

Realistic Features in Analysing the Effect of the Seismic Motion upon Localized Structures Considering Base Isolation Influence on Their Dynamic Behaviour

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Abstract. The effects of the earthquakes on buildings and the concept of seismic base isolation are investigated by using the model of the vibrating bar embedded at one end. The normal modes and the eigenfrequencies of the bar are highlighted and the amplification of the response due to the excitation of the normal modes (eigenmodes) is computed. The effect is much enhanced at resonance, for oscillating shocks which contain eigenfrequencies of the bar. Also, the response of two linearly joined bars with one end embedded is calculated. It is shown that for very different elastic properties the eigenfrequencies are due mainly to the "softer" bar. The effect of the base isolation in seismic structural engineering is assessed by formulating the model of coupled harmonic oscillators, as a simplified model for the structure building-foundation viewed as two coupled vibrating bars. The coupling decreases the lower eigenfrequencies of the structure and increases the higher ones. Similar amplification factors are derived for coupled oscillators at resonance with an oscillating shock.

1. Introduction

The concept of seismic base isolation aims at formulating solutions for protecting buildings against earthquakes, by designing special couplings between buildings and their foundations. As a key element in earthquake engineering it aims at designing means of achieving to some extent a building-foundation decoupling, such that the response of the building to vibrations be not (too) damaging. Usually, the dynamics of the structure building-foundation is approached by means of the model of two coupled oscillators. It examines here the formulation of this model starting with coupled elastic bars, [1-7]. To this end, it is useful to assess first the response to the ground excitation of a vibrating bar with one end embedded in the ground. At resonance, the response of the bar exhibits amplification factors which may attain large values. These amplification factors arise as a consequence of the excitation of the normal modes in the bar. Two coupled bars are also studied, excited at their lower end; for bars with very different elastic properties it is shown that the eigenfrequencies of this system are given mainly by the "softer" bar. Such an information may throw light upon the elasticity of composite structures, like, for instance, those including voids. Making use of the information gained from the study of the vibrating bars, is formulated the model of coupled harmonic oscillators and investigate its response to an oscillating shock. It is shown that the lower frequency of the system is lowered by the coupling, while the higher frequency is



raised. At resonance the coupled oscillators exhibit amplification factors, similar with the vibrating bar.

2. Embedded bar

The most convenient model for investigating the response of a building to ground vibrations is the bar embedded at one end. Let us assume that a vertical elastic bar with uniform cross-section is fixed in the ground at one end, having a length l above the ground surface; the bar end above the ground is free. Under the action of the seismic waves the buried end of the bar is set in motion. It is assumed the cross-sectional dimensions of the bar being much smaller than the bar length, so it can be considered only to the z -dependence of the displacement, where z is the vertical coordinate (along the bar). At the same time, it is considered the length of the bar and the excitation sufficiently small, such that the bar does not enter the regime of flexural elasticity (bending). The strain tensor reduces to $u_{xz} = u_{zx} = \frac{1}{2} \frac{\partial u_x}{\partial z}$, $u_{yz} = u_{zy} = \frac{1}{2} \frac{\partial u_y}{\partial z}$, $u_{zz} = \frac{1}{2} \frac{\partial u_z}{\partial z}$; the stress tensor is $\sigma_{xx} = \sigma_{yy} = \mu \frac{\partial u_z}{\partial z}$, $\sigma_{zz} = (\lambda + 2\mu) \frac{\partial u_z}{\partial z}$, $\sigma_{xz} = \sigma_{zx} = \mu \frac{\partial u_x}{\partial z}$ and $\sigma_{yz} = \sigma_{zy} = \mu \frac{\partial u_y}{\partial z}$, where λ and μ are the Lamé coefficients. It follows an elastic force density $f_x = \frac{\partial \sigma_{xz}}{\partial z} = \mu \frac{\partial^2 u_x}{\partial z^2}$, $f_y = \frac{\partial \sigma_{yz}}{\partial z} = \mu \frac{\partial^2 u_y}{\partial z^2}$, $f_z = \frac{\partial \sigma_{zz}}{\partial z} = (\lambda + 2\mu) \frac{\partial^2 u_z}{\partial z^2}$. The equations of the elastic motion are [8]

