigma_r = -p_0, \quad \tau_{r\theta} = 0. \tag{1}$$

Figure 1. Tunnel with a radius a and the Cauchy stress vector and displacement vector preset as the boundary conditions.

Rock mass around the tunnel is in elastic state. It is required to find distribution of stresses, strains and displacement around the tunnel.

The problem is solved by integration of a second-order differential equation under the boundary conditions (1). The differential equation in terms of stresses is given by:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \tag{2}$$

the Hooke low for the plain strain:

$$\begin{cases} \sigma_r - \sigma_\theta = 2\mu(\varepsilon_r - \varepsilon_\theta), \\ \sigma_r + \sigma_\theta = 2k(\varepsilon_r + \varepsilon_\theta), \end{cases} \tag{3}$$

$$2\mu = E/(1-\nu), \ 2k = E/(1-2\nu)(1+\nu).$$

The Cauchy relations:

$$\varepsilon_r = \frac{du_r}{dr}, \ \varepsilon_\theta = \frac{u_r}{r}. \tag{4}$$

Placement of (4) in (3), (3) in (2) brings a second-order differential equation in terms of radial displacement u_r:

$$\frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0. \tag{5}$$

The unknown is Eq. (5). Its solving needs setting the function $(u_r = -u_0)$ and the normal derivative (coordinate r) at the boundary; the coordinate derivative r is:

$$\frac{du_r}{dr} = \frac{\sigma_r}{k+\mu} - \frac{k-\mu}{k+\mu}\varepsilon_\theta. \tag{6}$$

In view of (1), the condition (6) is re-written as:

$$\left.\frac{du_r}{dr}\right|_{r=a} = -\frac{p}{k+\mu} + \frac{k-\mu}{k+\mu}\frac{u_0}{a}. \tag{7}$$

The general solution of (5) is presented as a sum $u_r = C_1 r + \dfrac{C_2}{r}$, where C_1, C_2 —arbitrary constants.

From the initial conditions (1), (7), the constants:

$$\begin{aligned} C_1 &= -\frac{1}{2(k+\mu)}\left(p + 2\mu\frac{u_0}{a}\right), \\ C_2 &= -au_0 - C_1 a^2. \end{aligned} \tag{8}$$

Then:

$$\begin{cases} u_r = C_1 r\left(1 - \dfrac{a^2}{r^2}\right) - \dfrac{au_0}{r}, \\ \varepsilon_r = -\dfrac{p}{2(k+\mu)}\left(1 + \dfrac{a^2}{r^2}\right) - \dfrac{\mu u_0}{(k+\mu)a} + \dfrac{k}{k+\mu}\dfrac{u_0}{a}\dfrac{a^2}{r^2}, \\ \sigma_r = -\dfrac{p}{k+\mu}\left(k + \mu\dfrac{a^2}{r^2}\right) - \dfrac{2k\mu}{(k+\mu)}\dfrac{u_0}{a}\left(1 - \dfrac{a^2}{r^2}\right). \end{cases} \tag{9}$$

It follows from (8), (9) that when $r \geq a$ the displacement u_r is negative. At the same time, the strain ε_r can always be positive, e.g. at $p = 0$, $k = \infty$. This fact seems unusual as under longitudinal and transversal compression of rock mass along the axis r the strain ε_r is positive, i.e. tensile. Actually, there is nothing unusual as the strain $\varepsilon_\theta = \dfrac{u_r}{r}$ is negative (for u_r is negative) while for an incompressible material $\varepsilon_r + \varepsilon_\theta = 0$, thus, $\varepsilon_r = -\varepsilon_\theta$.

To sum up this Section, it follows from (1), (7) that setting the stress vector at the tunnel boundary is equal to setting the normal derivative of the function of displacement at this boundary.

The equation (5) yields the formula to determine σ_r at 'infinity':

$$\sigma_r\big|_{r \to \infty} = -\frac{p\,k}{k+\mu} - \frac{2k\,\mu}{k+\mu}\frac{u_0}{a}$$

(as $k \to \infty$ $\sigma_r\big|_{r \to \infty} = -p - 2\mu u_0/a$).

Near the tunnel, in the radial direction, always tension takes place, which is the source of the zonal disintegration phenomenon [6, 7].

3. Problem solving with due regard to inhomogeneous distribution of shear modulus

The Cauchy problem is formulated so that to take into account inhomogeneous distribution of shear modulus around a cylindrical tunnel. Let a rock mass around a tunnel with a radius r_0 have inhomogeneous distribution of shear modulus:

$$2\mu = f\left(\frac{r}{r_0}\right), \tag{10}$$

where f —arbitrary function of the radial coordinate.

