Asymptotic Estimation and Existence of A Coupled System Model Based on Symmetric Space of Riemannian Manifold

Shuxian Deng, Hongen Li*

Department of Basic Courses, Zhengzhou University of Industrial Technology, Zhengzhou 451150, China

*Corresponding author e-mail: 857931663@qq.com

Abstract. In this paper, we will discuss the fully nonlinear elliptic and parabolic equations related to classical Euclidean geometry and conformal geometry. Some algebraic and analytic properties of concave symmetric functions and Garding's theory of hyperbolic polynomials are collected in the appendix. According to the classification theory of Riemann symmetry space, we use transformation to convert the subalgebra of a very tight part to a very noncompact subalgebra. By calculating the projection, we calculate the section curvature of all irreducible Riemann symmetric spaces. Using the classification theory of the maximal subfamily of abstract roots and the control chart, we calculate the partial positive partial negative values of the curvature of all irreducible Riemann symmetric spaces.

1. Introduction

A new model describing immiscible, compressible two-phase flow is considered. Motivation for the following mathematical problem arises in the area of modeling multiphase flow in porous media. The main feature of this model is the introduction of a new global pressure and the full equivalence to the original equations. In the case of immiscible two-phase flows with more compressible fluids with exchange between the phases, i.e. a multicomponent model, existence of weak solutions to these equations under some assumptions on the compressibility of the fluids has been recently established. We use a small parameter $h > 0$ and construct approximate solutions with a time discretization. We show the weak controllability with the help of differential inequalities by estimating the relationship between energy inequalities and attenuating property of weak solutions.

2. Construction and Analysis of Mathematical Model

The theory of symmetric space has a great influence on the development of differential geometry. On the one hand, symmetric space has extremely rich geometric properties, providing us with a typical reference. Symmetric space has a special place in the development of modern differential geometry, so a thorough study of its properties helps us to understand geometry problems.

$$p_{tt} - \gamma \Delta p_t + \nabla P = \vartheta(x,t) \tag{1}$$

$$div \rho P = 0 \tag{2}$$

$$\vartheta(x,0)=\vartheta_0(x),\ \ P(x,0)=p_0(x) \qquad in\Omega \tag{3}$$

where p_0, p_1, ϑ are given functions, ΔP is a Laplacian with respect to the variable $x \in \Omega$, $p = p(t, x)$ is an unknown function, $\gamma > 0$ is a fixed positive number, $\vartheta \in L^2(\Omega)$, $p_0 \in W_p^2(\Omega)$ are given external forces, and satisfying the following conditions:

$$\| p_0\|_{W_p^2} \cdot \| w_0\|_{W_p^2} \leqslant 1 \tag{4}$$

Now let us define an operator, we consider

$$f \in L_2,\ u \in L_q\left(R^+, L_p\right) \ \ and \ \ \frac{2}{p}+\frac{n}{q} \le 1 \tag{5}$$

$$u = v - \overline{v} - u_0 \text{ and } w = \rho - \overline{\rho} - w_0 . \tag{6}$$

One can easily verify that (u, w) satisfies the following system:

$$d + a\,|\,x\,|^p \le f'(x,t) \le d(1+|x|^q), \forall s \in \Omega \tag{7}$$

And

$$u_{tt} - div\lambda\nabla u^2 - \Delta u_t = f(x,\ t) \tag{8}$$

For some positive $a, p \in [1/2, 2), C > 0,$ and $q > 0$, we have a weak solution, which fulfills additionally

$$\partial_{x_i} p - \lambda\Delta p - (p + p_0) = F(p,w) \quad in\ \Omega, \tag{9}$$

$$(w + w_0)divp + \mu(w+v) \cdot \nabla w = G(u,w) \qquad on\ \ \partial\Omega \tag{10}$$

We have several techniques to prove the existence of weak decay solutions with respect to the phase space $F_\varepsilon(u,w)$ and $G_\varepsilon(u,w)$; and have additional nice properties with energy inequality for almost all times or solutions with weak decay properties for $t \to \infty$, this has been studied recently by several people, e.g. YM. Qin, Ebihara, Xin Liu [1-6] etc.

