On the Diophantine equations $x^2 - 7y^2 = 1$ and $y^2 - Dz^2 = 9$

Jianhong Zhao[a] Lixing Yang[b]

Department of Teachers and Education, Lijiang Teachers College

Lijiang, Yunnan 674199, China

* Corresponding author

[a] E-mail:312508050@qq.com [b] E-mail: 375734990@qq.com

Abstract. The integer solution of diophantine equations $x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z)$ and $y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z)$ is a matter of great concern. Researchers study for different m,n and D_1, D_2 , and obtain some correlation results as follows.
When $m = 1$,$n = 1$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 1$.At present, there are only a few conclusions on it, see Ref [1] and [2].
When $m = 1$,$n = 4$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 4$. For even numbers D_1, D_2, the integer solution see Ref [3] - [9].
When $m = 1$,$n = 16$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 16$ .The previous conclusions see Ref [10].
When $m = 1$,$n = 9$, the diophantine equations turns into $x^2 - D_1y^2 = 1$and $y^2 - D_2z^2 = 9$. Up to now, there is no relevant result on the integer solution of (5), this paper mainly discusses the integer solution of (5) when $D_1 = 7, D_2$ is an even number.

1. Introduction

The integer solution of diophantine equations

$$x^2 - D_1y^2 = m, (D_1 \in Z^+, m \in Z) \text{ and } y^2 - D_2z^2 = n, (D_2 \in Z^+, n \in Z) \tag{1}$$

is a matter of great concern. Researchers study for different m,n and D_1, D_2 , and obtain some correlation results as follows.

When $m = 1$,$n = 1$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and} \quad y^2 - D_2z^2 = 1 \tag{2}$$

At present, there are only a few conclusions on (2), see Ref [1] and [2].

When $m = 1$,$n = 4$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and} \quad y^2 - D_2z^2 = 4 \tag{3}$$

For even numbers D_1, D_2, the integer solution of (3), see Ref [3] - [9].

When $m = 1$,$n = 16$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and} \quad y^2 - D_2z^2 = 16 \tag{4}$$

The previous conclusions on (4), see Ref [10].

When $m = 1$,$n = 9$, diophantine equations (1) turns into:

$$x^2 - D_1y^2 = 1 \text{and} \quad y^2 - D_2z^2 = 9 \tag{5}$$

Up to now, there is no relevant result on the integer solution of (5), this paper mainly discusses the integer solution of (5) when $D_1 = 7, D_2$ is an even number.

2. Key lemma

Lemma 1[11] Let p be an odd prime number, there is no integer solution of the diophantine equation $x^4 - py^2 = 1$ except $p = 5, x = 3, y = 4$ and $p = 29, x = 99, y = 1820$.

Lemma 2[12] There is 1 sets of solutions of the diophantine equation $ax^4 - by^2 = 1$ at most when a is a square number which is greater than 1.

Lemma 3[13] Let D be a square-free positive integer, then the equation $x^2 - Dy^4 = 1$ has two sets of positive integer solutions (x, y) at most. Furthermore, the necessary and sufficient condition of it is $D = 1785$ or $D = 28560$, or $2x_0$ and y_0 are square numbers where (x_0, y_0) is the basic solution of $x^2 - Dy^4 = 1$.

Lemma 4[14] Suppose that all the integer solution on Pell equation $x^2 - 7y^2 = 1$ could be $(x_n, y_n), n \in Z^+$, let $m, k \in Z^+$ and $gcd(m, k) = d$, then the following conclusions are established:

(I) $gcd(x_m, y_k) = y_d$.

(II) $gcd(x_m, x_k) = 1$ if $2|\frac{mk}{d^2}$, or else $gcd(x_m, x_k) = x_d$ when $2 \nmid \frac{mk}{d^2}$.

(III) $gcd(x_k, y_m) = 1$ if $2 \nmid \frac{m}{d}$.

Lemma 5 Suppose that all the integer solution on Pell equation $x^2 - 7y^2 = 1$ could be $(x_n, y_n), n \in Z$, for the arbitrary $n \in Z$, it has the following properties on (x_n, y_n):

(I) x_n is a square number if and only if $n = 0$.

(II) $\frac{x_n}{8}$ is a square number if and only if $n = 1$ or $n = -1$.