$$\begin{aligned} \rho \ddot{u}_x - \mu \frac{\partial^2 u_x}{\partial z^2} &= 0, \\ \rho \ddot{u}_y - \mu \frac{\partial^2 u_y}{\partial z^2} &= 0, \\ \rho \ddot{u}_z - (\lambda + 2\mu) \frac{\partial^2 u_z}{\partial z^2} &= 0 \end{aligned} \quad (1)$$

where ρ is the density of the bar. It may be limited to only one equation of motion, which writes in the generic form

$$\ddot{u} - c^2 \frac{\partial^2 u}{\partial z^2} = 0 \quad (2)$$

where u and c stand for $u_{x,y,z}$ and, respectively, $c_l = \sqrt{(\lambda + 2\mu)/\rho}$, $c_t = \sqrt{\mu/\rho}$; $c_{l,t}$ are the velocities of the longitudinal and, respectively, transverse waves in the bar.

3. Shock-type excitation

The equation (2) is solved for a free upper end of the bar, while the lower end has the prescribed motion $u_0(t)$ of the ground; therefore, are imposed the boundary conditions

$$\left. \frac{\partial u}{\partial z} \right|_{z=l} = 0, \quad u \Big|_{z=0} = u_0(t); \quad (3)$$

the motion is limited to $t > 0$ and $0 < z < l$. Using time Fourier transform, equation (2) and the boundary conditions (3) read

$$u'' + \kappa^2 u = 0, \quad u'_l = 0, \quad u_0 = u_0(\omega) \quad (4)$$

where $\kappa^2 = \omega^2/c^2$ and the prime denotes the derivation with respect to z . The solution for the limited interval $0 < z < l$ can also be obtained by extending the equation to the whole space and limiting ourselves to the restriction of the solution to the interval $0 < z < l$; as it is well

known, this is achieved by multiplying the equation by $\theta(z)\theta(l-z)$ and absorbing the step functions θ in the derivatives (the method of generalized functions). Unfortunately, it should be use in this case the Green function which implies wave propagation in both directions, in order to satisfy the boundary conditions at infinity; this complicates the technical procedure (in contrast with the half-line, where one-direction Green function is needed). The natural "initial" condition which requires the vanishing of the solution for past times ($t < 0$) is treated most conveniently by integrating over frequency ω in the lower half-plane (the causality condition). The solution of the equation (4) has the form

$$u = A \cos \kappa z + B \sin \kappa z \quad (5)$$

where the constants A and B are determined by the boundary conditions; it is obtained

$$A = u_0, \quad B = u_0 \tan \kappa l \quad (6)$$

and

$$u(z, \omega) = u_0(\omega)(\cos \kappa z + \tan \kappa l \cdot \sin \kappa z) \quad (7)$$

The reverse Fourier transform gives

$$u(z, t) = \frac{1}{2\pi} \int d\omega u_0(\omega) e^{-i\omega t} \cos \kappa z + \frac{1}{2\pi} \int d\omega u_0(\omega) e^{-i\omega t} \tan \kappa l \cdot \sin \kappa z, \quad (8)$$

or

$$u(z, t) = \frac{1}{2} [u_0(t - z/c) + u_0(t + z/c)] + \frac{1}{2\pi} \int d\omega u_0(\omega) e^{-i\omega t} \tan \kappa l \cdot \sin \kappa z \quad (9)$$