$$n \cdot \mu D(p) \cdot \tau + f \cdot p\tau = 0 \quad on\ \ \partial\Omega \tag{11}$$

$$n \cdot p = 0; \quad p \cdot w = 0 \quad on\ \ \partial\Omega \tag{12}$$

As a model of nonlinear function equation, for $(u,w) \in w_p^2(\Omega) \times w_p^1(\Omega)$, equation (1)-(3)admits a global weak decay solution as large initial data, which was proved by Y.M. Qin, Xin Liu, X.G. Yang, Lan Huang etc [2-3]. where (6) -(8) are given functions and $F_\varepsilon\left(u,\ w\right)$ *and* $G_\varepsilon\left(u,\ w\right)$ F(u, w) and G(u, w) are regularizations to F(u, w) and G(u, w). It can be obtained by replacing the functions u_0 and w_0 by their regular approximations u_0^ε and w_0^ε , In two, three-dimensional case, they describe the viscoelastic solid of anti-plane shear action. While $B = b - 2\mu n \cdot P \cdot \tau - f\tau$ and $n \cdot u_0|_{\Gamma} = w - p$ *and* $w_0|_{\Gamma} = \delta_{in} - 1$.

Ulteriorly, X.G. Yang gave the proof of global controllability with smooth solution in the case of small initial data. Make use of combining L^p -theorem of Sobolev space and semigroup theorem of operators, In order to prove the following theorem it is enough to prove the existence of a solution (u, w) to the system (3)-(3) provided that $\|u_0\|_{W_p^2}, \|w_0\|_{W_p^2}$ and $\|P\|_{W_p^{1-1/p}(\Gamma)}$ are small enough.

$$d'(t) + k_1 \|w_1(t) - w_2(t)\|_p^{\upsilon} d(t) \le k_2 d(t)^{p+q-1} + \kappa d(t)^{\mu} \|w_1(t)\|_2^{\upsilon+1} \tag{13}$$

As we already mentioned, the presence of the term u·∇w in the continuity equation makes it impossible to show the compactness of a solution operator if we try to apply fixed point methods directly to the system (11).

Let us define an operator $P_\epsilon : P \subset W_p^2(\Omega) \times W_p^1(\Omega) \to W_p^2(\Omega) \times W_p^1(\Omega)$, where $(p_\varepsilon, w_\varepsilon)$ is a solution to (1)-(3),

If $\|w(t)\| \le k_0 (1+t)^{-\frac{\lambda}{2}}$, and $\|w(t)\|_p \le \tau (1+t)^{(\frac{\tau}{2} + \frac{n}{2}(\frac{1}{2} - \frac{1}{p}))}$. Here, Ω is a bounded domain in R^n with a smooth, $\partial\Omega$ is said to be C^2 class boundary, which satisfies the following uniform hyperbolic assumption.

Definition 1. By an elliptic regularization to the system we mean a system $(p_\epsilon, w_\varepsilon) = P_\varepsilon(p_\varepsilon, w_\epsilon)$.

For some constants $\rho_0, M > 0$, $\tau \in H^4[0, +\infty)$ satisfies:

$$x'(t) + \tau_0 (1+t)^{(\ell + \frac{d}{2})} x(t)^{(2+\frac{\gamma}{p})} \le \sigma y(t)^{\frac{\upsilon}{2}} + \tau_1 (1+t)^{\delta} y(t)^{\gamma} + a_0(w) \tag{14}$$

$$\tau(v^2) \ge \sigma_0 > 0 \text{ and } \tau(v^2) + 2\sigma_0(v) v^2 \ge \delta_0 > 0 \tag{15}$$

$$\tau(v^2) \le 2\sigma_0(v^2) v + \tau''(v^2) \le M < \infty \tag{16}$$

3. Main Results

we shall make a remark concerning the term $f(x, t) \in H^1([0, \omega], L^2(\Omega))$, that is rather unexpected in an energy estimate. Its presence is due to the functions $u(t)$ on the (13)-(16). However, this term does not cause any problems when we apply (15) to interpolation, since it is multiplied by a small constant.

