A power series expansion method based on frequency response function matrix for sensitivity analysis of viscously damped system

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Abstract. The modal method represents the derivative of the eigenvector as a superposition of all modes. Although this method is relatively effective, it has a problem of modal truncation error. In this paper, based on the two kinds of calculation methods of frequency response function matrix, the transformation relationship between system matrices and eigensolutions is proposed. Meanwhile the transition from the high-order modes to the low-order modes is emphasized. Then the conversion relationship is applied to reduce the modal truncation error, and finally a more accurate method for sensitivity analysis is derived.

1. Introduction

Sensibility reflects that the change of design variables has an effect on objective function and constraint function strongly. It is widely used to calculate the sensitivity of damped eigensystems in system identification, model modification, vibration control and structural optimization. Since the problem of sensitivity has been proposed, many reseachers in the field developed a large amount of methods, which can be mainly divided into three different groups: the algebraic method, Nelson's method and the modal method.

In 1968, Fox and Kapoor [1] developed an algebraic method. According to this, Garg[10] and Rudisill[11,12] investigated the method for general asymmetric eigensystems. Lee [5,6] has extended the algebraic method to viscously damped systems. Later, Choi et al.[7] improved the algebraic method for asymmetric systems. However, they did not take left eigenvectors into account and were restricted to first-order derivatives. Based on previous methods, Li et al.[8, 9] and Brandon[19] improved the method for the left eigenvectors and second-order derivatives. Merely requiring modes of interest, the algebraic method is accurate and compact. But we will face a matrix decomposition problem when solving the algebraic equations of the system.

By summing a particular solution and a homogeneous solution, Nelson [2] then suggested a method for the eigenvector derivatives of undamped systems. Then the method was extended by Ojalvo[3] and Dailey[4] to the eigenvalue problem with multiple natural frequencies. Mills-Curran [14] and Tang [15,16]applied it to deal with eigensensitivity of the symmetric eigensystems. Friswell and Adhikari[13] modified it to symmetric and asymmetric damped systems. Although Nelson's approach is accurate and only requires modes of interest, there remain matrix decomposition problems when the particular solution is obtained, which results in heavy CPU computation time.

Fox and Kapoor [1] also derived a modal method by applying modal superposition to express the derivative of every eigenvector as a linear combination of all eigenvectors. However, this method will be computationally expensive if a large number of modes are needed to guarantee the accuracy of sensitivities. By the way, it is often difficult to obtain all the modes. By estimating the contribution of unavailable higher modes, Wang [17] put forward an improved approximate methods with residual-static modes. Zeng [18] proposed a highly accurate modal method with shifted-poles. Later, Moon et al.[20] proposed a modified modal methods eigenpair sensitivity of asymmetric damped system. Obviously, these approaches are less accurate for truncating the higher modes.

This paper extends the modal method to a more efficient one to calculate the sensitivity of damped eigensystems. First, a relationship between system matrices and eigensolutions is proposed based on the two kinds of calculation methods of frequency response function matrix. According to the relationship, the transition from the high-order modes to the low-order modes is established. At last, by applying the conversion relationship and Neumann series with shifted frequency q to reduce the modal truncation error, a more accurate method for sensitivity analysis is derived.

2. Transformation from higher-order modes to lower-order modes

2.1. Preparatory theory

Lemma (Neumann series) For any matrix $A \in C^{N\times N}$, the inverse matrix of the matrix $(I_N - A)$ can be expressed as the following power series expansion

$$(I_N - A)^{-1} = I_N + A + A^2 + A^3 + A^4 + \cdots \tag{1}$$

The convergence condition of this formula is that the maximum eigenvalue of matrix A is less than 1.

2.2. Two ways to expand frequency response function matrix

The free vibration differential equation of the viscous damped linear system with N DOF is as follows

$$M\ddot{x} + C\dot{x} + Kx = f(t) \tag{2}$$

Where M, C and $K \in R^{N\times N}$ are, respectively, the mass, damping and stiffness matrices. We only consider symmetric system matrices with respect to design parameter p in our study.

Considering the eigenvalue problem of equation (1)

$$(\lambda_i^2 M + \lambda_i C + K)u_i = 0 \tag{3}$$

Where λ_i is the i-th eigenvalue of the system, and u_i is the corresponding eigenvector.

