IOP Conf. Series: Earth and Environmental Science **82** (2017) 012031 doi:10.1088/1755-1315/82/1/012031

Higher moments method for generalized Pareto distribution in flood frequency analysis

C R Zhou[1], Y F Chen[1,3], Q Huang[1] and S H Gu[2]

[1]College of Hydrology and Water Resources Hohai University, No. 1 Xikang Road, Gulou District in Nanjing, Jiangsu Province, China
[2]Shanghai Hydrological General Station, No. 241 East Longhua Road, District Xuhui in Shanghai, China

E-mail: 19870056@hhu.edu.cn

Abstract. The generalized Pareto distribution (GPD) has proven to be the ideal distribution in fitting with the peak over threshold series in flood frequency analysis. Several moments -based estimators are applied to estimating the parameters of GPD. Higher linear moments (LH moments) and higher probability weighted moments (HPWM) are the linear combinations of Probability Weighted Moments (PWM). In this study, the relationship between them will be explored. A series of statistical experiments and a case study are used to compare their performances. The results show that if the same PWM are used in LH moments and HPWM methods, the parameter estimated by these two methods is unbiased. Particularly, when the same PWM are used, the PWM method (or the HPWM method when the order equals 0) shows identical results in parameter estimation with the linear Moments (L-Moments) method. Additionally, this phenomenon is significant when $r \geq 1$ that the same order PWM are used in HPWM and LH moments method.

1. Introduction

Peak Over Threshold (POT) series is an alternative to the annual maximum series when studying characteristics of extreme events. Several distributions can be used to describe extreme events of discharge observations (annual floods). For POT series, the generalized Pareto distribution (GPD) has been proven to be a perfect candidate for modeling POT samples [1, 2]. The GPD can describe the partial duration series (or POT series) which are obtained by letting a threshold select the highest values from an ordinary hydrological time series [3].

However, the POT approach remains under-employed recently, mainly due to the complexities associated with its usage [4], such as the choice of thresholds, the selection of criteria for retaining floods peaks, and so on.

Parameter estimation is a crucial part of flood frequency analysis (FFA). Many types of estimators have been developed to improve the efficiency and robustness of the fitting techniques. Moment methods, including the method of moments (MOM), the probability weighted moments (PWM) method, and the linear moments (L-Moments) method, use sample moments to estimate population moments. The law of large numbers is the basic discipline of these methods. The PWM method showed advantages for small datasets due to its smaller uncertainty than ordinary moments [3]. This characteristic suggested that the PWM method was a more suitable method for relatively short length of flood series. The L-Moments method was first put forward by Hosking [5] and it has been widely applied to estimate various distributions. L-Moments, which are linear combinations of the PWM, tend to share similar characteristics with PWM. Additionally, Wang [6] developed LH moments which are generalizations of L-Moments. Higher L-Moments (LH moments) rely on the linear combinations of higher-order statistics with the capability of a more detailed analysis of annual flood peak data [7].

Previous researchers have concluded that higher moments gave more consideration to significant flood peaks and extracted information existing in the POT series to the greatest possible extent. Several studies have attempted to deduce calculation procedures for ordinary moments, such as the emergence of , LH moments or higher PWM for common distributions (generalized Extreme Value, generalized Pareto, Pearson Type-III, Gumbel and Weibull distributions [8], to name a few).

This paper is structured as follows: in the first section, a brief introduction to GPD and estimators based on PWM will be presented; in section 2, statistical experiments were given to investigate the performances of all methods studied in this paper. In section 3, these methods were applied to a case study of the POT series of the Yangtze River, China.

2. Backgrounds

The Pareto distribution (PD) was first put forward by Vilfredo Pareto (an Italian civil engineer, economist, and sociologist). Pickands [9] extended it to the generalized Pareto distribution (GPD), and revealed its great importance to extreme value theory. In this study, GPD with three parameters, μ,σ,ζ , which refer to the location, scale and shape of GPD respectively, was analyzed. The shape of GPD differs with the value of shape parameter ζ . The definition of GPD is given by:

$$G(\mu,\sigma,\zeta)=\begin{cases}1-(1+\zeta\dfrac{x-\mu}{\sigma})^{-\frac{1}{\zeta}},\zeta\neq 0\\ 1-\exp(-\dfrac{x-\mu}{\sigma}),\zeta=0\end{cases} \tag{1}$$

Considering practical application, adequate flood quantile estimates are crucial. The quantile function is given by equation (2).

$$x_{(G)}=\begin{cases}\mu-\dfrac{\sigma}{\zeta}\left[1-(1-G)^{-\zeta}\right],\zeta\neq 0\\ \mu+\sigma\left[-\ln(1-G)\right],\zeta=0\end{cases} \tag{2}$$

The probability weighted moments (PWM) method is a widely used estimator in parameter estimation for hydrological distributions, including GPD. However, one should note that the PWM method is only applicable when $\zeta<0.5$. Considering X is the random variable and its distribution is $F(x)$, the PWM of X is given by:

$$M_{p,r,s}=\int_0^1[x_{(F)}]^p F^r(1-F)^s dF \tag{3}$$

Where $x_{(F)}$ is the quantile function; p,r,s are real numbers; and r refers to the order of PWM. To obtain excellent statistics characteristics of GPD, $p=1,s=0$ were used. The PWM of X shown as equation (4).