(III) $\frac{y_n}{3}$ is a square number if and only if $n = 0$ or $n = 1$.

Proof: (I) Let $x_n = a^2$, we will get $a^4 - 7y^2 = 1$, from Lemma 1 we can get there are only 2 integer solution $(a, y) = (\pm 1, 0)$ on $a^4 - 7y^2 = 1$, so $x_n = 1, n = 0$. On the contrary, it also holds.

(II) Let $\frac{x_n}{8} = a^2$, we will get $64a^4 - 7y^2 = 1$, from Lemma 2 we can get there are only 4 integer solution $(a, y) = (\pm 1, \pm 3)$ on $64a^4 - 7y^2 = 1$, so $x_n = 8, n = 1$ or $n = -1$. On the contrary, it also holds.

(III) Let $\frac{y_n}{3} = b^2$, we will get $x^2 - 63b^4 = 1$, from Lemma 3 we can get there are only 6 integer solution $(x, b) = (\pm 1, 0), (\pm 8, \pm 1)$ on $x^2 - 63b^4 = 1$, so $y_n = 0$ or $y_n = 3$. $n = 0$ or $n = 1$. On the contrary, it also holds.

3. Proof of main theorem

By using elementary method such as congruence, the integer solution of the diophantine equations on $x^2 - 7y^2 = 1$ and $y^2 - Dz^2 = 9$ can be obtained.

3.1 Theorem

Let $p_s (1 \le s \le 4)$ are diverse odd primes, $D = 2^k p_1^{a_1} \cdots p_s^{a_s} (a_i = 0 \text{ or } 1, 1 \le i \le 4, k \in Z^+)$, then the diophantine equations

$$x^2 - 7y^2 = 1 \text{ and } y^2 - Dz^2 = 9 \qquad (6)$$

(i) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 48)$ when $D = 2 \times 127$.

(ii) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 24)$ when $D = 2^3 \times 127$.

(iii) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 12)$ when $D = 2^5 \times 127$.

(iv) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 6)$ when $D = 2^7 \times 127$.

(v) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 3)$ when $D = 2^9 \times 127$.

(vi) has only nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 48)$ when $D \neq 2^\alpha \times 127 (\alpha = 1,3,5,7,9)$.

3.2 Proof of main theorem

3.2.1 Primary analysis

It is easily shown that $(x_1, y_1) = (2,1)$ is the minimal solution of the Pell equation $x^2 - 7y^2 = 1$, therefor all integer solution of it will be $x_n + \sqrt{7} y_n = (8 + 3\sqrt{7})^n, n \in Z$. and the following recursive sequence will be established:

$$y_{n+2} = 16 y_{n+1} - y_n, y_0 = 0, y_1 = 3 \tag{7}$$

$$x_{n+2} = 16 x_{n+1} - x_n, x_0 = 1, x_1 = 8 \tag{8}$$

Using modulo 2 on (7), we will get residue class sequence:0,1,0,1……, and $y_n \equiv 1 (mod 2)$ only when $n \equiv 1 (mod 2)$, $y_{2n} \equiv 1 (mod 2)$ only when $n \equiv 0 (mod 2)$. as a result $y_{2n} \equiv 0 (mod 2)$ and $y_{2n+1} \equiv 1 (mod 2)$.

Using modulo 2 on (8), we will get residue class sequence:1,0,1,0……, and $x_n \equiv 0 (mod 2)$ only when $n \equiv 1 (mod 2)$, $y_{2n} \equiv 1 (mod 2)$ only when $n \equiv 0 (mod 2)$. as a result $x_{2n} \equiv 1 (mod 2)$ and $x_{2n+1} \equiv 0 (mod 2)$.

Suppose $(x, y, z) = (x_{n+1}, y_{n+1}, z), n \in Z$ is the integer solution of (6), then $y_{n+1}^2 - 9 = y_{n+1}^2 - 9(x_{n+1}^2 - 7y_{n+1}^2) = 64 y_{n+1}^2 - 9 x_{n+1}^2 = (8 y_{n+1} + 3 x_{n+1})(8 y_{n+1} - 3 x_{n+1}) = y_n y_{n+2}$, it is equivalent to:

$$D z^2 = y_n y_{n+2} \tag{9}$$

Obviously $DZ^2 = 0$ when $n = -2$ or $n = 0$, here we will get the common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ on (6).