Lemma 1. Assume that $f(x, t) \in H^1([0, \omega], L^2(\Omega))$ are small enough and f is large enough. Then for sufficiently smooth solutions to system (1)-(3) the following estimate is valid

$$C^2(p,w) \cap F(p,w) \cap G(p,w) \tag{17}$$

$$\left\{ \|u_t(t)\|^2 + \|\nabla u_{tt}(t)\|^2 \right\} + \int_0^{\omega} \|u_{tt}(s)\|^2 ds \le C\left(M_0^2 + M_1^2\right)_{T, G \in [0,\omega]} \tag{18}$$

$$\int_{\Omega} (\sigma P^2(u) + div^2 p) dt + \int_{\Omega} \left(p + \frac{1}{2} u\right) |u| d\tau = \int_{\Omega} (F + C(x,t)) dx \tag{19}$$

Where C, F, G is the dual space of P.

Lemma 2. Suppose that $f(x,\ t)\in H^1(I,\ L^2(\Omega))$ is a solution to (1)-(3). Then the following estimate is valid provided that the data $u_0,\ u_1\in H^2(\Omega)\bigcap H_0^1(\Omega)$ are small enough and σ is large enough $\|f(t)\|^2+\|f_t(t)\|^2=O(e^{-\alpha t})$, and $\|u_{tt}\|^2+\|\Delta u\|^2+\|\nabla u_t(t)\|^2+\|\Delta u_t\|^2+\int_t^\infty\|\nabla u_{tt}\|^2 ds\le Ce^{-\beta t}$ where the constant C depends on the data but does not depend on $\|u_0\|_{H^2}+\|u_1\|_{H^2}$ for some constant $\alpha>0$, and

$$\int_{\Omega}(\alpha_1(w)\cdot\triangle wp)dx\leqslant\int_{\Omega}pw\nabla p_1(t)dt+\int_{\Gamma}G(u,p)Fds \tag{20}$$

$$\int_{\Gamma}(\alpha_2(w))w^2div(w+w_0)dx\leqslant\epsilon\int_{\Omega}w^2\big(\partial_{x_i}p(w)+w+w_0\nabla G(w)\big)dx+\int_{\Gamma}uw\nabla F(u,p)dt \tag{21}$$

Under assumption H[1], the problem admits a unique solution $u(t)$ in the class and further, for some constant $\|F(t)\|^2\ +\|F_t(t)\|^2=o(e^{-\alpha t})$, as $M_i^2=\int_0^\omega\left\|\frac{\partial^i}{\partial t^i}f(s)\right\|^2 ds,\ i=\ 0,1,3$, the estimate. Holds (22) for some constant $\rho_0, M>0$, where $C>0$ is a constant depending on $\|u_0\|_{H^2}+\|u_1\|_{H^2}$.

Theorem. Assume that ω, $\omega_0(t)=e^{-t\Delta}$ are small enough and f is large enough. Then there exists a solution $(v,\rho)\in W$, $\|\omega_0(t)\|_2\ \le\lambda_0(1\ +t)^{-\tau/2}$, $\omega_0\in H^2(\Omega)\cap H_0^4(\Omega)$, $\omega_1\in H_0^4(\Omega)$, $f(x,\ t)\in H_0^2(\Theta,$ $L^2(\Omega))$ and $f'(x,\ t)\in W_0^2\ (\Theta,\ L^2(\Omega))$, then problem(1)-(3) hold a unique local weak solution $\omega(t)$ with the following estimate to the system (1)-(3):

$$\|u-\overline{u}\|_{W_p^2}+\|p-\overline{p}\|_{W_p^1}\leqslant E \tag{22}$$

$$y'(t)+\lambda_0(1+t)^{\frac{\kappa}{p}}y(t)^{(1+\frac{\varepsilon}{2}\kappa)}\le\lambda_1 y(t)^{\frac{\gamma}{2}\mu}+\lambda_2(1+t)^{-q}y(t)^{\sigma}\cdot\nabla x\omega(t) \tag{23}$$

$$\int_{\Omega}[G(w)]w\Delta wdx\leqslant\int_{\Omega}[G(w)]w|\Delta w|^2dx+\int_{\Omega}w\nabla wF(\overline{w})\,(\overline{u}+u_0)dx \tag{24}$$

where E is a constant depending on the data, i.e. on (E,ρ)in, b, the constants in the equation and the domain, that can be arbitrarily small provided that the data is small enough.