The solution range is $r \geq r_0$. We have the second-order linear differential equation with the variable coefficients to determine u ($u = u_r$):

$$\frac{d^2 u}{dr^2} + \frac{d\mu}{(k+\mu)dr}\left(\frac{du}{dr} - \frac{u}{r}\right) + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0. \tag{11}$$

We introduce (11) with a logarithmic coordinate ξ and a function $P(\xi)$ given by:

$$\xi = \ln\left(\frac{r}{r_0}\right), \quad P(\xi) = \frac{d}{d\xi}\ln(k+\mu). \tag{12}$$

In this case, from (11) we have:

$$\frac{d^2 u}{d\xi^2} + P(\xi)\left(\frac{du}{d\xi} - u\right) - u = 0. \tag{13}$$

Reduce the order of the differential equation (11) by introducing a new function:

$$v = \frac{du}{d\xi} - u. \tag{14}$$

From (13) with regard to 914), we have the system of the first-order linear differential equations for finding the functions v and u:

$$\begin{cases} \dfrac{dv}{d\xi} + (1+P(\xi))v = 0, \\[2mm] \dfrac{du}{d\xi} - u = v. \end{cases} \tag{15}$$

The system (15) is solved analytically. First, from (13), it is found that:

$$v = v_0\, e^{-\int\limits_0^\infty [1+P(\xi)]d\xi},$$

where v_0 —the value of v at the boundary $r = r_0$. Later on, from the second equation of (13), we find the function u:

$$u = u_0\, e^{\xi} + v_0 \left\{ \int\limits_0^\xi e^{\int\limits_0^\xi [1+P(\xi)]d\xi} e^{-\xi} d\xi \right\} e^{\xi}, \tag{16}$$

where u_0 —the value of the displacement u at the boundary $r = r_0$.

It is noteworthy that (16) is a general solution of Eq. (13), where u_0, v_0 —constants. Let us analyze the case when the shear modulus 2μ in rock mass around a tunnel changes by the law:

$$k + \mu = (k + \mu_0)(1 + \aleph \sin(2\pi n\xi)); \tag{17}$$

here, n defines the number of 'blocks' in the range of the coordinate ξ from 0 to 1 (the radius r varies from r_0 to $r = e\,r_0$, where $e \approx 2.718$). Substitution of (17) in (12) yields $P(\xi) = \dfrac{2\pi \aleph \cos(2\pi n\xi)}{1 + \aleph \sin(2\pi n\xi)}$. Having integrated the first equation (15), we have:

$$v = v_0 \frac{e^{-\xi}}{1 + \aleph \sin(2\pi n\xi)}. \tag{18}$$

Finding $u(\xi)$ needs using (16), i.e. taking integral of the function $e^{-2\xi}/(1 + \aleph \sin(2\pi n\xi))$. Having that the parameter $\aleph$ in (17) changes from 0 to 1 and $|\sin(2\pi n\xi)| \leq 1$, we expand the function $1/(1 + \aleph \sin(2\pi n\xi))$ in an infinitely decreasing progression. In this case:

$$\frac{e^{-2\xi}}{1 + \aleph \sin(2\pi n\xi)} \approx e^{-2\xi}(1 - \aleph \sin(2\pi n\xi) + \aleph^2 \sin^2(2\pi n\xi) - \aleph^3 \sin^3(2\pi n\xi) + ...).$$

The unknown integral is presented as a sum of integrals: $\int e^{-2\xi}\, d\zeta$, $\int e^{-2\xi} \sin(2\pi n\xi)\, d\zeta$, $\int e^{-2\xi} \sin^2(2\pi n\xi)\, d\zeta, ...$ with the coefficients $1, -\aleph, \aleph^2, -\aleph^3, ...$ Each integral is calculated using the formula of integration by parts [8].

Then, we find the connection between the constants u_0 and v_0 in (16). It is assume that the tunnel boundary is stress-free, i.e. $\sigma_r|_{\xi=0} = 0$. In this case:

$$r_0 \sigma_r = (k + \mu_0)\frac{du}{d\xi} + (k - \mu_0)u = 0, \quad u = u_0, \quad \left(\frac{du}{d\xi}\right)\Bigg|_{\xi=0} = u_0 + v_0. \tag{19}$$

In the end, we have: $(k + \mu_0)[u_0 + v_0] + (k - \mu_0)u_0 = 0$, where $v_0 = -\dfrac{2k\,u_0}{k + \mu_0} = -\dfrac{u_0}{1 - v}$, v —

Poisson's ratio. The final expression for $u(\xi)$ is given by:

$$\frac{u(\xi)}{u_0} = e^{\xi}\left(1 - \frac{1}{1-v}\int_0^{\xi} e^{-2\xi}(1 - \aleph\sin(2\pi n\xi) + \aleph^2\sin^2(2\pi n\xi) - \aleph^3\sin^3(2\pi n\xi) + ...)d\xi\right). \quad (20)$$