$$\beta_r=M_{1,r,0}=\int_0^1 x_{(F)}F^r dF;(r=0,1,2,...,n) \tag{4}$$

Let $\{x_1,x_2,x_3,...x_n\}$ be random samples selected from X . Sort the series ascending, then it will become a new series $\{x_{(1)},x_{(2)},...,x_{(n)}\}$ (note $x_{(1)}\leq x_{(2)}\leq...\leq x_{(n)}$). Therefore, PWM can be calculated by equation (5). The b_r of samples are estimators to the β_r of population.

$$b_r = \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)(i-2)...(i-r)}{(n-1)(n-2)...(n-r)}x_{(i)} \tag{5}$$

If $\zeta \neq 0$, the quantile function of X is $x_{(G)} = \mu - \frac{\sigma}{\zeta}[1-(1-G)^{-\zeta}]$. Thus, β_r is given by:

$$\beta_r = \int_0^1 x_{(G)}G^r dG = \int_0^1 [\mu - \frac{\sigma}{\zeta}(1-(1-G)^{-\zeta})]G^r dG = \frac{\mu}{r+1} - \frac{\sigma}{\zeta(r+1)} + \frac{\sigma}{\zeta}\int_0^1 G^r(1-G)^{-\zeta}dG \tag{6}$$

According to the GPD-based method of moments (MOM), the equations below (see equations (7) to (9)) put a constraint condition on the use of moment methods. Consequently, they are theoretically applicable only when $\zeta \neq \frac{1}{3}$, and ζ must not exceed 0.5 for the reason that when $\zeta \to 0.5$, the variance will tend to be infinite. However, researchers have previously suggested that moments' estimators for ζ <0.25.

$$E(X) = \mu + \frac{\sigma}{1-\zeta} \tag{7}$$

$$Var(X) = \frac{\sigma^2}{(1-\zeta)^2(1-2\zeta)} \tag{8}$$

$$Skew(X) = \frac{2(1+\zeta)\sqrt{1-2\zeta}}{1-3\zeta} \tag{9}$$

If $\zeta < 1$, then $\int_0^1 G^r(1-G)^{-\zeta}dG = B(r+1, 1-\zeta)$, where $B(r+1,1-\zeta)$ is the Beta-function which can be expressed as $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$. Equation (6) can be changed into equation **(**10).

$$\beta_r = \frac{\mu}{r+1} - \frac{\sigma}{(r+1)\zeta} + \frac{\sigma}{\zeta}B(r+1,1-\zeta) \tag{10}$$

3. Higher moments methods

3.1. Higher probability weighted moments method

Higher PWM (HPWM) give more weights to larger flows [10] because the power function F^r increases with the values of samples. The HPWM method has been developed to be an estimator for many hydrological distributions [11, 12]. These applications have further proven the availability and robustness of HPWM method. In this paper, the HPWM method was developed for GPD, and the estimator to each parameter was listed.

According to the descriptions above, if $\zeta < 1, \zeta \neq 0$, the PWM for GPD can be calculated as equation (11).

$$(r+1)\beta_r = \mu - \frac{\sigma}{\zeta} + (r+1)\frac{\sigma}{\zeta}B(r+1,1-\zeta) \tag{11}$$

If $\zeta < 1$, one can calculate as follows according to the HPWM method based on GPD.

$$(r+2)\beta_r = \mu - \frac{\sigma}{\zeta} + (r+2)\frac{\sigma}{\zeta}B(r+2,1-\zeta) \tag{12}$$

$$(r+2)\beta_{r+1} - (r+1)\beta_r = (r+2)\frac{\sigma}{\zeta}B(r+2,1-\zeta) - (r+1)\frac{\sigma}{\zeta}B(r+1,1-\zeta) \tag{13}$$

$$\frac{(r+2)\beta_{r+1} - (r+1)\beta_r}{(r+3)\beta_{r+2} - (r+2)\beta_{r+1}} = \frac{(r+2)B(r+2,1-\zeta) - (r+1)B(r+1,1-\zeta)}{(r+3)B(r+3,1-\zeta) - (r+2)B(r+2,1-\zeta)} \tag{14}$$

$$\sigma = \frac{\zeta[(r+2)\beta_{r+1} - (r+1)\beta_r]}{(r+2)B(r+2,1-\zeta) - (r+1)B(r+1,1-\zeta)} \tag{15}$$

$$\mu = (r+1)\beta_r + \frac{\sigma}{\zeta} - (r+1)\frac{\sigma}{\zeta}B(r+1,1-\zeta) \tag{16}$$

The key, clearly, is to establish a method to estimate ζ. To calculate the estimator $\hat{\zeta}$ of ζ, let the left side of equation (14) be Z; then, its estimator can be expressed by $\hat{Z} = \frac{(r+2)b_{r+1} - (r+1)b_r}{(r+3)b_{r+2} - (r+2)b_{r+1}}$. Then, the relationship between Z and ζ can be investigated. Considering that ζ is a random number within 0~1, so equation (17) is given below.

$$Z = \frac{(r+2)B(r+2,1-\zeta) - (r+1)B(r+1,1-\zeta)}{(r+3)B(r+3,1-\zeta) - (r+2)B(r+2,1-\zeta)} \tag{17}$$

If r and ζ are given, Z in equation (17) can be easily calculated. Relationship between Z and ζ were drawn, and then these curves were fitted by polynomial. The results apparently showed linear relationships between Z and ζ (see figure 1). The linear fitting coefficients and linear fitting formulas were shown in table 1.

