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Techniques for Computation of Frequency Limited H∞ Norm

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Abstract. Traditional H_∞ norm depicts peak system gain over infinite frequency range, but many applications like filter design, model order reduction and controller design etc. require computation of peak system gain over specific frequency interval rather than infinite range. In present work, new computationally efficient techniques for computation of H_∞ norm over frequency limited interval are proposed. Proposed techniques link norm computation with maximum singular value of the system in limited frequency interval. Numerical examples are incorporated to validate the proposed concept.

1. Introduction

Consider a continuous linear time invariant dynamical system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \tag{1}$$

where $A \in R^{\{n\times n\}}, B \in R^{\{n\times m\}}, C \in R^{\{p\times n\}}, D \in R^{\{p\times n\}}$ with *n*= system order, *m*= number of inputs and *p*= number of outputs. The system (1) is called stable if all eigenvalues of A have negative real parts. The transfer matrix of (1) given as $G(j\omega) = C(j\omega\, I - A)^{\{-1\}}B + D$ is of order $p \times m$. For system (1), among many variants of norms, H_∞ norm is the most popular and extensively used for system analysis and design applications like model order reduction, cost function minimization, filter and controller design.

Definition 1: For stable system (1), the H_∞ norm of the system over infinite frequency range is given by [1]:

$$||G(j\omega)||_{\{H_\infty\}} = sup_{\{\omega\in R\}}\, \sigma_{\{max\}}(G(j\omega))$$
$$||G(j\omega)||_{\{H_\infty\}} = sup_{\{\omega=[-\infty,+\infty]\}}\sigma_{\{max\}}(G(j\omega)) \tag{2}$$

where $\sigma_{\{max\}}(.)$ is the largest singular value (SV) of the system over infinite frequency interval.

Using Definition 1, infinity norm can be computed via largest SV of the system transfer matrix. Many techniques that link norm with SV computation have been developed such as SV computation for time delay system [2], Routh table [3], characterization of polynomial [4], state space formulation [5], bisection [1], [6] etc. In [1] and [6] imaginary eigenvalues of Hamiltonian matrix are computed and linked with largest SV the system. Also in [7] extremum SVs are computed for strong $H_{\{\infty\}}$ norm computation. Similarly, largest SV can be computed using two sided Jacobi's method and Householder bidiagonalization method. Jacobi's method apply plane rotations to diagonalize transfer matrix [8]. Largest diagonal entry is the largest SV of the system. Although two sided Jacobi's method is computationally loaded, it is optimally accurate and guarantee stability even if transfer matrix

elements contain small relative errors [9]. Householder method (computationally less complex but less accurate as compared to Jacobi's technique) biadiagonalize transfer matrix by applying Householder transformation. Further it diagonalize system with orthogonal projections to yield largest SV [9].
Existing schemes [1]-[7] by definition, compute norm over infinite frequency range. However, many applications like frequency limited model reduction [10]-[13], filter design [14], signal reconstruction [15] etc. require analysis or design for the system in limited frequency interval. To the author's knowledge, no scheme to compute frequency limited H_∞ norm appear in literature (although concept of frequency limited $H_{\{2\}}$ norm has been developed [16]). Therefore in present work, techniques to compute frequency limited infinity norm via largest SV (by introducing frequency limited versions of two sided Jacobi and Householder methods for system (1)) are proposed.

2. Proposed techniques

Definition 2: Given the stable system (1), frequency limited infinity norm is defined as:

$$\left||G(j\omega)|\right|_{\{H_{\{\infty\}},\delta\}} = sup_{\{\omega\in\delta\}}\, \sigma_{\{max,\delta\}}(G(j\omega)) \tag{3}$$

where $\delta = [-\omega_i, -\omega_{\{i-1\}}]U[\omega_{\{i-1\}}, \omega_i]$, $\omega_{\{i-1\}}$ is the lower frequency and ω_i is the higher frequency, U represents union and i is the frequency interval index.
Remark 1: When $\delta = [-\infty, +\infty]$, $\sigma_{\{max,\delta\}}(G(j\omega)) = \sigma_{\{max\}}(G(j\omega))$.
Remark 2: δ may contain multiple frequency intervals as $\delta = \left[-\omega_{\{i\}}, -\omega_{\{i-1\}}\right]U\ldots U[-\omega_2, -\omega_1]U[\omega_1, \omega_2]\ldots U\left[\omega_{\{i-1\}}, \omega_i\right] = \delta_1\, U\ldots U\delta_{\{i-1\}}$. Consequently, $\left||G(j\omega)|\right|_{\{H_\infty,\delta\}} = max\, (\,||G(j\omega)|_{\{H_\infty,\delta_1\}},\ \left||G(j\omega)|\right|_{\{H_\infty,\delta_2\}}, \ldots \left||G(j\omega)|\right|_{\{H_\infty,\delta_{\{i-1\}}\}})$.