For the viscously damped system expressed by equation (1), the dynamic equation of the system in Laplace domain can be expressed as

$$(s^2 M + sC + K)X(s) = F(s) \tag{4}$$

Where $F(s)$ is the forcing vector and $X(s)$ is the displacement vector. $s = i\omega$, ω (rad/s) is the forcing frequency. Equation (4) can be simplified as

$$D(s)X(s) = F(s) \tag{5}$$

Where $D(s)$ is called the dynamic stiffness, and $D(s) = s^2 M + sC + K$. $H(s) = D(s)^{-1}$ is called the frequency response function (FRF) matrix. In viscously damped system, H(s) can also be expressed in terms of the eigenvalues and eigenvectors as [21, 22]

$$H(s) = \sum_{j=1}^{2N} \frac{u_j u_j^T}{\theta_j (s - \lambda_j)} \tag{6}$$

Where $\theta_j = u_j^T (2\lambda_j M + C)u_j$.

2.2.1. *Power series expansion with shifted frequency q and eigensolutions.* H(s) can be casted into the matrix form:

$$H(s) = -U\Theta^{-1}(\Lambda - sI)^{-1}U^T \tag{7}$$

Where Λ，Θ are diagonal matrices, and

$$\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_{2N}]\text{,}\quad \Theta = diag[\theta_1, \theta_2, \cdots, \theta_{2N}] \text{ and } U = [u_1, u_2, \cdots, u_{2N}] \tag{8}$$

By using shift frequency *q* and Neumann expansion, we can rewrite the inverse matrix in equation (7)

$$(\Lambda - sI)^{-1} = [(\Lambda - qI) - (s-q)I]^{-1} = [I - (s-q)(\Lambda - qI)^{-1}]^{-1}(\Lambda - qI)^{-1} = \sum_{r=0}^{\infty}(s-q)^r(\Lambda - qI)^{-r-1} \tag{9}$$

Where *q* represents a constant of shifted frequency. Substituting the expansion into *H(s)*,

$$H(s) = -\sum_{r=0}^{\infty}(s-q)^r U\Theta^{-1}(\Lambda - qI)^{-r-1}U^T \tag{10}$$

According to what mentioned above, we get a formulation about *H(s)* with system matrices and shifted frequency *q*.

2.2.2. Power series expansion with system matrices. We can reform the dynamic stiffness matrix $D(s) = s^2 M + sC + K$ as

$$D(s) = K[I_N + sK^{-1}(C + sM)] \tag{11}$$

So the inverse matrix of *D(s)* is

$$D(s)^{-1} = [I_N + sK^{-1}(C + sM)]^{-1}K^{-1} \tag{12}$$

We can expand it through Neumann expansion,

$$[I_N + sK^{-1}(C + sM)]^{-1} = \sum_{k=0}^{\infty}(-s)^k(K^{-1}(C + sM))^k = \sum_{k=0}^{\infty}(-s)^k \Xi_k \tag{13}$$

Where $\Xi_0 = I_N, \Xi_1 = K^{-1}C$ and $\Xi_K = K^{-1}C\Xi_{k-1} - K^{-1}M\Xi_{k-2}$

Therefore, substituting the equation above into equation $H(s) = D(s)^{-1}$, we can obtain

$$H(s) = \sum_{k=0}^{\infty}(-s)^k \Xi_k K^{-1} = \sum_{k=0}^{\infty}s^k \Gamma_k \tag{14}$$

Where, $\Gamma_0 = K^{-1}, \Gamma_1 = -K^{-1}CK^{-1}$ and $\Gamma_K = -K^{-1}C\Gamma_{k-1} - K^{-1}M\Gamma_{k-2}$

Here we obtain the FRF matrix only involving system matrices.