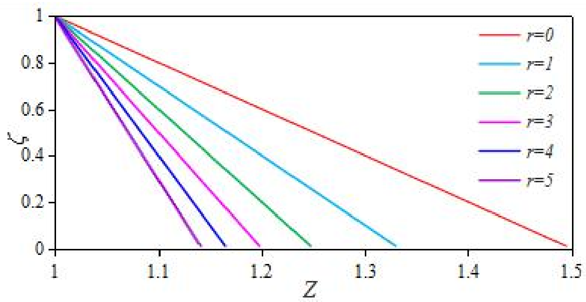

Figure 1. Relationship diagram of $Z \sim \zeta$.

Table 1. Linear fitting coefficients and linear fitting formulas.

Order *r*	Fitting coefficients and correlation coefficients			Linear fitting formulas
	α_1	α_0	R^2	
0	-2.0000	3.0000	0.9999990	$\zeta = -2.0000Z + 3.0000$
1	-3.0000	4.0000	0.9999998	$\zeta = -3.0000Z + 4.0000$
2	-4.0000	5.0000	0.9999451	$\zeta = -4.0000Z + 5.0000$

3	-5.0001	6.0001	0.9999975	$\zeta = -5.0001Z + 6.0001$
4	-6.0000	7.0000	0.9999994	$\zeta = -6.0000Z + 7.0000$
5	-6.9999	7.9999	0.9999996	$\zeta = -6.9999Z + 7.9999$

In conclusion, when $\zeta < 1, \zeta \neq 0$, HPWM can be used to estimate the parameters of GPD according to the following steps:

- Step 1: calculate PWM of the samples (b_r, b_{r+1}, b_{r+2}), and apply them to compute $\hat{Z}$ as the estimator of Z ;
- Step 2: substitute $\hat{Z}$ into the linear fitting formulas in table 1 to calculate $\hat{\zeta}$, such as we can calculate the estimate of $\hat{\zeta}$ when the order of the HPWM equals 0: $\hat{\zeta} = -2.0000\hat{Z} + 3.0000$.
- Step 3: substitute $\hat{\zeta}$ into equation (15) to compute $\hat{\sigma}$;
- Step 4: use $\hat{\zeta}, \hat{\sigma}$ to calculate $\hat{\mu}$ by equation (16).

3.2. L-Moments

Hosking introduced L-Moments as an alternative to Moments Method for characterizing distributions and data [10]. They are linear combinations of PWM and have shown excellent performances as estimators [13-15]. PWM are essential components of L-Moments, and PWM (β_r with $r = 0,1,2$) can be calculated by equation (18).

$$\beta_0 = \mu - \frac{\sigma}{\zeta} + \frac{\sigma}{\zeta} B(1,1-\zeta)$$

$$\beta_1 = \frac{\mu}{2} - \frac{\sigma}{2\zeta} + \frac{\sigma}{\zeta} B(2,1-\zeta) \tag{18}$$

$$\beta_2 = \frac{\mu}{3} - \frac{\sigma}{3\zeta} + \frac{\sigma}{\zeta} B(3,1-\zeta)$$

Given the relationship between L-Moments and PWM, L-Moments can be expressed by equation (19).

$$\lambda_1 = \beta_0$$

$$\lambda_2 = 2\beta_1 - \beta_0 \tag{19}$$

$$\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$$

Referring to the introduction of PWM and HPWM, the L-moments of samples can be estimated by its PWM as $\hat{\lambda}_1 = b_0; \hat{\lambda}_2 = 2b_1 - b_0; \hat{\lambda}_3 = 6b_2 - 6b_1 + b_0$.

Therefore, according to equations (18) and (19), L-M of the GPD samples with $r = 1,2,3$ can easily lead to the expression of $\lambda_1, \lambda_2, \lambda_3$ (see equation (20)).

$$\lambda_1 = \mu - \frac{\sigma}{\zeta} + \frac{\sigma}{\zeta} B(1,1-\zeta) = \mu + \frac{\sigma}{1-\zeta}$$

$$\lambda_2 = \frac{\sigma}{\zeta}[2B(2,1-\zeta) - B(1,1-\zeta)] = \frac{\sigma}{(1-\zeta)(2-\zeta)} \tag{20}$$

$$\lambda_3 = \frac{\sigma}{\zeta}[6B(3,1-\zeta) - 6B(2,1-\zeta) + B(1,1-\zeta)] = \frac{\sigma(1+\zeta)}{(3-\zeta)(2-\zeta)(1-\zeta)}$$