2.1. *Two Sided Jacobi's Technique for Frequency Limited H-Infinity Norm Computation:*

Definition 3: Jacobi's two sided transformation is defined as [17]:

$$Z = J^T G(j\omega_\delta)J \tag{4}$$

where $G(j\omega_\delta) = C(j\omega_\delta\, I - A)^{\{-1\}}B + D$, $\omega_\delta \in \delta$, J is the Jacobi rotation matrix given by:

$$J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{5}$$

where $c = cos(\theta)$, $s = sin(\theta)$ and θ is the rotation applied. On applying Jacobi rotation to (p, q) block of $G(j\omega_\delta)$, we obtain:

$$\begin{bmatrix} Z_{\{pp\}} & Z_{\{pq\}} \\ -Z_{\{qp\}} & Z_{\{qq\}} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{G}(\mathrm{j}\omega_\delta)_{\{\mathrm{pp}\}} & \mathrm{G}(\mathrm{j}\omega_\delta)_{\{\mathrm{pq}\}} \\ \mathrm{G}(\mathrm{j}\omega_\delta)_{\{\mathrm{qp}\}} & \mathrm{G}(\mathrm{j}\omega_\delta)_{\{\mathrm{qq}\}} \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{6}$$

To reduce to diagonal form, set $Z_{\{pq\}} = Z_{\{qp\}} = 0$. Manipulation and comparison yield:

$$G(j\omega_\delta)_{\{pq\}}\,(c^2 - s^2) + (G(j\omega_\delta)_{\{pp\}} - G(j\omega_\delta)_{\{qq\}})cs = 0$$
$$\frac{G(j\omega_\delta)_{\{qq\}} - G(j\omega_\delta)_{\{pp\}}}{G(j\omega_\delta)_{\{pq\}}} = \frac{(c^2 - s^2)}{2cs} = cot(2\theta) = \zeta$$

Let $t = tan\,(\theta)$ satisfy $t^2 + 2\zeta\, t - 1 = 0$ whose solution give:

$$t = -sig(\zeta)(|\zeta| + \sqrt{(\{1+\zeta^2\})} = sig(\zeta)/(|\zeta| + \sqrt{(\{1+\zeta^2\})})$$
$$c = 1/\sqrt{(\{1+\zeta^2\})}s = ct \tag{7}$$

where sig is the signum function, c and s are invoked in (5) and (4) to obtain diagonal form and consequent largest SV. In following summary of method discussed above is presented.

Algorithm 1: Frequency Limited H_∞ Norm Computation using Jacobi's Technique
Input: Given the system (1), frequency interval $\delta = [\omega_1, \omega_2]$ and tolerance $\in$:
Output: Frequency limited H_∞ Norm.
for δ
$|G(j\backslash omega\_\backslash delta)| = |C(j\backslash omega\_\backslash delta\, I - A)\hat{}\{-1\}B + D|$
repeat
for all pairs $p < q$
Compute J from (7) and (5)
Compute Z from (4)
Update $|G(j\omega_\delta)|$ $by\ solving$ (6)
Compute $\sigma_{\{max,\delta\}}(G(j\omega_\delta)) = max(G(j\omega_\delta)_{\{pp\}}, G(j\omega_\delta)_{\{qq\}}$
until(all $\frac{|G(j\omega_\delta)_{\{pq\}}|}{(G(j\omega_\delta)_{\{pp\}}G(j\omega_\delta)_{\{qq\}})^{\{\frac{1}{2}\}}} \leq \in$)
end for

