Modeling the External Structure of a Fractals

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Abstract. The article describes the main provisions of the theory of fractal geometry, two- and three-dimensional fractals. The possibility of using the fractal theory in the design of buildings and structures, as well as individual elements of the building structures is proposed. The fractal structure has been developed by parametric methods using program «3D modeling of Fractals» with the help of finite element method.

1. Introduction
One of the benefits of the development of information technologies can be regarded as the emergence of fractal objects, the study of which is progressing and gaining popularity over the last decades. Wherein, the sphere of application of fractals is not yet fully researched.

Currently, fractals are used in the field of computer design for the generation of various kinds of artificial objects. The theory of fractals is used not only in the virtual world, but also in the creation of real objects that are maximally adapted to the design of the frame elements of the building or the entire structure as a whole. The peculiarity of this work is the study of the external structure of the fractal.

2. Theoretical section
The concept of fractal geometry appeared in the late 70-ies of XX century. Benua Mandelbrot is considered to be the founder of fractal geometry with his scientific work «The Fractal Geometry of Nature» [1]. The research results of Anri Puancare, Pierre Fatou, Guston Maurice Julia, George Cantor, Felix Hausdorff are of great importance in fractal geometry.

The main elements of fractals are inaccessible to direct observation. In this case, they are fundamentally different from the usual objects of Euclidean geometry, such as a straight line or a circle. Fractals are expressed not in geometric forms, but in algorithms, sets of mathematical procedures. To visualize fractals, algorithms are transformed into geometric shapes using a computer.
Fractal geometry of Mandelbrot describes well the objects with cracks and holes, the surfaces of which are not smooth. The geometry of natural formations, such as mountains, clouds, coastlines is overwhelmingly incorrect and distorted [2]. Fractals are structures that have two important properties - brokenness and self-similarity. Any suitable part of the fractal line contains a small copy of the entire line, i.e. this is no longer a Euclidean line, but a kind of «thick line».

Fractals can be geometric, algebraic and stochastic [3]. Algebraic and geometric fractals are deterministic, i.e. absolutely reproducible; they give identical images regardless of the number of repetitions. Stochastic fractals are considered to be nondeterministic.

Geometric fractals in the two-dimensional case are obtained with the help of some broken line, called generator. In one step of the algorithm, each of the segments constituting the broken line is replaced by

a polygonal generator, on the appropriate scale. As a result of an infinite repetition of this procedure, a geometric fractal is obtained, for example, Cantor dust, Koch star, Sierpinsky carpet, etc.

Regular geometric fractal «Cantor dust» is constructed as the following: the unit interval is divided into 3 equal parts; the middle part is thrown out, the other two remain. A similar operation is performed on each of the remaining parts (figure 1). In the limit, a point set of unbound fragments is obtained - the «Cantor dust».
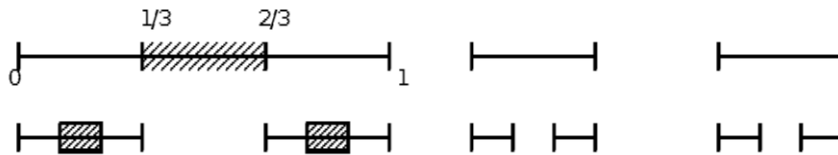

Figure 1. Fractal «Cantor dust»

As an initial object of the «Koch star» fractal, an equilateral triangle with sides of unit length is chosen, each unit segment is divided into 3 equal parts, the middle part is thrown out, and in its place two sides of an equilateral triangle are constructed on its base. This operation is repeated for all sides of the original triangle, and then, for each reduced side of the resulting star, repeatedly (figure 2).
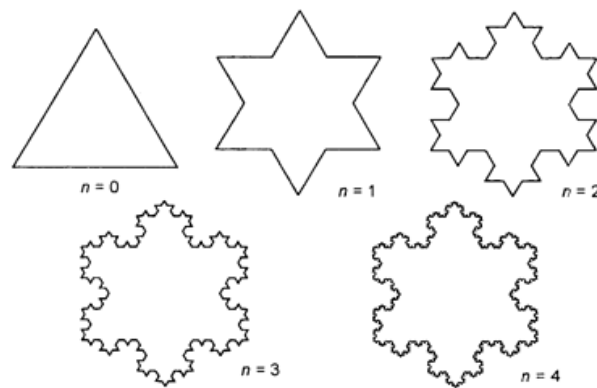

Figure 2. Four iterations for constructing «Koch Star» fractal

Algebraic fractals are obtained by iterative processes using expressions of the type

$$z_{n+1} = f(z_n).$$

The most vivid representations of quadratic algebraic fractals are the Julia and Mandelbrot sets. Both types of fractals are obtained on the complex plane during iteration using (1):

$$z_{n+1} = z_n^2 + c. \tag{1}$$

Let $c = 0$, then at each iteration the exact square of a complex number

$$z_0 \rightarrow z_0^2 \rightarrow z_0^4 \rightarrow \dots$$

For this sequence, depending on c_0, there are key points:

- the number is getting smaller and smaller, their sequence tends to zero, i.e. zero is an attractor;
- numbers continuously increase - infinity is also an attractor;
- the points remain at a distance of 1 from zero. The unit circle is the boundary of two attractors (zero and infinity).