where is taken the real part. It can be seen that half of the displacement $u_0(t)$ applied to the grounded end propagates along the bar with velocity c , while the other half propagates in the opposite direction (it "comes from the future"). It is considered seismic excitations which have a general aspect of shocks, *i.e.* they are concentrated in at the initial moment of time. This is valid for both the primary P and S waves, as well as for the main shock produced by the so-called surface waves. Consequently, it is assumed first a shock-like ground motion $u_0(t) = u_0 T \delta(t)$, $u_0(\omega) = u_0 T$, where T is a measure for the duration of the shock. The second integral in equation (9) implies the contribution of the poles arising from the zeroes of the denominator of $\tan \kappa l$: $\cos \kappa l = \cos \omega_n l / c = 0$, $\omega_n = (2n + 1)\pi c / 2l$, where n is any integer. Results

$$u(z, t) = \frac{1}{2} u_0 T [\delta(t - z/c) + \delta(t + z/c)] + u_0 \frac{cT}{l} \sum_n \sin \omega_n t \cdot \sin \omega_n z / c \quad (10)$$

It can be see that half of the δ -pulse applied at the fixed end at the initial moment propagates along the bar up to $z = l$, while the other half, "coming from the future", brings no contribution to the motion of the bar ($0 < z < l$), except for $z = 0$; in addition, vibrations given by the normal modes with the eigenfrequencies ω_n are excited in the bar. The summation over n in equation (10) gives a pulse going forth and back along the bar. Any approximation to the series in equation (10) (*e.g.* a truncated series) gives normal modes extending over the length of the bar.

The amplitude of the pulse is of the order u_0 , while the amplitude of the normal modes is of the order $u_0 c T / l$; is introduced the parameter

$$g = \frac{cT}{l} \quad (11)$$

and denote by $u_n(t,z)$ the contribution to the displacement of the n -th normal mode, *i.e.*

$$u_n = gu_0 \sin \omega_n t \cdot \sin \omega_n z / c \quad (12)$$

the corresponding velocity and acceleration are given by

$$\dot{u}_n = gu_0 \omega_n \cos \omega_n t \cdot \sin \omega_n z / c = g \frac{u_0}{T} (\omega_n T) \cos \omega_n t \cdot \sin \omega_n z / c \quad (13)$$

and, respectively,

$$\ddot{u}_n = -gu_0 \omega_n^2 \sin \omega_n t \cdot \frac{\sin \omega_n z}{c} = -g \frac{u_0}{T^2} (\omega_n T)^2 \sin \omega_n t \cdot \sin \omega_n z / c \quad (14)$$

A δ -shock of the form $u(t) = u_0 T \delta(t)$ includes an overlapping of oscillations with equal weights for all frequencies. It can be seen from the above equations that the response of the bar is affected by the factor g and powers of $\omega_n T$; beside the ground displacement u_0 , the quantities u_0/T and u_0/T^2 may be viewed as ground velocity and, respectively, acceleration. Typical values of the velocity of the elastic waves in the bar are $c \approx 3 \times 10 \text{ m/s}$; for a short duration $T = 0.1 \text{ s}$ is obtained $g = 10$ for a length $l = 30 \text{ m}$. It can be observed that the displacement, velocity and acceleration amplitudes in the bar could be enhanced in comparison with their ground counterparts. This is why it can be called the parameter g the amplification factor.

However, a pulse with a finite duration T excites mainly frequencies ω_n up to $\approx \pi/T$; in the above formulae, a weight factor $f(\omega_n)$ should be inserted, which decreases appreciably for frequencies $\omega_n > \pi/T$; therefore, the amplification parameter is subject to the condition

$$\omega_n T = \frac{(2n+1)\pi}{2} g \leq \pi \quad (15)$$

which implies values for g of the order as high as unity, corresponding to the fundamental frequency $\omega_0 = \pi c/2l$ ($n = 0$). In addition, it is well known that the seismic spectrum includes a range of frequencies extending up to $\approx 10 \text{ s}$, which is far below a fundamental frequency of the order $c/l \approx 100 \text{ s}$ for $c \approx 3 \times 10 \text{ m/s}$ and $l = 30 \text{ m}$. Therefore, it is unlikely that a short pulse can excite normal modes which might lead to appreciable amplification factors in reasonable conditions. However, the situation is different if the pulse includes resonance frequencies.