2.2. Householder Transformation

Definition 4: Householder transformation for bidiagonalization is defined as [18]:

$$B(j\omega_\delta) = H_k G(j\omega_\delta) O_k \tag{8}$$

where $G(j\omega_\delta) = C(j\omega_\delta\, I - A)^{\{-1\}}B + \mathrm{D}$ is of order $p \times m, \omega_\delta \in \delta, k$ is the column or row of $G(j\omega_\delta)$ whose selected elements are to be zeroed out, H_k is premultiplier Householder that successively zero out elements below (k,k) entries of $G(j\omega_\delta)$ computed by:

$$H_k = I - 2w_{\{H_k\}}w_{\{H_k\}}^T \tag{9}$$

where

$$w_{\{H_k\}} = v_{\{H_k\}}/\left|\left|v_{\{H_k\}}\right|\right|_2$$
$$v_{\{H_k\}} = (0,\ldots,0, a_{\{kk\}}^k - \alpha_{\{H_k\}}, a_{\{k+1,k\}}^k, \ldots, a_{\{pk\}}^k)$$
$$\alpha_{\{H_k\}} = -sig(a_{\{kk\}}^k)||(0,\ldots,0, a_{\{kk\}}^k, a_{\{k+1,k\}}^k, \ldots, a_{\{pk\}}^k)||$$

$a_{\{kk\}}$ is the (k,k) element of $G(j\omega_\delta)$ and O_k is post multiplier that successively zero out transfer matrix elements past $(k, k+1)$ entries to yield bidiagonal form of system matrix.

$$O\_k = I - 2w\_\{O\_k\}\hat{}Tw\_\{O\_k\} \tag{10}$$

where

$$w_{\{O_k\}} = v_{\{O_k\}}/\left|\left|v_{\{O_k\}}\right|\right|_2$$
$$v_{\{O_k\}} = (0,\ldots,0, a_{\{kk+1\}}^k - \alpha_{\{O_k\}}, a_{\{k,k+2\}}^k, \ldots, a_{\{km\}}^k)$$
$$\alpha_{\{O_k\}} = -sig(a_{\{kk+1\}}^k)||(0,\ldots,0, a_{\{kk+1\}}^k, a_{\{k,k+2\}}^k, \ldots, a_{\{km\}}^k)||$$

Remark 3: Householder transformation maps one vector subspace into another by preserving the norms of initial and resulting vectors. Moreover $H_k = H_k^T, H_k^{\{-1\}} = H_k^T, H_k^2 = I$.

Remark 4: Householder bidiagonalization is a successive process i.e. $H_1\, G(j\omega_\delta) = S_1, S_1 O_1 = S_2, H_2 S_2 = S_3, \ldots, S_m O_m = B(j\omega_\delta)$ and

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$$G(j\omega\_\delta) = (H_1 \ldots H_p)B(j\omega_\delta)(O_m \ldots O_1).$$

After bidiagonal form $B(j\omega_\delta)$ is obtained, it's orthogonal matrices P and Q are computed such that $\Sigma = P^T B(j\omega_\delta)Q$ is diagonal and nonnegative. Largest diagonal entry qualify as infinity norm in given frequency interval. The columns of P and Q are right and left singular vectors respectively.

Algorithm 2: Frequency Limited H_∞ Norm Computation using Householder Technique
Input: Given the system (1), frequency interval $\delta = [\omega_1, \omega_2]$:
Output: Frequency limited H_∞ Norm.
for δ
 $G(j\omega_\delta) = C(j\omega_\delta I - A)^{\{-1\}}B + D$
 Set $S_{\{k-1\}} = S_0 = G(j\omega_\delta)$
 for $k = 1{:}m$
  Compute H_k from (9)
  Compute $S_{\{kH\}} = H_k S_{\{k-1\}}$
  Compute O_k from (10)
  Apply post multiplier $S_k = S_{\{kH\}} O_k$
 end for
 Set $B(j\omega_\delta) = S_k$
 Diagonalize to obtain $\Sigma = P^T B(j\omega_\delta)Q$
 Compute norm by $\sigma_{\{max,\delta\}} = max(\Sigma)$
end for

Remark 5: Techniques *A* and *B* can be used for computation of H_∞ norm for discrete time systems as well.

Remark 6: The proposed techniques *A* and *B* are applicable to arbitrarily constructed matrices. However as most applications of frequency limited H_∞ norm are related to systems having certain physical interpretation, therefore posteriori constructed system matrices are considered in present work.

3. Illustrative examples

Proposed techniques are applied to many continuous and discrete time systems out of which results for few examples are presented.