If in the iteration process (1) fix c_0 and change z_0, then the set of Julia set is obtained, while the inner attractor is no longer zero, the boundary is bent. If we fix z_0 and change c in (1), then the Mandelbrot set is obtained (figure 3).
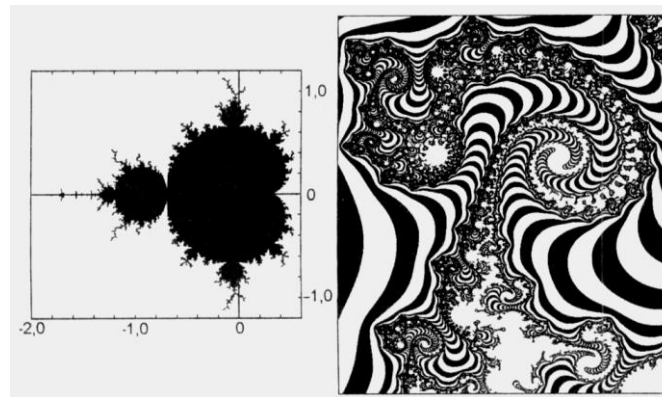
Figure 3. The Mandelbrot Set

In 1975 Benua Mandelbrot proposed a new non-Euclidean geometry. The fractal geometry of Mandelbrot studies nonsmooth, rough, foamy, pitted, cracked, punctured objects. Geometry of natural formations in the vast majority is just like that - irregular and distorted.

Topological dimension for geometric objects 0 for a point, 1 for a line, 2 for a surface, 3 for a space does not perceive tortuosity of the line, surface roughness, porosity of the space. German topologist Felix Hausdorf and Russian mathematician A.S. Bezikovich derived the fractional dimension of fractals.

For an infinite closed and bounded set of spheres of radius ε there is a finite subcovering - a finite number of spheres of radius ε such that each element of the set belongs to at least one of the spheres, not necessarily coinciding with its center. Let $N(\varepsilon)$ be the number of spheres in a finite subcovering. $N(\varepsilon)$ can be expanded in a Laurent series in integer powers of. The correct part of the Laurent series (Taylor) contains integer positive powers, the main part - negative integers.

Let $\dfrac{1}{\varepsilon^d}$ — the principal term of the Laurent part of the expansion of $N(\varepsilon)$ in powers of ε.

$$N(\varepsilon) \sim \frac{1}{\varepsilon^d}. \tag{2}$$

The exponent d is called the Hausdorff-Besicovitch dimension. Taking the logarithm of (2), we obtain:

$$\log N(\varepsilon) \approx -d \cdot \log \varepsilon \Rightarrow d \approx \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log\left(\dfrac{1}{\varepsilon}\right)}. \tag{3}$$

The numerator of the Hausdorff-Besicovitch dimension is the number of elements in the finite subcovering of the original compact set by spheres of radius ε, the denominator is a number that shows how many times the radius of the sphere ε is stacked in units of length.

For example, for the fractal «Cantor dust», the number of elements in the final subcover is 2, and in a unit of length three circles with the radius of 1/3 are stacked. Then

$$d = \frac{\log 2}{\log 3} \approx 0.6309\ldots$$

For a fractal «Koch star» the rough line of the contour has the dimension

$$d = \frac{\log 3}{\log 4} \approx 1.2618\ldots$$

The artificially introduced fractional dimension of regular fractals quantitatively characterizes the chaos that arises when they are constructed.

Another known class of fractals are stochastic fractals, which are obtained if, in the iteration process, randomly changing any of its parameters. In doing so, objects are obtained that are very similar to natural

asymmetric trees, rugged coastlines, etc. Two-dimensional stochastic fractals are used in modeling the terrain and sea surface.

All reviewed linear fractals as graphic constructions are flat. But if we assume that any flat figure is, say, an orthogonal projection of an object in space, then we can speak of the existence of three-dimensional fractal figures.

A new step in the development of fractal geometry are three-dimensional fractals. The Mandelbrot shell is a three-dimensional fractal, the analog of the Mandelbrot set created by Daniel White and Paul Nylander using hypercomplex algebra based on spherical coordinates.