Moreover, if $\|u_0\|_{H^2}+\|u_1\|_{H^2}$ and (v,ρ) are two solutions to (1) satisfying the estimate (2) then

$$\left|\int_{\Omega}uw\nabla[p(\overline{w})]^p dx\right|\leqslant C\|\nabla F(w)\|_{L_p}\|v\|_{W_2^{m,p}}\leqslant E\left(\|P\|_{W_2^1}\|G\|_{L_q}^2\right) \tag{25}$$

4. Proof of Main Result

4.1. Proof of Theorem.

Taking the two-dimensional vorticity of (2.4)1 we get

$$E(t) \equiv \frac{1}{2}\left\{\left\|w_t(t)\right\|^2 + \int_\Omega \int_0^{|\nabla\omega|^2} \tau(\sigma)\, d\sigma dx\right\} \tag{26}$$

$$(w_t,\ \mu)|_0^t \le \int_0^t \left\{(w_t,\mu_t) - (\mathrm{div}(\tau\nabla F),\mu) - (\Delta G_t,\mu) - (F,\mu)\right\} dx \tag{27}$$

$$\int_0^t (f, w(t))\, dsdy \le \int_0^t \left\|\nabla\omega_t(t)\right\|^2 ds + \frac{1}{2}\left\{\left\|\nabla w_1(t)\right\|^2 + \int_\Omega \int_0^{|\nabla\omega_0|^2} \tau(\xi)\, d\xi dy\right\} \tag{28}$$

Let us define
$H := (w+P)\, divF_\varepsilon + G(w) W_\varepsilon$
where $(w_\varepsilon, u_\varepsilon)$ is a solution to (1)-(3) and $G(w)$ is defined. Then $q = \dfrac{2p}{p-2} < +\infty$, $p^* = \dfrac{p}{p-1}$ and w_ε satisfies the following equation

$$\|\nabla G\|_{L_p} = C\left[\|F(u,v)\| + \|G\|_{W_p^{1-1/p}(\Omega)} + \|u\|_{W_p^q} + \|F\|_{W_q^1}\right] + E\|W\| \tag{29}$$

$$\frac{\partial}{\partial t}\left(\left\|\partial_t(\omega)\right\| + \lambda(\theta(\nabla u),1) + \sigma\left\|\nabla\omega\right\|_{L^\infty}^2 + \frac{1}{2}(G(v),1) + (\varphi,\varepsilon)\right) + \eta\left\|\nabla\partial_t h\right\|_{H_0^2}^2 = 0 \tag{30}$$

In order to prove the prior estimates on H^1-norm of the velocity and L^2-norm of the density for the system (1)-(3), we estimate the following on $U = \left\{u \in H^2(\Omega, R^3) : u \cdot p|_{\partial\Omega} = 0\right\}$

$$\delta'(t) + \lambda_1 (1+t)^{\kappa\frac{d}{2}} y(t)^{\lambda + \frac{\kappa}{p}} \le \lambda_2 \delta(t)^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{2}\right)} + \varepsilon(1+t)^{-(p-r)} \delta(t)^{\frac{1}{q}} \tag{31}$$

Where ε is a small positive number which will be fixed, Then we arrive at

$$\frac{\partial}{\partial t}\left(\delta\frac{\rho}{2}\left\|\partial_t(\omega)\right\|^2 + \lambda(\partial_t(\omega),\omega)\right) = \sigma\left\|\partial_t\omega\right\|_{L^\infty}^2 + \theta\left\|(\nabla u)\right\|^2 \tag{32}$$

using (3) and the Korn inequality, we get

$$\tau(\varphi'(\nabla w), \nabla w) + \frac{1}{2}\upsilon(G(v), v) \le \sigma(g, w) \tag{33}$$

Using Hölder inequality, obviously we let λ, α be small enough. Now we have to deal with the term with δ, τ. Due to the boundary conditions, we get

$$E(\xi_u(t)) = \frac{p-2}{p}\left\|\partial_t\omega\right\|_{L^2}^p - \frac{pn+4}{p+4-n}\left\|\nabla\omega\right\|_{L^2}^2 + \lambda(\partial_t\omega,\omega) + C\left\|\nabla u\right\|_{L_2}^2 \tag{34}$$