As a technical point, it should be noted that if the pulse is applied at some point on the bar, different from the bar ends, then it can be considered in fact two bars; the solution has four constants of the type A and B in equation (5) and the boundary conditions are the continuity of the displacement at the point of application of the excitation, the equality of the displacement with the excitation at that point and the conditions at the two ends; the resulting four equations determine the four constants of the solution.

4. Coupled oscillators

Here are examined the necessary conditions for two coupled vibrating bodies be approximated by two coupled dimensionless (point) harmonic oscillators. [9] The motion of the n -th normal mode is described by the equation

$$\rho \ddot{u}_n + \mu \kappa_n^2 u_n = \rho \ddot{u}_n + \rho \omega_n^2 u_n = 0 \quad (16)$$

and it can be seen that the z -dependence becomes irrelevant, and it could be view u_n as a global representation of the displacement of a point oscillator; this equation is written for the shear modes, but it has the same form, with μ replaced by $\lambda + 2\mu$, for the longitudinal modes. This is the equation of the harmonic oscillators with a set of eigenfrequencies. For frequencies near a certain eigenfrequency ω_1 (e.g., the fundamental frequency) there is the possibility of limitation to only one harmonic-oscillator equation, written as

$$\rho \ddot{u}_1 + \omega_1^2 u_1 = 0 \quad (17)$$

where is introduced the label 1 because a similar equation is written for another oscillator, denoted by 2, coupled to the former.

At the joining point $z=0$ bar 1 acts with a force density (per unit area) $\mu_1 u_1' \big|_{z=0}$ on bar 2, while bar 2 acts with a force $\mu_2 u_2' \big|_{z=0}$ on the former (for shear displacement). The forces which act upon the bars viewed as oscillators are $\mu_{1,2} u_{1,2}' \big|_{z=0} S$, where S is the area of the joining surface. The derivatives of the displacement can be represented as $u_{1,2}' \big|_{z=0} \cong u_{1,2}/d_{1,2}$, where $d_{1,2}$ are some fictitious distances, introduced for controlling the dimensionality of the equations. It follows that the interaction forces are of the order $\mu_{1,2} u_{1,2} S/d_{1,2}$. In order to conserve energy one must have $\mu_1 S/d_1 = \mu_2 S/d_2$; indeed, $\mu_1 S d_2 = \mu_2 S d_1$ is the interaction energy transferred between the two oscillators. This is a necessary condition for the two bars be approximated by oscillators. Therefore, is introduced the elastic interaction constant

$$K = \mu_1 S/d_1 = \mu_2 S/d_2 \quad (18)$$

and write the equations of motion for the two oscillators

$$m_{1,2} \ddot{u}_{1,2} + m_{1,2} \omega_1^2 u_{1,2} + K u_{2,1} = 0 \quad (19)$$

with the masses $m_{1,2}$ for each oscillator. It is worth comparing the interaction constants K with the oscillator constants

$$m_{1,2} \omega_{1,2}^2 \cong m_{1,2} \frac{c_{1,2}^2}{l_{1,2}^2} \alpha_{1,2n}^2 = (\mu_{1,2} S/l_{1,2}) \alpha_{1,2n}^2 \quad (20)$$

where $\alpha_{1,2n}$ are numerical factor from the eigenfrequencies $\omega_{1,2n} = (c_{1,2}/l_{1,2}) \alpha_{1,2n}$; these factors increase with increasing n . For lower frequencies and $l_{1,2}$ of the same order of magnitude as $d_{1,2}$, all the oscillation constants $m_{1,2} \omega_{1,2}^2$ and K are of the same order of magnitude. This implies a severe restriction upon the coupled-oscillators approximation, since, rigorously speaking, these energies are not equal; it originates in the circumstance that the model of coupled oscillators requires the eigenfrequencies and the coupling constant derive from different forces, while for elastic bars these quantities have the same common origin - the elastic force. However, if is gave up the assumption of a sharp joining surface and consider that the two bars are welded, then there exists a smooth joining and the difference between the two interaction forces and the transferred interaction energies is taken over by the welding; during motion there is a mechanical work dissipated in the welding. In these conditions it can be assumed $m_1 \omega_1^2 \neq m_2 \omega_2^2 \neq K$.