Thus, equations (21) to (23) can be used to calculate the estimation of each parameter.

$$\hat{\zeta} = \frac{3\hat{\tau}_3 - 1}{1+\hat{\tau}_3}; \hat{\tau}_3 = \frac{\hat{\lambda}_3}{\hat{\lambda}_2} \tag{21}$$

$$\hat{\sigma} = (1-\hat{\zeta})(2-\hat{\zeta})\hat{\lambda}_2 \tag{22}$$

$$\hat{\mu} = \hat{\lambda}_1 - (2-\hat{\zeta})\hat{\lambda}_2 \tag{23}$$

3.3. LH Moments

Wang [10] was the first to suggest LH moments as the generalization of L-Moments. Previous researches have applied it into fitting with hydrological distributions [16-18], which have revealed the limitation in the usage of the proposed LH moments. Many studies introduced the LH moments, but there are still rare studies about its corresponding calculation methods.

Similar to L-Moments, the first three orders of LH moments can be defined by equations (24) and (25).

$$\lambda_1^r = (r+1)\beta_r$$

$$\lambda_2^r = \frac{r+2}{2!}[(r+2)\beta_{r+1} - (r+1)\beta_r] \tag{24}$$

$$\lambda_3^r = \frac{r+3}{3!}[(r+4)(r+3)\beta_{r+2} - 2(r+3)(r+2)\beta_{r+1} + (r+1)(r+2)\beta_r]$$

$$\tau_3^r = \frac{r+3}{3(r+2)} \frac{(r+4)(r+3)\beta_{r+2} - 2(r+3)(r+2)\beta_{r+1} + (r+2)(r+1)\beta_r}{(r+2)\beta_{r+1} - (r+1)\beta_r} \tag{25}$$

According to the GPD-based PWM, β_r in equation (25) above can be easily obtained. So equations above can be transformed into equations (26) and (27).

$$\begin{aligned}
\lambda_1^r &= (r+1)\beta_r = \mu - \frac{\sigma}{\zeta} + \frac{\sigma}{\zeta}(r+1)B(r+1,1-\zeta) \\
\lambda_2^r &= \frac{r+2}{2}\frac{\sigma}{\zeta}[(r+2)B(r+2,1-\zeta) - (r+1)B(r+1,1-\zeta)] \\
\lambda_3^r &= \frac{r+3}{6}\frac{\sigma}{\zeta}[(r+4)(r+3)B(r+3,1-\zeta) - 2(r+3)(r+2)B(r+2,1-\zeta) + (r+2)(r+1)B(r+1,1-\zeta)]
\end{aligned} \tag{26}$$

$$\begin{aligned}
\tau_3^r &= \frac{\lambda_3^r}{\lambda_2^r} = \frac{r+3}{3(r+2)} \\
&* \frac{(r+4)(r+3)B(r+3,1-\zeta) - 2(r+3)(r+2)B(r+2,1-\zeta) + (r+2)(r+1)B(r+1,1-\zeta)}{(r+2)B(r+2,1-\zeta) - (r+1)B(r+1,1-\zeta)}
\end{aligned} \tag{27}$$

Referring to equation (27), ζ is the only unknown parameter corresponding to a given order r.

Because the calculation expression of ζ is not that ready to get, it leads one to seek the numerical methods to it, and find the approximate relationship between τ_3^r and ζ.

Let the value of ζ change within the range of [-2, 1), and one can calculate each τ_3^r corresponding to different ζ. Then draw the points (τ_i, ζ_i) corresponding to each given order r. Fit these points with the polynomial functions. Finally, cubic polynomial performs well in fitting with the points. So the approximate relationship between τ_3^r and ζ can be expressed as by equation (28).

$$\zeta = \alpha_3(\tau_3^r)^3 + \alpha_2(\tau_3^r)^2 + \alpha_1\tau_3^r + \alpha_0 \tag{28}$$

Figure 2 shows the results of polynomial fitting and the relationship between τ_3^r and ζ; Table 2 gives out the coefficients of polynomial fitting and correlation coefficients.

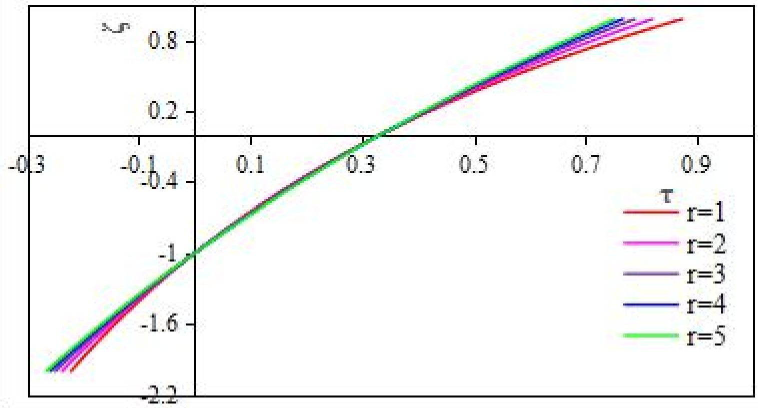

Figure 2. Approximate relationship between τ and ζ.