3.1. Continuous Time Systems

Example 1: Consider following $6^{\{th\}}$ order SISO system [19]:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -5.4545 & 4.5455 & 0 & -.0545 & .0455 & 0 \\ 10 & -21 & 11 & .1000 & -.2100 & .1100 \\ 0 & 5.5000 & -6.5000 & 0 & .0550 & -.0650 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & .0909 & .4 & -.5 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 2 & -2 & 3 & 0 & 0 & 0 \end{bmatrix}, D = 0$$

Example 2: Consider another $6^{\{th\}}$ order MIMO system [20]:

$$A = \begin{bmatrix} -20.02 & -0.124 & -0.203 & -0.254 & 0.203 & 0.3057 \\ 3.967 & -0.165 & 1.017 & 1.272 & -1.017 & -1.526 \\ -0.279 & -1.399 & -7.118 & -2.647 & 0.117 & 0.1766 \\ -0.349 & -1.749 & 0.9872 & -3.766 & 3.013 & 4.519 \\ 0.2798 & 1.399 & 0.2253 & 0.2816 & -5.225 & -3.338 \\ 0.4196 & 2.098 & 3.134 & 3.917 & -1.134 & -4.7 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1.67e-016 \\ 2.665e-015 & 8.352e-016 \\ 0.8296 & 2 \\ 1.037 & 1.665e-016 \\ -0.8296 & 2 \\ -1.244 & -2.22e-016 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.2378 & 1.189 & 0.6226 & 0.6533 & -0.122 & -0.183 \\ 0.3584 & 0.5419 & 0.5319 & 0.6648 & -0.031 & 0.7022 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3.2. Discrete Time Systems

Example 3: Consider following $4^{\{th\}}$ order discrete time SISO system [21]:

$$A_d = \begin{bmatrix} -1.1000 & 0.0100 & 0.2750 & 0.0600 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C_d = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, D_d = 0$$

Example 4: Consider following discrete time $6^{\{th\}}$ order system [21]:

$$A_d = \begin{bmatrix} -0.8750 & -0.75 & -0.5 & -0.3 & -0.25 & -0.1 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$C_d = \begin{bmatrix} 0.25 & 1.25 & 1.75 & 2 & 2.5 & 0.25 \end{bmatrix}, D_d = 0$$

SV plot for systems are shown in Fig. 1, Fig. 2, Fig. 3 and Fig. 2 respectively and computed infinity norms are given in Table 1 and Table 2 respectively, for various frequency intervals. The computed norms match with the maximum SV in given frequency interval that certify the correct development of the proposed techniques.

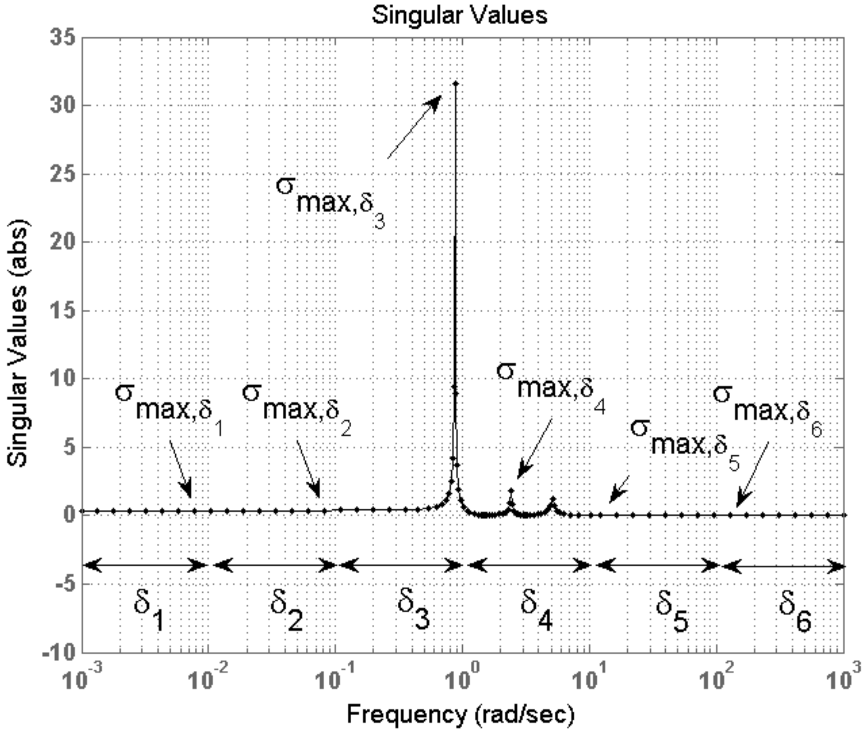

Figure 1. SV plot for example 1.