2.3. Transformation from higher-order modes to lower-order modes

Comparing the two forms of *H(s)*, we can discover there are some relationships between them. Obviously, from equation (10) and equation (14) , we have

$$-\sum_{r=0}^{\infty}(s-q)^r U\Theta^{-1}(\Lambda - qI)^{-r-1}U^T = \sum_{k=0}^{\infty}s^k\Gamma_k \tag{15}$$

From left side to right side of the equation, each item equals accordingly, we obtain

$$-(s-q)^k U\Theta^{-1}(\Lambda - qI)^{-k-1}U^T = s^k\Gamma_k \text{ for } k = 0,1,2,\cdots,\infty \tag{16}$$

H(s) can be separated into the lower modes and the higher modes as follows,

$$H(s) = -U_L{\Theta_L}^{-1}(\Lambda_L - sI_L)^{-1}{U_L}^T - U_H{\Theta_H}^{-1}(\Lambda_H - sI_H)^{-1}{U_H}^T \tag{17}$$

Where $\Lambda_L = diag[\lambda_1, \lambda_2, \cdots, \lambda_L]$, $\Theta_L = diag[\theta_1, \theta_2, \cdots, \theta_L]$ and $U_L = [u_1, u_2, \cdots, u_L]$

$\Lambda_H = diag[\lambda_{L+1}, \lambda_{L+2}, \cdots, \lambda_{2N}]$, $\Theta_H = diag[\theta_{L+1}, \theta_{L+2}, \cdots, \theta_{2N}]$ and $U_H = [u_{L+1}, u_{L+2}, \cdots, u_{2N}]$

Just like *H(s)* expanded in 2.2.1, here we can express *H(s)* as

$$H(s) = \sum_{r=0}^{\infty}[-(s-q)^r U_L{\Theta_L}^{-1}(\Lambda_L - qI_L)^{-r-1}{U_L}^T - (s-q)^r U_H{\Theta_H}^{-1}(\Lambda_H - qI_H)^{-r-1}{U_H}^T] \tag{18}$$

Comparing equation(16) and (18), we can obtain

$$-(s-q)^r U_H{\Theta_H}^{-1}(\Lambda_H - qI_H)^{-r-1}{U_H}^T = s^k\Gamma_k + (s-q)^k U_L{\Theta_L}^{-1}(\Lambda_L - qI_L)^{-k-1}{U_L}^T \tag{19}$$

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The arbitrariness of the eigenvectors can be removed when $\theta_j = 1$, which is also $\Theta = I$, then the equation(19) can be simplified as

$$-(s-q)^r U_H (\Lambda_H - qI_H)^{-r-1} U_H{}^T = s^k \Gamma_k + (s-q)^k U_L (\Lambda_L - qI_L)^{-k-1} U_L{}^T \quad k = 0,1,2,\cdots,\infty \tag{20}$$

Through the equation above, we have established a relationship between system matrices and eigensolutions successfully. Given that it is often difficult to get higher modes in engineering, this relationship can help to correct the modal truncation in sensitivity analysis.

3. Transformation from higher-order modes to lower-order modes

3.1. Approximate modal method

There has been already an accurate method for the calculation for sensitivity of eigenvalues. In this chapter, we will mainly analyze sensitivity of eigenvectors. Considering the eigenvalue problem derived by equation (3), the modal method express each eigenvector derivative as a linear combination of all the eigenvectors, which is

$$u_{i.p} = \sum_{k=1}^{2N} c_{ik} u_k \tag{21}$$

The coefficients are given by the following equation

$$c_{ik} = \begin{cases} \dfrac{1}{\lambda_k - \lambda_i} u_k{}^T (M_{.p} + \lambda_i C_{.p} + K_{.p}) u_i \\ -\dfrac{1}{2} u_i{}^T (2\lambda_i M_{.p} + C_{.p}) u_i \end{cases} \tag{22}$$

From the formula above we can see that the calculation for derivatives of the eigenvector is not as simple as eigenvalues, especially it is necessary to know all the eigenvalues and eigenvectors to get an exact value.

3.2. A power series expansion method based on FRF matrix

It can be seen that the modal method is used to calculate the eigenvector sensitivity by superposition of modal. In order to ensure that the exact derivative of each modal shape is obtained, the modal method requires the superposition of all modal shapes. For multi-degree of freedom (DOF) engineering problems, the computational complexity of the method will be large using this method. Usually, only a few lower modes in large-scaled system are available. When the eigenvectors being derivatived are previous order vectors of system, modes in higher order can be neglected, which means

$$u_{i.p} \approx \sum_{k=1}^{L} c_{ik} u_k \tag{23}$$

It can be seen from the above formula, this calculation only take the former *L*-order modes, directly ignoring the impact of high-order mode, is relatively simple modal truncation. However, only when the higher order modes have less contribution to the derivative of the low order mode,we can get a result of smaller error by this method. If this conditon is not satisfied, the method can only calculate the approximate value of the derivative of an eigenvector, then there will be the following form of modal truncation error

$$E_{iH} = \sum_{k=L+1}^{2N} c_{ik} u_k = \sum_{k=L+1}^{2N} \frac{u_k{}^T R_i u_k}{\lambda_k - \lambda_i} \tag{24}$$