Table 2. The coefficients of polynomial fitting and correlation coefficients.

Order r	**The coefficients of polynomial fitting** α_3	α_2	α_1	α_0	**Correlation coefficients** R^2
1	1.0552	-2.6431	3.7896	-1.004	0.999999
2	0.7668	-2.0926	3.624	-10018	0.999999
3	0.5745	-1.7192	3.5152	-1.0009	0.999999
4	0.4437	-1.4539	3.4387	-1.0005	0.999999
5	0.3518	-1.2574	3.3821	-1.0003	0.999999

The value of τ_3^r can be estimated by the PWM of samples (see equation (29)). Thus the estimation of ζ can be obtained by the polynomial relationship between them (equation (28)).

$$\hat{\tau}_3^r = \frac{r+3}{3(r+2)} \frac{(r+4)(r+3)b_{r+2} - 2(r+3)(r+2)b_{r+1} + (r+2)(r+1)b_r}{(r+2)b_{r+1} - (r+1)b_r} \tag{29}$$

When the estimation of ζ has been calculated, the other two unknown parameters σ, μ would be estimated by equations (30) to (31).

$$\hat{\sigma} = \frac{2\hat{\zeta}\lambda_2^r}{(r+2)[(r+2)B(r+2,1-\hat{\zeta}) - (r+1)B(r+1,1-\hat{\zeta})} \tag{30}$$

$$\hat{\mu} = \lambda_1^r + \frac{\hat{\sigma}}{\hat{\zeta}} - \frac{\hat{\sigma}}{\hat{\zeta}}(r+1)B(r+1,1-\hat{\zeta}) \tag{31}$$

4. Statistical experiments and results

Statistical experiments were used to evaluate and compare the methods of PWM, HPWM, L-Moments and LH moments. Cunnane [19] demonstrated that the unbiased plotting position, which was the mean of the order statistics in samples from the reduced variety population, had the smallest mean square error among all such estimates. In order to reduce sampling errors as much as possible, a random-number generator was used in Matlab, to generate random numbers to follow the GPD. Two random datasets, each with a size of 1000*1000 (parameters: $\zeta = 0.2$, $\sigma = 0.1$, $\mu = 0.1$, $\zeta = -0.1$, $\sigma = 1$, $\mu = 1$) are generated. The data was ascendinglysorted.

According to theory of order statistics, the mean of each order statistics was calculated as an unbiased plotting position. The parameters of the samples were estimated by the method given or deduced above. All results are given in table 3.

From table 3, it was clear that if the first three PWM were used, the results of different methods were identical to each other. This was clear to see from the results of PWM, HPWM ($r=0$), L-Moments and LH moments ($r=0$). Moreover, when the same PWMs were used, LH moments methods and HPWM method (when $r \geq 1$) showed almost the same performances.

Table 3. Estimators and their performances.

Distribution Function: GPD;		Parameters: $\zeta = 0.2, \sigma = 1, \mu = 1$					
Parameters Estimators	PWM	HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	0.2005	0.2005	0.2001	0.1998	0.1995	0.1992	0.1991
σ	0.9993	0.9993	1.0000	1.0008	1.0016	1.0023	1.0027
μ	1.0000	1.0000	0.9995	0.9988	0.9980	0.9972	0.9967
Parameters Estimators	L-M	LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	0.2005	0.2005	0.2019	0.2002	0.1995	0.1992	0.1989
σ	0.9993	0.9993	0.9961	0.9999	1.0016	1.0026	1.0034
μ	1.0000	1.0000	1.0002	0.9995	0.9998	0.9969	0.9959
Distribution Function: GPD;		Parameters: $\zeta = -0.1, \sigma = 1, \mu = 1$					
Parameters Estimators	PWM	HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.1002	-0.1002	-0.1002	-0.1002	-0.1002	-0.1002	-0.1000
σ	1.0006	1.0006	1.0006	1.0005	1.0006	1.0004	0.9997
μ	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001	1.0005
Parameters Estimators	L-M	LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.1002	-0.1002	-0.0950	-0.0977	-0.0988	-0.0993	-0.0996
σ	1.0000	1.0000	0.9916	0.9955	0.9974	0.9984	0.9989
μ	1.0006	1.0006	1.0045	1.0032	1.0023	1.0018	1.0014

5. Case study and results

Daily runoff series of Cuntan and Hankou gauged stations (Yangtze River in China) was selected for further study to investigate the applicability of GPD and HPWM to the POTS observations and test the performance of HPWM and LH moments method in parameter estimation. Cuntan gauged station was set up in the year of 1893. It monitors the fluxes and sand from the upper stream of Yangtze River, and this makes it be an important station to control the inflow of the Three George Project [20]. Hankou gauged station is set up in the year of 1865, but the daily runoff time series from 1952 to 2005 is included in this research due to they can be taken as stationary(pass the statistical test).