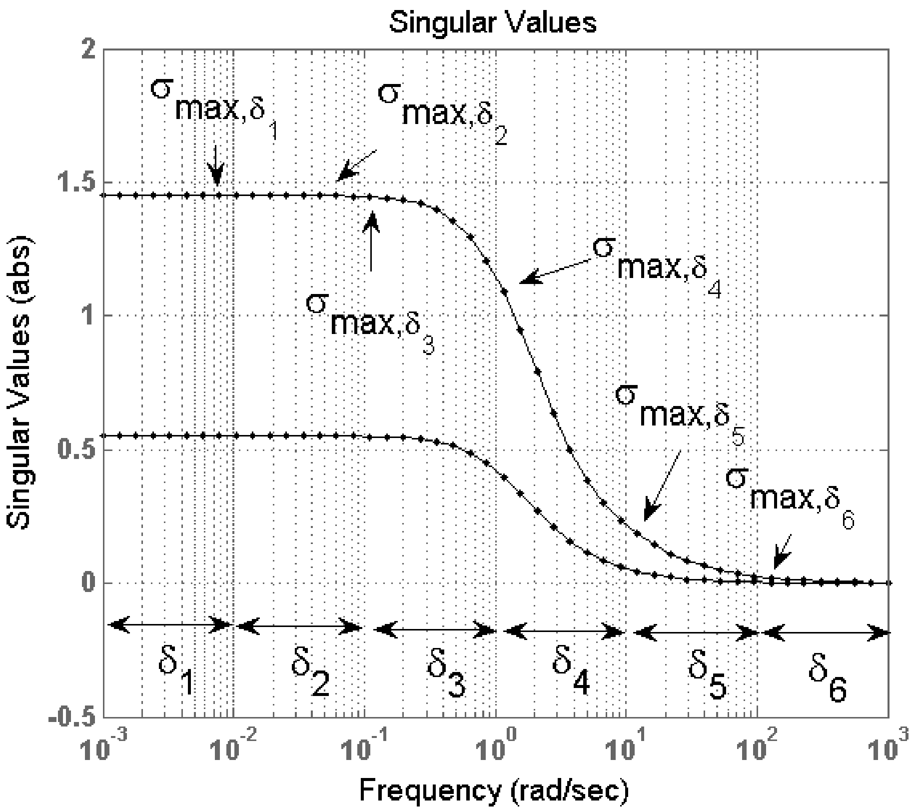

Figure 2. SV plot for example 2.

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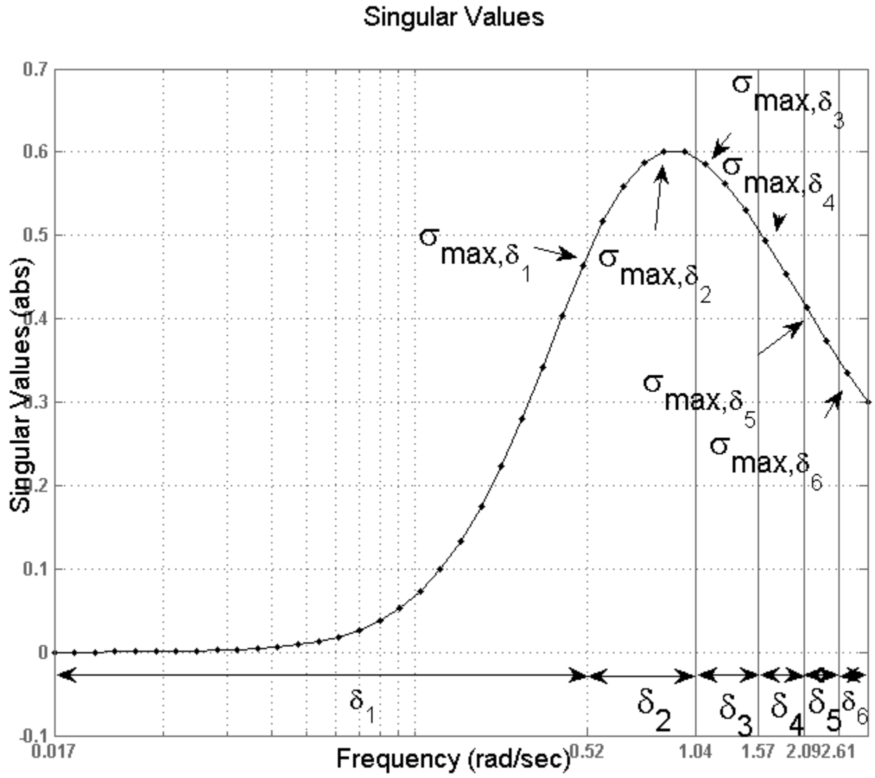

Figure 3. SV plot for example 3.