Because $y_{n+2} \equiv 1 (mod 2)$, $y_n \equiv y_{n+2} \equiv 1 (mod 2)$ when n is an positive odd number. Therefor $2(y_n y_{n+2}) = 0$, and $2(D) = 1$, we will get $2(DZ^2)$ is an odd number, it is self-contradiction, this shows that n is an nonnegative even number. Let $n = 2m, m \in Z^+$, (9) is equivalent to:

$$D z^2 = 4 x_m x_{m+1} y_m y_{m+1} \tag{10}$$

As a result the equation (10) will be:

Case 1 m is an positive even number.

Case 2 m is an positive odd number.

3.2.2 Discussion on Case 1

Let $m = 2^t p (t \in Z^+, p$ is an positive odd number), (10) is equivalent to:

$$D z^2 = 4 x_{2^t p} x_{2^t p+1} y_{2^t p} y_{2^t p+1} \tag{11}$$

For $y_{2m} = 2 x_m y_m$, (10) is equivalent to:

$$D z^2 = 2^{2+t} x_{2^t p+1} x_{2^t p} x_{2^{t-1} p} \cdots x_{2p} x_p y_{2^t p+1} y_p \tag{12}$$

From (I) of Lemma 4, we can get $gcd(y_{2^t p+1}, y_p) = y_1 = 3$. From (II) of Lemma 4, we can get $gcd(x_{2^t p+1}, x_p) = x_1 = 8$, so (12) is equivalent to:

$$D z^2 = 2^{8+t} \cdot 3^2 \cdot x_{2^t p} x_{2^{t-1} p} \cdots x_{2p} \cdot \frac{y_{2^t p+1}}{3} \cdot \frac{y_p}{3} \cdot \frac{x_p}{8} \cdot \frac{x_{2^t p+1}}{8} \tag{13}$$

For $gcd(y_{2^t p+1}, y_p) = 3$, we can get $\left(\frac{y_{2^t p+1}}{3}, \frac{y_p}{3}\right) = 1$, for $gcd(x_{2^t p+1}, x_p) = 8$, we can get $\left(\frac{x_{2^t p+1}}{8}, \frac{x_p}{8}\right) = 1$. From (II) of Lemma 4, we can get $x_{2^t p}, x_{2^{t-1} p}, \cdots, x_{2p}, \cdots x_p$ pairwise coprime. From (III) of Lemma 4, we can get $x_{2^t p+1}, x_{2^t p}, x_{2^{t-1} p}, \cdots, x_{2p}, \cdots x_p$ is coprime with $y_{2^t p+1}$ and y_p. So

$\frac{x_{2^tp+1}}{8}, x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, \frac{x_p}{8}$ is coprime with $\frac{y_{2^tp+1}}{3}$ and $\frac{y_p}{3}$. It means that $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, \frac{y_{2^tp+1}}{3}, \frac{y_p}{3}, \frac{x_p}{8}, \frac{x_{2^tp+1}}{8}$ pairwise coprime.

Because $x_{2n} \equiv 1(mod2), y_{2n+1} \equiv 1(mod2)$, we will get $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, y_{2^tp+1}, y_p$ are odd numbers, so $\frac{y_{2^tp+1}}{3}, \frac{y_p}{3}$ are odd numbers. And Because $x_{2n+1} \equiv 8(mod16)$, we will get $\frac{x_p}{8}, \frac{x_{2^tp+1}}{8}$ are odd numbers. $2(D) = 1$, so $2(2D^2)$ is odd number, but $D\left(2^{8+t} \cdot x_{2^tp} \cdot x_{2^{t-1}p} \cdots x_{2p} \cdot \frac{y_{2^tp+1}}{3} \cdot \frac{y_p}{3} \cdot \frac{x_p}{8} \cdot \frac{x_{2^tp+1}}{8}\right) = 8 + t$, t must be positive odd number.