Formula for the n-th power of the three-dimensional hypercomplex number $\{x, y, z\}$ is the following:

$$x, y, z^n = r^n \cos(n\theta)\cos(n\phi), \sin(n\theta)\cos(n\phi), \sin(n\phi), \tag{4}$$

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{5}$$

$$\theta = \arctan(y/x); \tag{6}$$

$$\phi = \arctan(z/\sqrt{x^2 + y^2}) = \arcsin(z/r). \tag{7}$$

We use the iteration $z_0 \to z_0^2 \to z_0^4 \to \dots$, where z and c are three-dimensional hypercomplex numbers on which the operation of raising to the natural power is performed as indicated above. For $n > 3$, the result is a three-dimensional fractal. The eighth degree is used most often.

Many objects of architecture are already fractal structures to some extent. For example, the Eiffel Tower is constructed using only one type of metal rod, which is used in various sizes throughout the tower to minimize the weight of the entire structure. This execution can be called one of the first ways to implement fractal geometry in the structure of a building.

Currently, some studies focus on creating designs that take advantage of fractal properties. The process of using fractal geometry to create structures capable of withstanding the required weight and pressure while having a much smaller mass is described in the paper [6]. Firstly, a hollow metal beam changes in diameter and thickness by means of re-engineering. Next, an iterative process of manufacturing next-generation beams is created by means of replacing full-scale self-similar versions (figure 4).
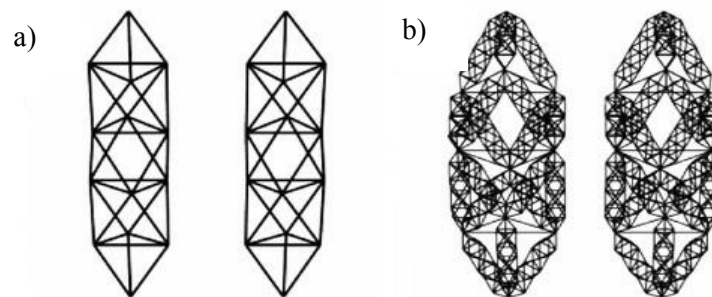

Figure 4. Fractal structures: a) zero generation; b) second generation

The resulting structures are more lightweight than those that are made of solid material. Consequently, the more iterations are used, the lighter the fractal structure becomes.

The general idea is to use fractal elements in building new structures, rather than borrowing fractal ideas and adding them unsystematically just to obtain lighter weight structures.

One of the options for applying fractal geometry in construction could be the stream in architecture called parametrism. Parametrism arose with the development of computer graphics and digital animation. The basis for the latest developments is advanced parametric design systems and various script methods (algorithms).

Parametric modeling of buildings and structures is fundamentally different from the usual two-dimensional image or three-dimensional modeling. The constructor in this case creates a kind of a

mathematical model of objects with parameters that change the configuration of the object when they vary. Parameters serve to determine the behavior of each element of the model and affect the relationship with other elements of the structure.

Working in parametric environments, the architect does not immediately see the final object, he/she considers only the algorithm of its creation in specialized software: Revit Architecture, Grasshopper, Rhinoceros.

The main difference between the parametric architecture and the others liess the attempt to combine the dynamics of natural forms in a complex and unusual spatial model and the almost complete absence of linearity of objects and symmetry.

An example of a parametric architecture is the «Fractal Pavilion», developed in the architectural school Unit 2 of London (figure 5).
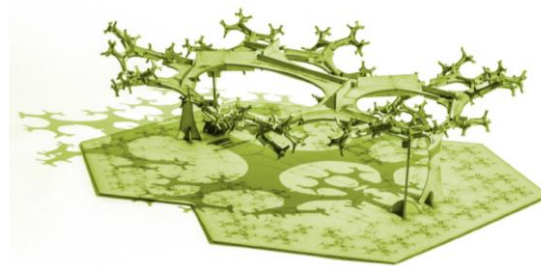

Figure 5. The structural model of the Fractal Pavilion (London, 2005)

It has practically no decoration and it is a structure based on the principle of the golden section, recursive repetition of crests and spiral rays directed from the top to the smallest branches with geometric progression. The resulting fractal design is designed and mounted to the 9-th iteration.

The pavilion has been generated by the software algorithm in the form of motion vectors in parallel, thereby creating a dynamic parametric shape reminiscent of a moving on the river forest.

3. Experimental section

To model the external structure of a fractal, the author has developed a computer program «3D Fractal Modeling» [7] by programming language C # in environment Visual Studio 2013.

The program generates points of space belonging to the three-dimensional fractal. The program provides input of initial data, including the number of points, the number of iterations of the set, and the power of the fractal. The results of the calculations are the sets of points that form the layers of the fractal set. The program interface allows entering the input data, the number of counted points and layers of the fractal set. The implementation of the point generator is performed in the environment of the SCAD.