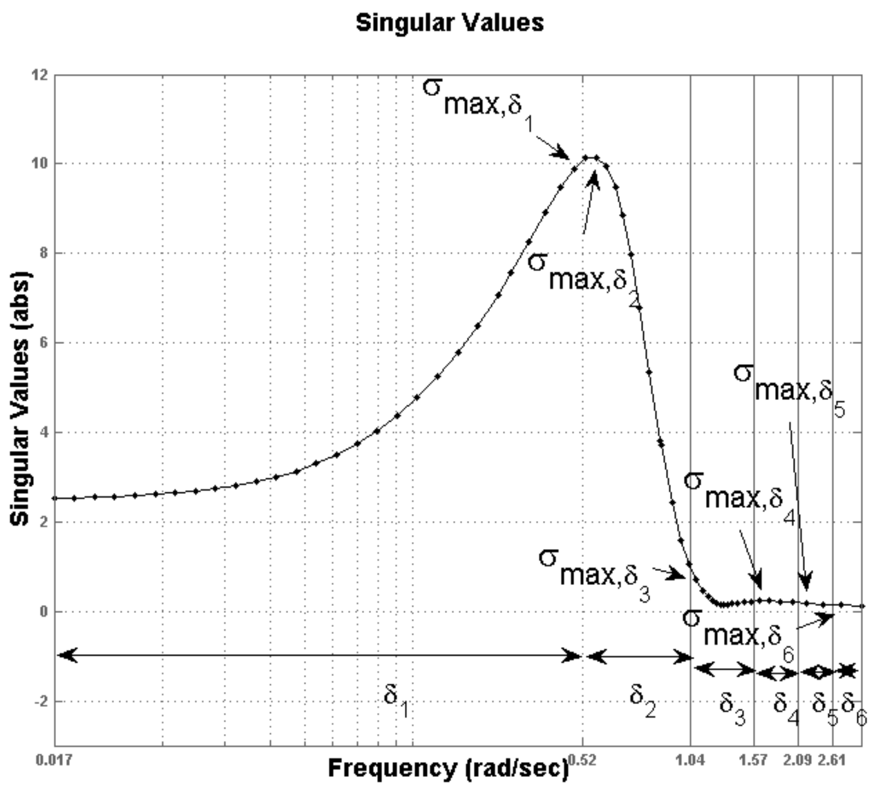

Figure 4. SV plot for example 4.

Table 1. Frequency limited infinity norms for continuous time systems.

Frequency Interval	Frequency Limited Norm	
	Example 1	Example 2
$\delta_1 = [10^{-3}, 10^{-2}]$	0.3810	1.4512
$\delta_2 = [10^{-2}, 10^{-1}]$	0.3847	1.4512
$\delta_3 = [10^{-1}, 10^{0}]$	31.5564	1.4465
$\delta_4 = [10^{0}, 10^{1}]$	1.8019	1.1542
$\delta_5 = [10^{1}, 10^{2}]$	0.0272	0.2166
$\delta_6 = [10^{2}, 10^{3}]$	2.1227e-04	0.0261
$Overall = [10^{-3}, 10^{3}]$	31.5564	1.4512

Table 2. Frequency limited infinity norms for discrete time systems.

Frequency Interval	Frequency Limited Norm	
	Example 3	Example 4
$\delta_1 = [\pi/180, \pi/6]$	0.4747	10.1704
$\delta_2 = [\pi/6, \pi/3]$	0.6020	10.1691
$\delta_3 = [\pi/3, \pi/2]$	0.5941	1.0026
$\delta_4 = [\pi/2, 2\pi/3]$	0.5060	0.2333
$\delta_5 = [2\pi/3, 5\pi/6]$	0.4177	0.2000
$\delta_6 = [5\pi/6, \pi]$	0.3502	0.1495
$Overall = [\pi/180, \pi]$	0.6020	10.1704

4. Concluding remarks

In order to emphasize system analysis, design and optimization in limited frequency interval, two techniques to compute frequency limited infinity norm are proposed. In both techniques namely Jacobi's and Householder, largest SV of the system is computed at each frequency point and maximum of these values is taken over limited interval. The computed values match with maximum SV in limited frequency interval that certify the successful development of the proposed schemes.

5. References

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