From (ii) of Lemma 5, we can get $\frac{x_p}{8}$ is square number only when $p = 1$ or $p = -1$. From (iii) of Lemma 5, we can get $\frac{y_p}{3}$ is square number only when $p = 1$ or $p = 0$. From Lemma 5, we can get $\frac{x_{2^tp+1}}{8}, x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, y_{2^tp+1}, \frac{y_{2^tp+1}}{3}$ are non-square numbers for the arbitrary positive odd number p, therefor, $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, y_{2^tp+1}, y_p, \frac{x_p}{8}, \frac{x_{2^tp+1}}{8}$ are non-square numbers.

When $p > 1$ is an positive odd number, $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, \frac{y_{2^tp+1}}{3}, \frac{y_p}{3}, \frac{x_p}{8}, \frac{x_{2^tp+1}}{8}$ are $t + 4$ odd numbers which is not equal to 1, so they provide $t + 4$ odd prime divisors at least for D. Further more, t is an positive odd number, $t + 4 \geq 5$, it means that the right half part of (13) provide 5 odd prime divisors at least for D, it is self-contradiction.

When $p = 1, t \neq 1$, $\frac{y_p}{3}$ and $, \frac{x_p}{8}$ are square numbers, and $\frac{x_p}{8} = \frac{x_1}{8} = 1, \frac{y_p}{3} = \frac{y_1}{3} = 1$, here (13) is equivalent to:

$$Dz^2 = 2^{8+t} \cdot 3^2 \cdot x_{2^tp} x_{2^{t-1}p} \cdots x_{2p} \cdot \frac{y_{2^tp+1}}{3} \cdot \frac{x_{2^tp+1}}{8} \tag{14}$$

At present, $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, \frac{y_{2^tp+1}}{3}, \frac{x_{2^tp+1}}{8}$ provide $t + 2$ odd prime divisors at least for D. And $t \neq 1$, t is an positive odd number, so $t \geq 3$, $t + 2 \geq 5$. $x_{2^tp}, x_{2^{t-1}p}, \cdots, x_{2p}, \frac{y_{2^tp+1}}{3}, \frac{x_{2^tp+1}}{8}$ provide 5 odd prime divisors at least for D, it is self-contradiction.

When $p = 1, t = 1$, (14) is is equivalent to:

$$Dz^2 = 2^9 \cdot 3^2 \cdot x_2 \cdot \frac{y_3}{3} \cdot \frac{x_3}{8} = 2^9 \times 3^2 \times 5 \times 11 \times 17 \times 23 \times 127 \tag{15}$$

It shows that the right half part of (14) have 6 different odd prime, it is conflict with topic hypothesis, then (11) is might be wrong, (6) has no integer solution.

3.2.3 Discussion on Case 2

Let $m = 2^tp - 1(t \in Z^+, p$ is an positive odd number), (10) is equivalent to:

$$Dz^2 = 4x_{2^tp}x_{2^tp-1}y_{2^tp}y_{2^tp-1} \tag{16}$$

It could be proved by imitate 3.2.2 that (16) have nontrivial solution only when $p = 1, t = 1$, here (16) turns into $Dz^2 = 4x_2x_1y_2y_1 = 8x_1^2x^2y_1^2 = 2^9 \times 3^2 \times 127$, so $D = 2 \times 127, z = 48$, or $D = 2^3 \times 127, z = 24$, or $D = 2^5 \times 127, z = 12$, or $D = 2^7 \times 127, z = 6$, or $D = 2^9 \times 127, z = 3$, therefor equation (6) have nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 48)$ when $D = 2 \times 127$, $(x, y, z) = (\pm 2024, \pm 765, \pm 24)$ when $D = 2^3 \times 127$, $(x, y, z) = (\pm 2024, \pm 765, \pm 12)$ when $D = 2^5 \times 127$, $(x, y, z) = (\pm 2024, \pm 765, \pm 6)$ when $D = 2^7 \times 127$, $(x, y, z) = (\pm 2024, \pm 765, \pm 3)$ when $D = 2^9 \times 127$,

In conclusion, the diophantine equations (6) has common solution $(x, y, z) = (\pm 8, \pm 3, 0)$ only when $D = 2^\alpha \times 127(\alpha = 1,3,5,7,9)$, and nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 48)$. has when $D = 2 \times 127$. $(x, y, z) = (\pm 2024, \pm 765, \pm 24)$ when $D = 2^3 \times 127$. $(x, y, z) = (\pm 2024, \pm 765, \pm 12)$ when $D = 2^5 \times 127$. $(x, y, z) = (\pm 2024, \pm 765, \pm 6)$ when $D = 2^7 \times 127$.$(x, y, z) = (\pm 2024, \pm 765, \pm 3)$ when $D = 2^9 \times 127$.otherewise it has only nontrivial solution $(x, y, z) = (\pm 2024, \pm 765, \pm 48)$when $D \neq 2^\alpha \times 127(\alpha = 1,3,5,7,9)$.