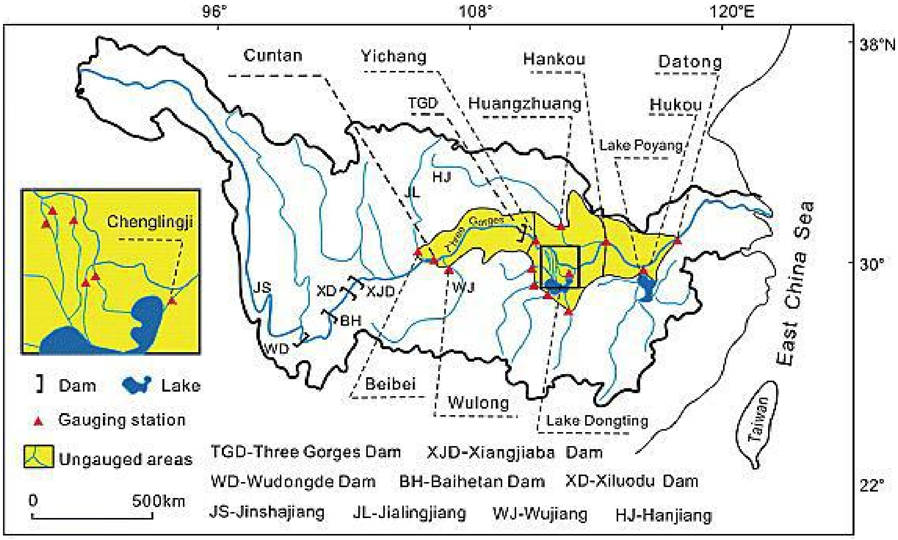

Figure 3. The location of Cuntan Station (Download from the paper of Yang, et al. 2007).

As a result of anthropogenic processes as well as the construction and operation of hydroelectrolical projects, the hydrologic properties of Cuntan station have dramatically changed. Thus, a consistency test must be studied before applying it to the FFA series.

Therefore, the runoff data from 1979 to 2014 was selected in this study. According to the results of statistical test (significance level=0.05), the series is stationary.

Previous studies [21] showed more reasonable results will be obtained when the number of annual average peaks to be 2 to 3. So the threshold value was determined to be 30000 m^3/s in this study. There are 95 peaks occurred in the series, so approximately 2.63 peaks per year. The same work was done to Hankou gauged station, the threshold is settled as 40000 m^3/s, and 133 peaks can be obtained in total, or 2.463 peaks each year (within 2 to 3)

All methods mentioned in this paper are used to fit with the peak runoff points. The results were listed in table 4, and all theoretical curves and empirical points are drawn in figure 4.

Table 4 shows that all of the methods performed well for parameter estimation. Particularly, the HPWM method and LH moments method seem to provide a broad range of selections because they can give out different results that correspond to different orders. Thus, one can select a better set of parameters to describe GPD in fitting with empirical points. The distributions used to fit with peak runoff points were drawn where the best fit was achieved. In order to describe the goodness of fit, sum relative square variation (WLS) is used in this study:

$$WLS = \sum_{i=1}^{n} \left(\frac{x_{ip} - x_p}{x_p}\right)^2 \times 100\% \tag{32}$$

Where WLS is the sum relative square deviation and x_{ip} is the estimator of x_p corresponding to the frequency p.

