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The Existence of the Solution to One Kind of Algebraic Riccati Equation

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Abstract. The matrix equation $A^T X + XA + XRX + Q = O$ is called algebraic Riccati equation, which is very important in the fields of automatic control and other engineering applications. Many researchers have studied the solutions to various algebraic Riccati equations and most of them mainly applied the matrix methods, while few used the functional analysis theories. This paper mainly studies the existence of the solution to the following kind of algebraic Riccati equation from the functional view point:

$$A^T X + XA + XRX - \lambda X + Q = O$$

Here, X, $A, R, Q \in \mathbb{R}^{n\times n}$, Q is a symmetric matrix, and R is a positive or negative semi-definite matrix, λ is arbitrary constants. This paper uses functional approach such as fixed point theorem and contraction mapping thinking so as to provide two sufficient conditions for the solvability about this kind of Riccati equation and to arrive at some relevant conclusions.

1. Introduction

The matrix equation $A^T X + XA + XRX + Q = O$ is called algebraic Riccati equation, which is very important in the fields of automatic control and other engineering applications.

In the field of numerical algebra and modern control theories, research on the solution to algebraic Riccati equation has made remarkable progress. In the field of algorithms, Professor Liu Xinguo explicated some representative numerical methods of the solution to algebraic Riccati equation[1], such as iterative method, invariant subspace method, symbolic method and the like. Professor Lu Linzhang summarized two categories of algebraic Riccati equation solutions based on the research development from the 1980s[2]. Meanwhile, theoretically, some scholars have constituted a special kind of algebraic Riccati fomular and have given its explicit solution[3]. In studying algebraic Riccati, many authors have mainly focused on matrix method while few of them paid attention to functional analysis.

Apparently, research on the solution to algebraic Riccati equation is practically meaningful. And the method of functional analysis will provide more widespread prospect for theoretical study on solutions to Riccati equation.

2. Existence of Riccati Type Equation Solutions Under the First Condition

2.1 Proof of Existence of Solution

Lemma 1 Suppose E is a Banach space, Ω is a bounded convex set in E, then $F:\overline{\Omega}\to E$ is a completely continuous operator. If $F(\partial\Omega)\subset\overline{\Omega}$, then F must have a fixed point in $\overline{\Omega}$ [4-5].

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Theorem 1 Suppose $\|A\| < \frac{1}{2}$ with $\|R\| \neq 0$, matrix equation $A^T X + XA + XRX - X + Q = O$ is in $\Omega = \left\{ X \in D, \|X\| < \frac{1-2\|A\|}{2\|R\|} \right\}$ where D is a set composed of all symmetric matrices on $R^{n\times n}$, where- upon a solution $X \in R^{n\times n}$ must exist. Here, $\|Q\| \leq \frac{(1-2\|A\|)^2}{4\|R\|}$.

Proof Suppose $f(X) = A^T X + XA + XRX + Q$, we prove that $f : D \to D$ is continuous mapping. Obviously, with $D \subset R^{n\times n}$, D is a Banach space. And also with

$$\begin{aligned}
&\|f(X+\Delta X) - f(X)\| \\
&= \left\|A^T \Delta X + \Delta XA + (X+\Delta X)R(X+\Delta X) - XRX\right\| \\
&= \left\|A^T \Delta X + \Delta XA + XRX + \Delta XRX + \Delta XR\Delta X\right\| \\
&\leq (\left\|A^T\right\| + \|A\| + \|XR\| + \|RX\| + \|R\Delta X\|)\|\Delta X\|
\end{aligned}$$

where $\lim_{\Delta X \to 0} \|f(X+\Delta X) - f(X)\| = 0$, so f is continuous mapping.

And again, with $D \subset R^{n\times n}$, so D is finite dimensional space. Besides, as $f : D \to D$ is continuous mapping, f is compacting mapping, $f : D \to D$ is then completely continuous operator.

Because of $\Omega = \{ X \in D, \|X\| < r \}$ and $r = \frac{1-2\|A\|}{2\|R\|} > 0$ in it, it's easy to prove that Ω is convex set in D. Therefore, Ω is D 's bounded convex set.

And then, in the case of $\|X_0\| = r$

$$\begin{aligned}
\|f(X_0)\| &= \left\|A^T X_0 + X_0 A + X_0 R X_0 + Q\right\| \\
&\leq \left\|A^T X_0\right\| + \|X_0 A\| + \|X_0 R X_0\| + \|Q\| \\
&\leq (\left\|A^T\right\| + \|A\|)\|X_0\| + \|R\|\|X_0\|^2 + \|Q\| \\
&= 2\|A\| r + \|R\| r^2 + \|Q\| \\
&\leq 2\|A\| \frac{1-2\|A\|}{2\|R\|} + \|R\| (\frac{1-2\|A\|}{2\|R\|})^2 + \frac{(1-2\|A\|)^2}{4\|R\|} \\
&\leq \frac{1-2\|A\|}{2\|R\|} = r
\end{aligned}$$

,

this evidences $f(\partial\Omega) \subset \overline{\Omega}$. Based on Lemma 1, it can be inferred that f must have a fix point on $\overline{\Omega}$, which means that there must exist a solution to matrix equation $A^T X + XA + XRX - X + Q = O$ in $\Omega = \{ X \in D, \|X\| < r \}$.