The potential energy associated with these two oscillators reads

$$V = \frac{1}{2}m_1\omega_1^2u_1^2 + \frac{1}{2}m_2\omega_2^2u_2^2 + Ku_1u_2 \quad (21)$$

It must have a minimum for $u_{1,2}=0$; this condition implies

$$K < m_1m_2\omega_1^2\omega_2^2 \quad (22)$$

With the notations introduced above this inequality reads

$$l_1l_2 < d_1d_2\alpha_{1n}^2\alpha_{2n}^2 \quad (23)$$

which can be satisfied, especially for higher eigenfrequencies.

It is convenient to introduce a parameter $0 < \gamma < 1$ through

$$K^2 = m_1m_2\omega_1^2\omega_2^2(1 - \gamma) \quad (24)$$

for $\gamma=1$ there is no coupling, for $\gamma=0$ the coupling is maximal. The parameter γ is a dimensionless coupling constant; the values of γ close to zero are of interest (maximal coupling). Also, it is introduced the notations $k_{1,2} = K/m_{1,2}$, such that the system of equations (19) can be written as

$$\ddot{u}_{1,2} + \omega_1^2u_{1,2} + k_{1,2}u_{2,1} = 0 \quad (25)$$

The eigenfrequencies of this system of equations are the roots $\Omega_{1,2}$ of the equation

$$\Delta = \Omega^2 - (\omega_1^2 + \omega_2^2)\Omega^2 + \omega_1^2\omega_2^2\gamma = 0; \quad (26)$$

and are obtained

$$\Omega_1^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 + \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2\gamma}\right] \cong \omega_1^2 + \omega_2^2 \quad (27)$$

and

$$\Omega_2^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 - \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2\gamma}\right] \cong \frac{\omega_1^2\omega_2^2}{\omega_1^2 + \omega_2^2}\gamma \quad (28)$$

One can see that Ω_1^2 increases from ω_1^2 to $\omega_1^2 + \omega_2^2$, while Ω_2^2 decreases from ω_2^2 ($\omega_2^2 < \omega_1^2$) down to zero for K^2/m_1m_2 going from zero to its maximum value $\omega_1^2\omega_2^2$ (for γ going from 1 to 0); the coupling lowers the low eigenfrequency and raises the high eigenfrequency.

For a realistic use of the coupled-oscillator model are considered the two oscillators as corresponding to a building (oscillator 2) and its foundation (oscillator 1). For a stiff foundation, such that $\omega_1 > \omega_2$ the eigenfrequencies of the building are reduced to an appreciable extent (down to zero), while the eigenfrequencies of the foundation are increased by the coupling. For a soft foundation ($\omega_1 < \omega_2$) the situation is reversed, the eigenfrequencies of the building are raised by the coupling and those of the foundation are reduced.

The solutions of the homogenous system of equations (25) are the real part of

$$u_{1,2}^0 = A_{1,2}e^{i\Omega_1 t} + B_{1,2}e^{i\Omega_2 t} \quad (29)$$

with complex constants $A_{1,2}$, $B_{1,2}$. These constants satisfy the system of equations (25) for $\Omega = \Omega_{1,2}$, respectively:

$$(\omega_1^2 - \Omega_1^2)A_1 + k_1A_2 = 0, \quad (\omega_2^2 - \Omega_2^2)B_1 + k_2B_2 = 0 \quad (30)$$

therefore

$$A_2 = \frac{\Omega_1^2 - \omega_1^2}{k_1} A_1 \cong \frac{\omega_2^2}{k_1} A_1, \quad B_1 = \frac{\Omega_2^2 - \omega_2^2}{k_2} B_2 \cong -\frac{\omega_2^2}{k_2} B_2; \quad (31)$$

the solution is the real part of

$$u_1^0 \cong A_1 e^{i\Omega_1 t} - \frac{\omega_2^2}{k_2} B_2 e^{i\Omega_2 t}, \quad u_2^0 \cong \frac{\omega_2^2}{k_1} A_1 e^{i\Omega_1 t} + B_2 e^{i\Omega_2 t} \quad (32)$$