Thus, the term will be negative provided that ε, λ be small enough.

$$\int g\left(|\upsilon|^2+|\omega_0|^4\right)|u|^{\frac{pn}{n-2}}\,dx \le \sigma\left(\|\upsilon\|_{L^2}^{\frac{p}{2}}+\|\omega_0\|_p^4\right)\ \|\omega\|_{H_0^2}^{\frac{p}{2}-1} \tag{35}$$

$$\|\nabla_x u\|_{L^p}^{n+1}+\|\upsilon\|_{L^q}^{\frac{n+1}{p}}+\|\partial_t \upsilon\|_{L^2}^{n-2}+\left(1+\ \|g\|_{L^2}^{n}\right)\le C\lambda(u)^{2+\frac{n}{q}}\left(\beta(u)^{\frac{p}{n}}+\|\upsilon_0\|_{L^p}^{p+1}\right)\|\upsilon\|_{L^p}^{n+1-p} \tag{36}$$

Where the constants $C_\varepsilon,\lambda_\varepsilon$ depend only on ε. The last term is the most inconvenient and it must be estimated by W_p^m-norm. Fortunately, it is multiplied by a small constant that turns out in the above lamma1 and lemma2. We have

$$E(\xi_u(t))=(\varphi(\nabla w),u)+\lambda(\partial_t v,w)+\frac{p-2}{p}\|\partial_t u\|_{L^2}^p-\frac{pn+4}{p+4-n}\|\nabla w\|_{L^2}^2 \tag{37}$$

$$C\|F(w)\|_{W_p^2}\|G(w)\|^2{}_{W_p^2}\leqslant E\|(u,v)\|^2{}_{W_p^2} \tag{38}$$

Now we integrate first and second component by parts. In the second component we use the fact that the integration interval does not depend on λ and τ. We get

$$\int_\Omega F\leqslant\left(\|G\|_{L^2}+C\|u\|_{H_2}\right)\|F\|_{L_2},\quad \int_\Omega G\leqslant\left(\|F\|_{L^2}+C\|u\|_{H_2}\right)\|G\|_{L_2} \tag{39}$$

The terms of above integrals of F and G can be estimated in a direct way:

$$0\leqslant E(\xi_u(t))\leqslant\int_\Omega F,\quad \int_\Omega G\leqslant E\|(u,v)\|_{L_2} \tag{40}$$

Combining this estimate, applying again the elliptic theory, substituting the Helmholtz decomposition, the proof is thus completed.

5. Conclusion

To calculate the upper bound or lower bound of the curvature of an irreducible symmetric space, its curvature is non-negative for a compact symmetric space, and its curvature is not positive for a non-compact symmetric space. We consider the non-compact irreducible symmetric space, the maximal non-compact subalgebra is the maximal exchange subspace, and the associated root rank is the rank of symmetric space. Therefore, we realized the asymptotic estimation and existence.

References

[1] O. Glass, P.B. Mucha, Inviscid limit for the 2-D stationary Euler system with arbitrary force in simply connected domains, Appl. Math. 35 (1) (2015) 49-67.
[2] M. Pokorny, Cauchy problem for the non-Newtonian incompressible fluids, Appl. Math. 41 (2016) 169–201.
[3] F. Bloom, W. Hao, Regularization of a non-Newtonian system in an unbounded channel: Existence and uniqueness of solutions, Nonlinear Anal. 44(2011) 281–309.
[4] P.B. Mucha, M. Pokorny, On a new approach to the issue of existence and regularity for the steady compressible Navier–Stokes equations, Nonlinearity 19 (2016) 1747–1768.
[5] G H. Kozono, T. Ogawa, Two-dimensional Navier–Stokes flow in unbounded domains, Math. Ann. 297 (2013) 1–31.

[6] H.O. Bae, B.J. Jin, Asymptotic behavior for the Navier–Stokes solutions in 2D exterior domains, J. Funct. Anal. 240 (2016) 508–529.
[7] D. Hoff, The zero-Mach limit of compressible flows, Comm. Math. Phys. 192 (1998) 543–554.
[8] R. Dascaliuc, C. Foias, M. Jolly, Some specific mathematical constraints on 2D turbulence, Physica D 237 (23) (2015) 3020–3029.