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2.2 Conclusion Promotion

Inference 1 Suppose $\|A\| < \frac{|\lambda|}{2}$ and $\|R\| \neq 0$, matrix equation $A^T X + XA + XRX - \lambda X + Q = O$ is in $\Omega = \left\{ X \in D, \|X\| < \frac{1-2\|A\|}{2\|R\|} \right\}$ where D is a set composed of all symmetric matrices on $R^{n\times n}$, whe-

reupon a solution $X \in R^{n\times n}$ must exist. Here, $\|Q\| \leq \frac{(1-2\|A\|)^2}{4\|R\|}|\lambda|$ and $\lambda \neq 0$.

Proof Because of $\lambda \neq 0$, both sides of equation $A^T X + XA + XRX - \lambda X + Q = O$ are divided by λ, and $(\lambda^{-1}A)^T X + X(\lambda^{-1}A) + X(\lambda^{-1}R)X - X + (\lambda^{-1}Q) = O$ is got. And then, based on theorem 1, we arrive at the proof.

Here, we make it clear that Riccati type equation in the paper is $A^T X + XA + XRX - \lambda X + Q = O$. In the case of $\lambda = 0$, it is transformed into Riccati equation.

3. Existence of Riccati Type Equation Solutions Under the SecondCondition

3.1 Proof of Existence of Solution

Definition 1 As for any $X \in \Omega, Y \in \Omega$,if $f : \Omega \to \Omega$ satisfies it and can realize $\|f(X) - f(Y)\| \leq c\|X - Y\| \quad (0 < c < 1)$, then we demonstrate that f is contraction mapping[6-7].

Theorem 2 Suppose $\|A\| < \frac{1}{2}$ and $\|R\| \neq 0$, matrix equation $A^T X + XA + XRX - X + Q = O$ is in $\Omega = \left\{ X \in D, \|X\| \leq \frac{1-2\|A\|}{4\|R\|} \right\}$ where D is a set composed of all symmetric matrices on $R^{n\times n}$, whereupon a solution $X \in R^{n\times n}$ must exist. Here, $\|Q\| \leq \frac{(1-2\|A\|)^2}{8\|R\|}$.

Proof We define $r = \frac{1-2\|A\|}{2\|R\|} > 0$ and suppose $\|X\| \leq \frac{r}{2}$, and then we get

$$
\begin{aligned}
\|f(X)\| &= \|A^T X + XA + XRX + Q\| \\
&\leq \|A^T X\| + \|XA\| + \|XRX\| + \|Q\| \\
&\leq (\|A^T\| + \|A\|)\|X\| + \|R\|\|X\|^2 + \|Q\| \\
&= 2\|A\|\frac{r}{2} + \|R\|(\frac{r}{2})^2 + \|Q\| \\
&\leq 2\|A\|\frac{1-2\|A\|}{2\|R\|} + \|R\|(\frac{1-2\|A\|}{2\|R\|})^2 + \frac{(1-2\|A\|)^2}{8\|R\|} \\
&\leq \frac{1-2\|A\|}{4\|R\|} = \frac{r}{2}
\end{aligned}
$$

In the following, suppose $\|Y\| \leq \frac{r}{2}$, because of

$$
\begin{aligned}
\|f(X)-f(Y)\| &= \left\|(A^T X + XA + XRX + Q) - (A^T Y + YA + YRY + Q)\right\| \\
&\le \left\|A^T (X-Y)\right\| + \|(X-Y)A\| + \|XRX - YRY\| \\
&\le \left\|A^T\right\|\|X-Y\| + \|X-Y\|\|A\| + \|XRX - XRY\| + \|XRY - YRY\| \\
&\le \|X-Y\|(\left\|A^T\right\| + \|A\|) + \|X-Y\|(\|XR\| + \|RY\|) \\
&\le \|X-Y\|(2\|A\| + r\|R\|) \\
&\le \|X-Y\|(2\|A\| + \frac{1-2\|A\|}{2\|R\|}\|R\|) \\
&\le \delta\|X-Y\|
\end{aligned} ,
$$

and because of $\|A\| < \frac{1}{2}$, there exists δ . Here, $0 < \delta < 1$.