Let us assume now that the foundation (oscillator 1) is subjected to a force $(t)f_0 e^{-\alpha t} \cos \omega_0 t$, $\alpha \ll \omega_0$, arising from the ground motion; the equations of motion of the two oscillators

$$\ddot{u}_1 + \omega_1^2 u_1 + k_1 u_2 = f \theta(t) e^{-\alpha t} \cos \omega_0 t, \quad \ddot{u}_2 + \omega_2^2 u_2 + k_2 u_1 = 0 \quad (33)$$

where $f = f_0/m_1$; a particular solution is the real part of

$$u_{1,2} = a_{1,2} e^{i\omega_0 t - \alpha t} \quad (34)$$

The constants $a_{1,2}$ are given by

$$a_1 = f \frac{\omega_2^2 - \tilde{\omega}_0^2}{\tilde{\Delta}}, \quad a_2 = -f \frac{k_2}{\tilde{\Delta}}, \quad (35)$$

where $\tilde{\Delta} = (\tilde{\omega}_0^2 - \Omega_1^2)(\tilde{\omega}_0^2 - \Omega_2^2)$ and $\tilde{\omega}_0 = \omega_0 + i\alpha$; adding the solution $u_{1,2}^0$ of the homogenous system of equations (equations (36)) the full solution is written

$$u_1 = A_1 e^{i\Omega_1 t} - \frac{\omega_2^2}{k_2} B_2 e^{i\Omega_2 t} + f \frac{\omega_2^2 - \tilde{\omega}_0^2}{\tilde{\Delta}} e^{i\tilde{\omega}_0 t}, \quad (36)$$

$$u_2 = \frac{\omega_2^2}{k_1} A_1 e^{i\Omega_1 t} + B_2 e^{i\Omega_2 t} - f \frac{k_2}{\tilde{\Delta}} e^{i\tilde{\omega}_0 t}$$

The (complex) constants A_1, B_2 are determined from the initial conditions $u_{1,2}(t=0) = 0$, $\dot{u}_{1,2}(t=0) = 0$.

By focusing on the resonance of the building, where $\omega_0 = \Omega_2$ ($\alpha \ll \Omega_2$) and $\tilde{\Delta} \cong \alpha \Omega_1^2 (\alpha - 2i\Omega_2)$, the initial conditions give $A_1 \cong 0$ and

$$B_2 \cong \frac{fk_2}{4\Omega_1^2 \Omega_2^2} \left(1 + i \frac{\Omega_2}{\alpha}\right); \quad (37)$$

the displacements are obtained

$$u_1 = -\frac{f\omega_2^2}{4\Omega_1^2 \Omega_2^2} \left(\cos \Omega_2 t - \frac{\Omega_2}{\alpha} \sin \Omega_2 t\right) (1 - e^{-\alpha t}) + O(\alpha), \quad (38)$$

$$u_2 = \frac{fk_2}{4\Omega_1^2 \Omega_2^2} \left(\cos \Omega_2 t - \frac{\Omega_2}{\alpha} \sin \Omega_2 t\right) (1 - e^{-\alpha t}) + O(\alpha)$$

The original damped excitation is lost in time and for long time both the building and the foundation oscillate with the resonance frequency Ω_2 of the building; the amplitudes of the oscillations are enhanced by the attenuation factor $1/\alpha$, as expected; the oscillation amplitude of the foundation is controlled by the exciting force, while the amplitude of the building is controlled by the coupling constant. It must be noted that was considered above oscillations without a damping factor; a damping factor affects the contribution of the normal modes and adds to the attenuation factor of the excitation.