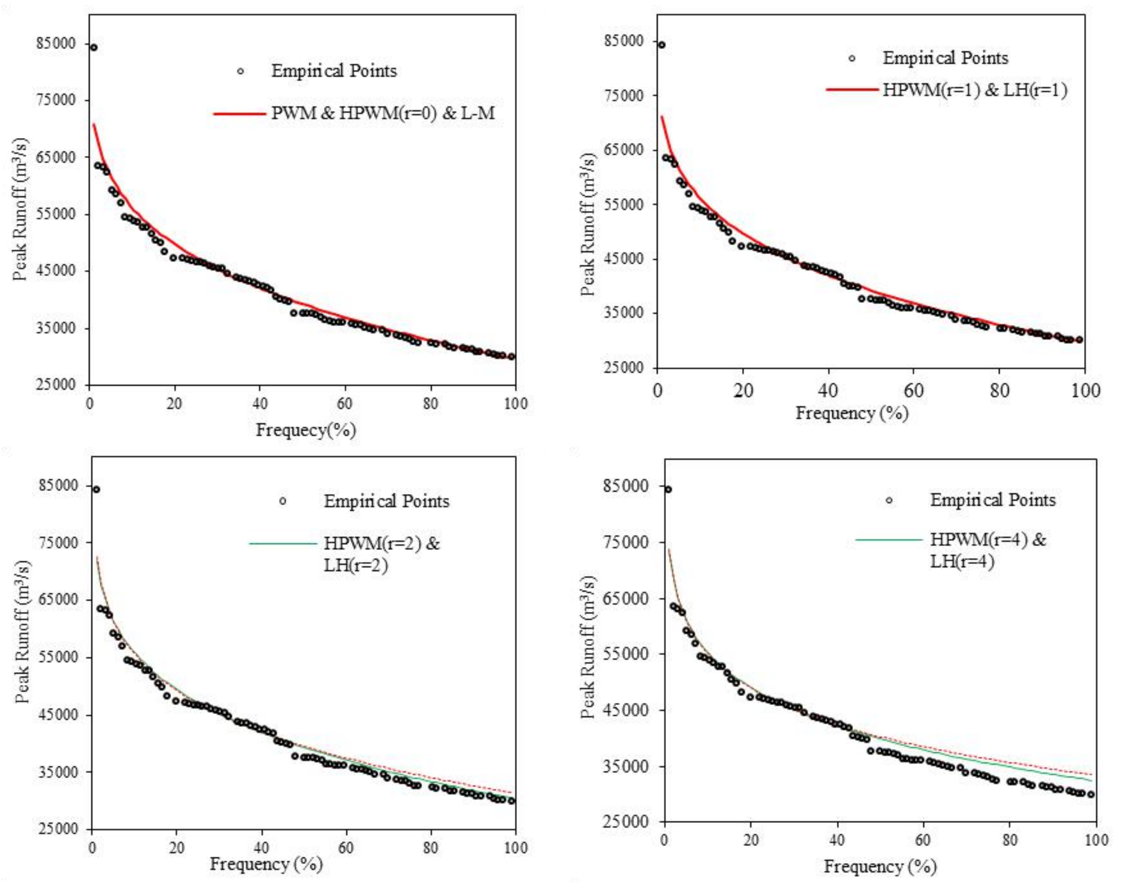

Figure 4. Relative diagrams of empirical points and theoretical curves, for which parameters estimate by all methods mentioned in this paper, take Cuntan gauged station as an example.

Table 4. Results of parameter estimation by several methods.

Parameters Estimators	PWM	Cuntan gauged station: HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.2547	-0.2547	-0.2414	-0.201	-0.1507	-0.101	-0.0575
σ	15280	15280	14966	13909	12540	11198	10051
μ	29566	29566	29710	30335	31305	32409	33478
Parameters Estimators	L-M	Cuntan gauged station: LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.2547	-0.2547	-0.2363	-0.1983	-0.1491	-0.1002	-0.0571
σ	15280	15280	14847	13838	12499	11175	10042
μ	29566	29566	29765	30377	31335	32428	33486
Parameters Estimators	PWM	Hnakou gauged station: HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.3297	-0.3297	-0.3025	-0.3065	-0.3146	-0.3235	-0.3325

σ	13921	13921	13367	13460	13669	13925	14203
μ	40874	40874	41118	41066	40932	40752	40542
Parameters Estimators	L-M	Hankou gauged station: LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
ζ	-0.3297	-0.3297	-0.2977	-0.3038	-0.3129	-0.3224	-0.3320
σ	13921	13921	13271	13397	13625	13894	14186
μ	40874	40874	41160	41101	40960	40773	40554

Table 5 lists the means of WLS between theoretical curves (for which the parameters are estimated by different methods) and empirical points, especially large empirical points (for which the frequency is not over 50% ($P(X) \le 50\%$)). According to the information shown in table 5 and figure 4, the performance of method does not significantly improve as higher PWM are used. It is clear that when the order ranges from 0 to 4, WLS decreases; however, it increases after the order is more than 4. The relative WLS shows the percentage of WLS of $P(X) \le 50\%$ takes in the sum WLS. Clearly to see that with the order increases, the relative WLS decreases dramatically. So the theoretical curves which parameters are estimated by higher moments' methods can fit with the large flood points better.

Table 5. Sum relative square variations (WLS) between flood peaks and theoretical values.

WLS Methods	PWM	Cuntan gauged station: HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
WLS ($P \le 50\%$)	0.058726	0.058726	0.056927	0.052042	0.04736	0.044855	0.045356
WLS (Summary)	0.077692	0.077692	0.077314	0.088834	0.137434	0.238788	0.38688
Relative WLS	75.59%	75.59%	73.63%	58.58%	34.46%	18.78%	11.72%
WLS Methods	L-M	Cuntan gauged station: LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
WLS ($P \le 50\%$)	0.058726	0.058726	0.056262	0.05173	0.046637	0.044823	0.045373
WLS (Summary)	0.077692	0.077692	0.0773178	0.090045	0.13554	0.240924	0.388189
Relative WLS	75.59%	75.59%	72.77%	57.45%	34.41%	18.6%	11.69%
WLS Methods	PWM	Hankou gauged station: HPWM					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
WLS ($P \le 50\%$)	0.02999	0.02999	0.02713	0.02742	0.02774	0.02776	0.02746
WLS (Summary)	0.03513	0.03513	0.03377	0.03357	0.03277	0.03184	0.03134
Relative WLS	85.37%	85.37%	80.34%	81.68%	84.68%	87.19%	87.62%
WLS Methods	L-M	Hankou gauged station: LH moments					
		$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$
WLS ($P \le 50\%$)	0.02999	0.02999	0.02667	0.02723	0.02649	0.02776	0.02746
WLS (Summary)	0.03513	0.03513	0.03367	0.03370	0.03137	0.03192	0.03137
Relative WLS	85.37%	85.37%	79.21%	80.80%	84.44%	86.97%	87.54%

Note: Relative WLS = WLS ($P \le 50\%$) / WLS (summary).