References

[1] Ljunggren W. Litt om Simultane Pellske Ligninger [J]. Norsk Mat. Tidsskr.,1941,23:132-138.
[2] Pan Jia-yu, Zhang Yu-ping, Zou Rong. The Pell Equations $x^2 - ay^2 = 1$ and $y^2 - Dz^2 = 1$ [J]. Chinese Quarterly Journal of Mathematics,1999,14(1):73-77.
[3] Guan Xun-gui. On the integer solution of Pell's equation $x^2 - 2y^2 = 1$ and $y^2 - Dz^2 = 4$ [J]. Journal of Huazhong Normal University (Natural Sciences):2012,46(3):267-269+278.
[4] Hu Yong-zhong, Han Qing. On the integer solution of the simultaneous equations $x^2 - 2y^2 = 1$ and $y^2 - Dz^2 = 4$ [J]. Journal of Hua Zhong Normal University (Natural Sciences):2002,36(1):17-19.
[5] Du Xian-cun, Guan Xun-gui, Yang Hui-zhang, On the system of Diophantine equations $x^2 - 6y^2 = 1$ and $y^2 - Dz^2 = 4$ [J] . Journal of Hua Zhong Normal University (Natural Sciences):2014.48(3):5-8.
[6] Zhao jian-hong, Wan Fei. On the Common Solution of Pell Equations $x^2 - 3y^2 = 1$ and $y^2 - 2^n y^2 = 16$ [J]. Journal of Hubei University for Nationalities (Natural Science Edition), 2016,2(6):146-148.
[7] He La-rong, Zhang Shu-jing, Yuan Jin. Diophantine Equation $x^2 - 6y^2 = 1$, $y^2 - Dz^2 = 4$ [J]. Journal of Yunnan University of Nationalities (Natural Sciences Edition),2012,21(1):57-58.
[8] Ran Yan-ping. Integer Solution of the Simultaneous Diophantine Equations $x^2 - 10y^2 = 1$ and $y^2 - Dz^2 = 4$ [J]. Journal of Yanan University (Natural Science Edition),2012,31(1):8-10.
[9] Guo Jing, Du Xian-cun. On the System of Pell Equations $x^2 - 30y^2 = 1$ and $y^2 - Dz^2 = 4$[J] . Mathematics in Practice and Theory.2015,45(1):309-314.
[10] Wan Fei, Du Xian-cun. On the Simultaneous Diophantine Equations $x^2 - 5y^2 = 1$ and $y^2 - Dz^2 = 16$ [J] . Journal of Chongqing Normal University (Natural Science),2016,33(4):107-111.
[11] Sun Qi, Yuan Ping-zhi, ON THE DIOPHANTINE EQUATION $x^4 - Dy^2 = 1$ [J]. Journal Of Sichuan University (Natural Science Edition),1997:34(3):265-267.
[12] Le Mao-hua. A Family of Binary Quartic Diophantine Equations [J]. Journal of Yunnan Normal University (Natural Sciences Edition),2010,30(1):12-17.
[13] Walsh G. A note on a theorem of Ljunggren and the Diophantine equations $x^2 - kxy^2 + y^4 = 1$ or 4[J]. Arch. Math.1999,73(2):504-513.
[14] Chen Yong-gao. Pell Equation $x^2 - 2y^2 = 1$ and $y^2 - Dz^2 = 4$ [J]. Acta Scientia rum Naturahum Universitatis Pekinesis,1994,30(3):298-302.
[15] Wang Guan-min, Li Bing-rong. The Integer Solution of The Pell's Equations $x^2 - 6y^2 = 1$ and $y^2 - Dz^2 = 4$ [J]. Journal of ZhangZhou Teachers College, 2002,15(4):9-14.