Inserting K and $\Omega_{1,2}$ from equations (24), (27) and (28) the leading contributions to the oscillation amplitude of the building are obtained

$$u_{20} = \frac{fk_2}{4\alpha\Omega_1^2\Omega_2^2} = \frac{f_0}{4\alpha} \sqrt{\frac{1-\gamma}{\gamma}} \frac{1}{\sqrt{m_1m_2(\omega_1^2+\omega_2^2)}}; \quad (39)$$

comparing it with the amplitude $u_{20}^{is} = f_0/m_2\omega_2^2$ of an isolated building (without foundation) subject to the same force and oscillating with the same frequency $\Omega_2 \ll \omega_2$, is obtained

$$u_{20} = u_{20}^{is} \frac{\omega_2}{4\alpha} \cdot \sqrt{\frac{1-\gamma}{\gamma}} \cdot \frac{\omega_2}{\sqrt{\omega_1^2+\omega_2^2}}; \quad (40)$$

it can be seen the enhancement factors ω_2/α arising from resonance and $\sqrt{\frac{1-\gamma}{\gamma}}$ arising from the coupling.

It is worth connecting the force f_0 to the amplitude u_0 of the ground displacement, in order to compare the displacement of the building with the ground displacement. According to the discussion above, assuming a good coupling between ground and foundation and a relatively homogeneous structure ground + foundation + building it can be used $f_0 = Ku_0$ (since, for a shear coupling, f_0 is of the order $\mu_g u'_0$, where μ_g is the rigidity modulus of the soil); the oscillation amplitude of the building:

$$u_{20} \cong \frac{\omega_1\omega_2}{4\alpha\sqrt{\omega_1^2+\omega_2^2}} \cdot \frac{1-\gamma}{\gamma} u_0, \quad (41)$$

where appears again the occurrence of an amplification factor $\cong \omega_c/\alpha$, where

$$\omega_c \cong \omega_1\omega_2(1-\gamma) \frac{\gamma}{4\alpha} \sqrt{\omega_1^2+\omega_2^2} \text{ is a characteristic frequency of the structure.}$$

Finally is obtained the solution of a coupled oscillators subjected to a damped force (shock) without oscillations, *i.e.* for $\omega_0 = 0$:

$$u_1 = \frac{f\omega_2^2}{4\Omega_1^2\Omega_2^2} \left(e^{-\alpha t} - \cos\Omega_2 t - \frac{\alpha}{\Omega_2} \sin\Omega_2 t \right) + O(\alpha^2), \quad (42)$$

$$u_2 = -\frac{fk_2}{\Omega_1^2\Omega_2^2} \left(e^{-\alpha t} - \cos\Omega_2 t - \frac{\alpha}{\Omega_2} \sin\Omega_2 t \right) + O(\alpha^2);$$

it should be noticed that the shock excites the oscillations with the lowest frequency (Ω_2), similar with an oscillating shock), and there is no enhancement of the oscillations, as expected.

5. Results and Discussion

The vibrations of an elastic bar extending above the ground surface with one end embedded in the ground are described and the response of the bar to various ground excitations applied to its lower end is calculated. Two bars coupled along their length are also considered; it is shown that for bars with very different elastic properties the eigenfrequencies of the system are given mainly by the "softer" bar. Such an information may be useful for composite structures, including, for instance, voids. Making use of the information gained from the vibrating bars are examined the formulation of the model of two coupled harmonic oscillators, and its application to the structure building-

foundation. The coupling lowers the low frequency of the system and raise the upper frequency. The response of two coupled oscillators to an oscillating shock is calculated, and amplification factors similar with the vibrating bars are highlighted.

6. Conclusions

Conclusion derived from the present study is that the model of embedded elastic bar can be used for investigating the earthquakes effects upon constructions. At the same time, the model provides a useful tool for studying the vibrations of two coupled oscillators, with relevance for building base isolation concept. It is shown in this paper that this model, although promising, does not lead to a simple, unique, practical design conclusion.

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