The purpose of the algorithm is to determine the points belonging to the surface of the fractal sphere. The coordinates of the points are determined by checking their belonging to the surface of the fractal shell after a given number of iterations. Verification of the belonging of points is carried out in spherical coordinates, changing cyclically first distance from the center r, then horizontal angle φ and vertical angle θ. If the current point is outside the surface, then the previous one belonged to it.

The points selected as a result of calculations are saved in a text file in *txt format, which is suitable for reading in the SCAD. The creation of triangular finite elements by the point numbers is recorded in the same file. As a result, in the SCAD, it is possible to read this text file and get a three-dimensional fractal sphere consisting of isoparametric triangular finite elements (figure 6).
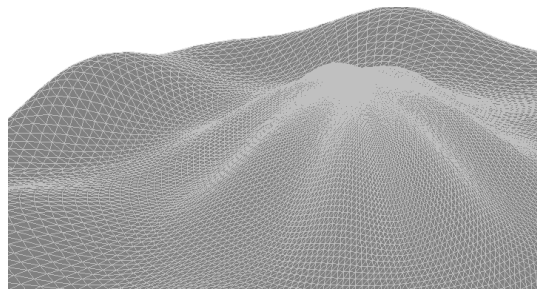
Figure 6. Fragment of the finite element model

The calculation of the coordinates of a large number of points with smaller steps of the angles makes it possible to obtain a more detailed and smooth surface of the fractal. This increases the accuracy of the calculation, but it has a strong effect on the time of generation of points. Therefore, it is important to correctly set the initial steps for r, φ and θ to obtain a better finite element mesh in the future.

For instance, the following initial data is used: power level of the fractal is eight; the number of iterations is three. The first iteration is a sphere. As the number of iterations increases, the surface becomes more complicated due to repeated fractal elements. The result of the calculations may be saved in a text file.

Figure 7, 8 demonstrates the fractal of 8-th and 44-th power accordingly. On the pole appears the formation of a volcanic species; the main surface is rough.
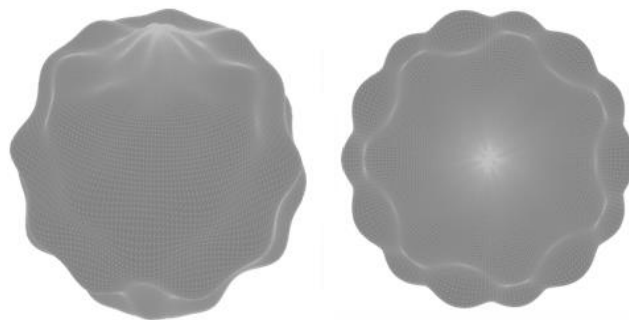

Figure 7. 3D model of the fractal, developed using finite element method
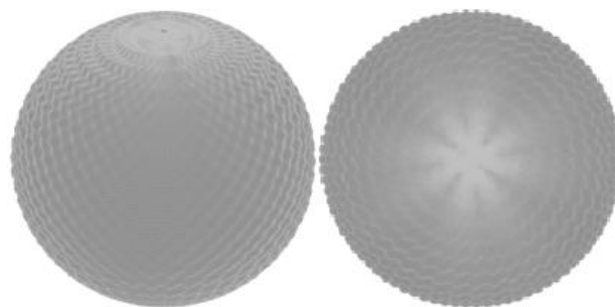with 8-th level of power

Figure 8. 3D model of the fractal with 44-th level of power

The program «3D Fractal Modeling» allows not only visualizing complex fractal geometry, but also obtaining a finite element model of a fractal.

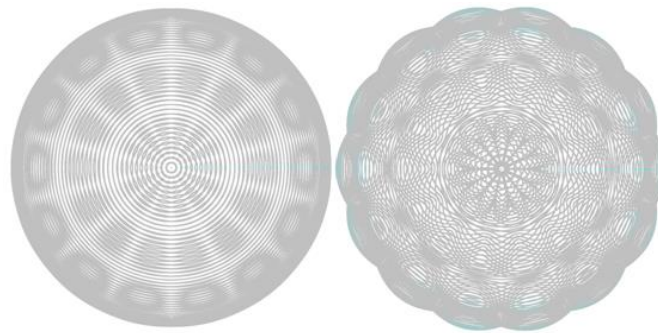Figure 9 shows the third and fourth layers of the fractal set in the projection onto the *XOY* plane.

b)

Figure 9. Layers of the fractal: the third layer and the fourth layer

4. Conclusions

Development of computer technologies provides the ability to create complex shapes. The method of structural designing and their individual elements changes with the use of mathematics, algorithms and innovative construction technologies.

The program «3D fractal modeling» has been developed; it performs the generation of fractal nodes. Fractal geometry allows creating completely new objects, light in weight and having a unique structure.

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