Where $R_i = (\lambda_i^2 M_{.p} + \lambda_i C_{.p} + K_{.p}) u_i$. The modal truncation error can also be transformed as

$$E_{iH} = U_H (\Lambda_H - \lambda_i I_H)^{-1} U_H{}^T R_i \tag{25}$$

Then the error item E_{iH} can be expanded as

$$E_{iH} = \sum_{k=0}^{\infty} (\lambda_i - q)^k U_H (\Lambda_H - qI_H)^{-k-1} U_H{}^T R_i \tag{26}$$

Where q is the frequency shift constant. The convergence of the series is

$$\left| \frac{\lambda_i - q}{\lambda_r - q} \right| < 1 \quad (r = L+1, L+2 \cdots 2N) \tag{27}$$

Which is $|\lambda_i| < |\lambda_{L+1}|$. Now, we will transform the higher-order modes in E_{iH} to lower-order modes using the relationship derived in 2.3. Substituting equation (20) into equation (26) , we can obtain

$$E_{iH} = -\sum_{k=0}^{\infty} (\lambda_i - q)^k [(\frac{s}{s-q})^k \Gamma_k + U_L (\Lambda_L - qI_L)^{-k-1} U_L{}^T] R_i \tag{28}$$

For computational efficiency, we can only consider the first n items in the error.

$$\begin{aligned} E_{iH} &= -\sum_{k=0}^{n-1} (\lambda_i - q)^k [(\frac{s}{s-q})^k \Gamma_k + U_L (\Lambda_L - qI_L)^{-k-1} U_L{}^T] R_i \\ &= -\sum_{k=0}^{n-1} [(\lambda_i - q)^k (\frac{s}{s-q})^k \Gamma_k R_i + \sum_{j=1}^{L} (\frac{\lambda_i - q}{\lambda_j - q})^k \frac{u_j{}^T R_i u_j}{\lambda_j - q}] \end{aligned} \tag{29}$$

The more accurate calculation of eigenvector derivative is obtained.

$$\begin{aligned} u_{i.p} &= \sum_{k=1}^{L} c_{ik} u_k + E_{iH} \\ &\approx \sum_{\substack{k=1 \\ k \neq i}}^{L} \frac{u_k{}^T (M_{.p} + \lambda_i C_{.p} + K_{.p}) u_i u_k}{\lambda_k - \lambda_i} - \frac{1}{2} u_i{}^T (2\lambda_i M_{.p} + C_{.p}) u_i u_i - \sum_{k=0}^{n-1} (\lambda_i - q)^k [(\frac{s}{s-q})^k \Gamma_k R_i + \sum_{j=1}^{L} \frac{u_j{}^T R_i u_j}{(\lambda_j - q)^{k+1}}] \end{aligned} \tag{30}$$

As the formula above shows, we have derived a power series expansion method based on frequency response function matrix for sensitivity analysis of viscously damped system.

4. Conclusion

According to Neumann series and the shifted frequency constant q, we expand the frequency response function in two ways. Then a transformation relationship between system matrices and eigensolutions is proposed from the two expansions. Using the relationship, we derived a more accurate and efficient method for calculation of eigensensitivity. The method takes into account the influence of the higher order mode and can calculate the more accurate truncation error as long as the eigenvalues falling within the convergence domain. Therefore, when applied to calculate the derivative of the eigenvector, the method is obviously more accurate than the modal truncation method with only the previous L terms. When the shift frequency q=0, the method is consistent with the method of Li et al.[23], so the method in this paper can be seen as an extension of the method in paper[23]. In the engineering problem, for the multi-degree-of-freedom vibration system, this method can be used to give a more accurate result when it is necessary to consider the sensitivity of many modal vectors.

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