Figure 2a shows the curve of the relative displacement $u(\xi)/u_0$ and the logarithmic coordinate ξ: up to the value $\xi \approx 0.8$ the derivative $u(\xi)/u_0$ of the coordinate ξ is negative, i.e. the material experiences tension in the line of the radial coordinate in the range of ξ from 0 to 0.8. Later on, after $\xi \approx 0.8$, he derivative becomes positive, which means compression of the material in this domain ($\xi \geq 0.8$).
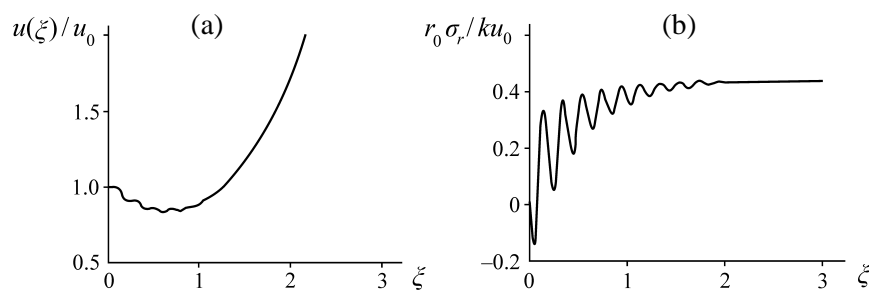

Figure 2. (a) Relative displacement $u(\xi)/u_0$ versus logarithmic coordinate ξ at $\aleph = 0.5$, $v = 0.3$, and $n = 5$; (b) radial stress $r_0\sigma_r/ku_0$ versus logarithmic coordinate ξ at $\aleph = 0.5$, $v = 0.3$, $n = 5$.

On the interval $\xi \in [0;\ 0.8]$, the derivative can be both positive and negative, i.e. at the tunnel boundary, the zones of tensile and compressive strains alternate—this is a zonal disintegration of rocks around the tunnel [6].

With the known displacements, the radial stress is found using the Hook law. In this case, the expression for the radial stress will take on form:

$$\frac{r_0\sigma_r}{u_0 k} = e^{-\xi}\left(\frac{\dfrac{du}{u_0}}{d\xi} + \frac{u}{u_0}\right) + (2(1-v)(1 + \aleph\sin 2\pi n\xi) - 1)\left(\frac{\dfrac{du}{u_0}}{d\xi} - \frac{u}{u_0}\right)e^{-\xi}. \quad (21)$$

Below in the article, the numerical solution of Eq. (15) is presented. The second-order differential scheme is given by:

$$\frac{u_{\xi+2} - 2u_{\xi+1} + u_{\xi}}{h^2} + P(\xi)\frac{u_{\xi+2} - u_{\xi}}{2h} - P(\xi)u_{\xi+1} - u_{\xi+1} = 0. \quad (22)$$

The initial conditions are described by (19).

Figure 3 depicts the relative displacement $u(\xi)/u_0$ versus the logarithmic coordinate ξ at $\aleph = 0.5$, $v = 0.3$, and the number of blocks is $n = 5$ (similarly to Figure 2a). The solid line is the analytical solution using (19), the points show the numerical solution using (22). Figure 3 illustrates total coincidence of the analytical and numerical solutions.
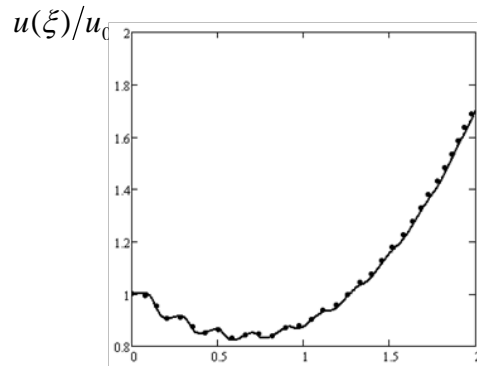
Figure 3. Relative displacement $u(\xi)/u_0$ versus logarithmic coordinate ξ at $\aleph = 0.5$, $\nu = 0.3$, and $n = 5$. Solid line–analytical solution (19); points—numerical solution (22).

In the same manner, the problem on the Young modulus versus radius is solved. In this case, instead of (11), we have:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} + \frac{1}{E}\frac{dE}{dr}\left(\frac{du}{dr} + \frac{\nu}{1-\nu}\frac{u}{r}\right) = 0 . \tag{23}$$

The Cauchy conditions:

$$u\big|_{r=a} = -u_0 , \qquad \sigma_r\big|_{r=a} = -p_0 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}\left(\frac{du}{dr} + \frac{\nu}{1-\nu}\frac{u}{r}\right)\Big|_{r=a} . \tag{24}$$

The problem (23), (24) is solved numerically.

4. Conclusion

The problem on stress state in rock mass around a cylindrical tunnel with the Cauchy conditions and inhomogeneous distribution of shear modulus has been analytically and numerically solved.

It is shown that zones of tensile and compressive strains alternate in adjacent rock mass, which the zonal disintegration phenomenon in rocks.

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