6. Conclusions

Several estimators, like PWM and L-Moments, were first introduced in this paper. In addition, the GPD-based approximate calculation methods of HPWM and LH moments were mainly deduced. A statistical experiment and a case study were applied to investigate and the WLS was the feasible measurement of these methods. Following conclusions can be drawn from the results of this study.

- Both the statistical experiment and case study proved that the GPD-based HPWM and LH moments deduced in this paper were feasible and applicable in parameter estimation.
- When the same order PWM were used in HPWM and LH moments methods, the results were identical. For example, the HPWM method (when $r=0$) showed similar performance to the methods of PWM, L-Moments and the LH moments (when $r=0$); the LH moments method (when $r \geq 1$) performed as well as the HPWM method (when $r \geq 1$) when the same PWM were used in calculation.
- However, from the standpoint of WLS which was used to measure the goodness-of-fit of each method, the order of the PWM affected the performances of the higher moments' methods. For example, when the order r increased from 0 to 4, the performances of the HPWM method and LH moments method became better. But when r exceeded 4, the performances of either the HPWM method or LH moments method became noticeably worse.
- The methods with higher moments tended to show more flexibility in fitting flood series on the up tail for example, the theoretical curves fit well with the empirical points which frequency were less than 50%.
- Comparing all of the methods outlined in this paper, it became clear that the HPWM method showed the same performance as the LH moments method and they used much more information in the flood series than the PWM method and L-Moments method. However, unlike LH moments method, the HPWM method was much easier to calculate. So HPWM method is a more preferable one.

References

[1] Wang J and Song S 2010 Application of generalized Pareto distribution in flood series frequency analysis *J. Northwest A. & F. Univ. (Nat. Sci. Ed.)* **38** 191-6 (in Chinese)
[2] Bermudez P D Z and Kotz S 2010 Parameter estimation of the generalized Pareto distribution part - II *J. Stat. Plan. Infer.* **140** 1374-88
[3] Diedda R and Puliga M 2009 Performances of some parameter estimators of the generalized Pareto distribution over round-off samples *Phys. Chem. Earth* **34** 626-34
[4] Lang M, Ouarda T B M J and Bobée B 1999 Towards operational guidelines for over- threshold modeling *J. Hydrol.* **225** 103-17
[5] Hosking J R M 1990 L-moments: Analysis and estimation of distributions using linear combinations of order statistics *J. Roy. Stat. Soc. B.* **52** 105-24
[6] Wang Q J 1997 LH moments for statistical analysis of extreme events *Water Resour. Res.* **33** 2841-8
[7] Meshgi A and Khalili D 2009 Comprehensive evaluation of regional flood frequency analysis by L- and LH moments I. A re-visit to regional homogeneity *Stoch. Environ. Res. Risk Assess.* **23** 119-35
[8] Busababodhin P, Seo Y A, Park J, *et al* 2016 LH-moment estimation of Wakeby distribution with hydrological applications *Stoch. Environ. Res. Risk Assess.* **30** 1757-67
[9] Pickands J 1975 Statistical inference using extreme order statistics Ann Stat **3** 119-31
[10] Wang Q J 1997 Using higher probability weighted moments for flood frequency analysis *J. Hydrol.* **194:** 95-106
[11] Song S, Li Y and Kang Y 2012 Calculation higher probability weighted moments for generalized Extreme value in hydrology *Adv. Mater. Res.* **518-523** 4015-21 (in Chinese)
[12] Li Y and Song S 2012 Application of higher probability weighted moments of P-III distribution to flood frequency analysis *J. Hydroele. Eng.* **33** 10-8 (in Chinese)

[13] Pilon P J, Adamowski K and Alila Y 1991 Regional analysis of annual maxima precipitation using L-moments *Atmos. Res.* **27** 81-92

[14] Hosking J R M and Wallis J R 1993 Some statistics useful in regional frequency analysis *Water Resour. Res.* **29** 271-81

[15] Chen Y, Wang Q, Sha Z, *et al* 2002 Application of L-moments based regional flood frequency analysis method to middle and lower Yangtze River *J. Hohai Univ. (Nat. Sci. Ed.)* **31** 207-11 (in Chinese)

[16] Bhuyan A, Borah M and Kumar R 2010 Regional flood frequency analysis of North-bank of the river Brahmaputra by using LH-Moments *Water Resour. Manag.* **24** 1779-90

[17] Hewa G A, Wang Q J, McMahon T A, *et al* 2007 Generalized extreme value distribution fitted by LH moments for low-flow frequency analysis *Water Resour. Res.* **430** 1-10

[18] Murshed M S, Park B J, Jeong B Y, *et al* 2009 LH-Moments of some distributions useful in hydrology *Communications of the Korean Statistical Society* **16** 647-58

[19] Cannane C 1978 Unbiased plotting positions - a review *J. Hydrol.* **37** 205-22

[20] Yang S L, Zhang J and Xu X J 2007 Influence of the Three Gorged Dam on downstream delivery of sediment and its environmental implications, Yangtze River *Geographs. Res. Lett.* **34** Doi: 10.1029/2007GL029472

[21] Fang B, Guo S L, Chai X L, *et al* 2005 Flood frequency analysis based on FPOT method *Water Power* **31** 9-12 (in Chinese)