Therefore, suppose $\overline{\Omega} = \left\{X \in D, \|X\| \le \frac{r}{2}\right\}$ with $r = \frac{1-2\|A\|}{2\|R\|} > 0$, then f is a contraction mapping. Therefore, randomly get X_0 to satisfy $X_0 \in \overline{\Omega}$ and $X_0{}^T = X_0$ with $\|X_0\| \le \frac{r}{2}$. Define $X_1 = f(X_0)$, then $X_1{}^T = X_1$ with $\|X_1\| \le \frac{r}{2}$ is got and $X_1 \in \overline{\Omega}$ obtained. In the same way, with $X_n = f(X_{n-1})$, we will get $X_n \in \overline{\Omega}$ with $\forall n \ge 1, n \in \mathbf{N}$. And then,

$$
\begin{aligned}
\|X_{k+1} - X_k\| &= \|f(X_k) - f(X_{k-1})\| \\
&\le \delta\|X_k - X_{k-1}\| \\
&\le \delta^2\|X_{k-1} - X_{k-2}\| \\
&\le \cdots\cdots \le \\
&\le \delta^k\|X_1 - X_0\|
\end{aligned} ,
$$

so

$$
\begin{aligned}
\|X_{k+m} - X_k\| &\le \sum_{i=1}^{m}\|X_{k+i} - X_{k+i-1}\| \\
&\le \sum_{i=1}^{m}\delta^{k+i-1}\|X_1 - X_0\|. \\
&\le \frac{\delta^k}{1-\delta}\|X_1 - X_0\|
\end{aligned}
$$

And again, because of $0 < \delta < 1$, in the case of $k \to \infty$, we get $\|X_{k+m} - X_k\| \to 0$. That is, $\{X_k\}$ is a Cauchy sequence. Thus, $\{X_k\}$ has a limit ξ , to wit, $\lim_{k\to\infty} X_k = \xi$. Because of $X_k \in \overline{\Omega}$ and $\overline{\Omega}$ as a closed set,

$\xi \in \overline{\Omega}$. And because of continuous function f, take the limits of both sides of $X_{k+1} = f(X_k)$, we get $\xi = f(\xi)$, which means that ξ is the solution sought.

3.2 Conclusion Promotion

Inference 2 There exists a single symmetric solution to equation in theorem 3.2.

Proof If both ξ_1 and ξ_2 are solutions to the equation, then $\xi_1 = \xi_2$. Otherwise, it will not satisfy contraction mapping; and again, if ξ_1 is the solution to the equation, then ξ_1^T is its solution, too.

Inference 3 Suppose $\|A\| < \frac{|\lambda|}{2}$ with $\|R\| \neq 0$, then Equation $A^T X + XA + XRX - \lambda X + Q = O$ is in $\Omega = \left\{ X \in D, \|X\| \leq \frac{1-2\|A\|}{4\|R\|} \right\}$, whereupon there must exist a single symmetric solution $X \in R^{n\times n}$. Here, $\|Q\| \leq \frac{(1-2\|A\|)^2}{8\|R\|}|\lambda|$ with $\lambda \neq 0$.

Proof Combine theorem 2 and inference 2, and we prove the above.

In the case of $\|Q\| = 0$, we find it easy to arrive at the following conclusions.

Inference 4 Suppose $\|A\| < \frac{|\lambda|}{2}$ with $\|R\| \neq 0$, then in $\Omega = \left\{ X \in D, \|X\| \leq \frac{1-2\|A\|}{4\|R\|} \right\}$, matrix equation $A^T X + XA + XRX - \lambda X = O$ has the sole solution zero.

4. Conclusion

The paper applies functional analysis, which means utilizing fixed point theorem and contraction mapping thinking to furnish sufficient conditions for the solution to a kind of algebraic Riccati equation under two different conditions and to arrive at some conclusions.

We find that in the case of $\|Q\| \leq \frac{(1-2\|A\|)^2}{8\|R\|}$, Riccati type equation has a single symmetric solution in $\Omega = \left\{ X \in D, \|X\| \leq \frac{1-2\|A\|}{4\|R\|} \right\}$; In the case of $\|Q\| \leq \frac{(1-2\|A\|)^2}{4\|R\|}$, Riccati type equation has solutions in $\Omega = \left\{ X \in D, \|X\| \leq \frac{1-2\|A\|}{2\|R\|} \right\}$, but we aren't sure how many solutions there are.

We can further discuss the more general Riccati type equation[8]:

$$A^T X + XA + XRX - G(X) = O$$

Here, with $X, A, R, Q \in \mathbb{R}^{n\times n}$, Q as a symmetric matrix and R a positive or negative semi-definite matrix, $G(X)$ is a continuous function related to X. In this situation, we can use functional analysis theory to study the existence of the solution to the equation.

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