A multigrid preconditioned algorithm for 8-node hexahedron combined hybrid element

Huiling Wang, Yufeng Nie*

Research Center for Computational Science, Xi'an 710129, P. R. China

*Corresponding author, email: yfnie@nwpu.edu.cn

Abstract. Multigrid preconditioned conjugate gradient method is proposed for the algebraic systems resulting from two 8-node hexahedron combined hybrid elements with high performance for the linear elasticity problem. Numerical computations are presented, which demonstrate the convergence and effectiveness of the method.

1 Introduction

Combined hybrid element method, a stable finite element discrete method, was firstly put forward by Zhou [7, 8]. The method was firstly applied to two-dimensional elasticity problem in [9] and four types of combined hybrid quadrilateral elements have been constructed there. Moreover, Nie further developed the method to three-dimensional problem, and two kinds of 8-node hexahedral elements with high performance were obtained in [10]. But there is few corresponding work for the efficient solvers of the discrete system from the method.

Recently multigrid methods have been considered for the linear elasticity equation, using the combined hybrid quadrilateral elements. On the basis of the proof idea in [2], [6] has proved that W-cycle, with sufficiently many smoothing steps on each level, converges in L_2 norm. In addition, variable V-cycle multigrid preconditioners have been proposed in [5], based on two simple intergrid transfer operators. And [5] testified that the condition numbers of the preconditioned systems are all uniformly bounded, according to the abstract theory in [1]. The objective of this paper is to develop the preconditioned conjugate gradient algorithm for two kinds of 8-node hexahedron combined hybrid elements with high performance in numerical aspects. That is to say, we construct various multigrid preconditioners for the linear systems discretized by the elements, based on the trilinear interpolation operator. The cantilever beam problem is given and the numerical results show that the method has a good convergence.

The remainder of this paper is organized as follows. The basic concept of combined hybrid element method is briefly stated in Section 2. The preconditioned conjugate gradient method is described in Section 3. In Section 4, several multigrid preconditioners are applied to the linear systems arisen from the two 8-node hexahedron combined hybrid elements of a simple cantilever beam problem. A conclusion is given in Section 5.

2 Combined hybrid element

Combined hybrid element method is based on combined hybrid variational principle which is formulated by the weighted average of domain decomposed Hellinger-Reissner principle and its dual. An important feature of this method is that Babuska-Brezzi condition [3] can hold automatically if the displacement space is weakly compatible [9]. Hence, it is a stable hybrid element method. Because the restriction of the condition is avoided, it offers a wider range of the optimization spaces for field variables.

Give an initial hexahedron subdivision $\Gamma_1 = \{\Omega\}$. By connecting the midpoints of the corresponding edges of the hexahedrons in Γ_{k-1}, a hexahedron subdivision Γ_k is obtained. Let $h_k = \max\{diam\Omega_j : \Omega_j \in \Gamma_k\}$ denote the mesh size. By this way, we have a family of hexahedron subdivisions of Ω: $\{\Gamma_k : 1 \le k \le J\}$.

Assume U_k and V_k be displacement and stress discrete spaces, respectively. The discrete formulation of linear elasticity problem by combined hybrid element is as follows: find $(\sigma_k, u_k) \in V_k \times U_k$ such that

$$\begin{cases} \alpha s_k(\sigma_k,\tau) - \alpha b_{2,k}(\tau,u_k) + b_{1,k}(\tau, u_k - T_c(u_k)) = 0, \\ \qquad \forall \tau \in V_k, \\ \alpha b_{2,k}(\sigma_k, v) - b_{1,k}(\sigma_k, v - T_c(v)) + (1-\alpha) d_k(u_k, v) \\ \qquad = (f, v), \forall v \in U_k, \end{cases} \tag{1}$$

where

$$s_k(\sigma,\tau) = \sum_{\Omega_j \in \Gamma_k} \int_{\Omega_j} \sigma \cdot D^{-1}[\tau] d\Omega_j,$$

$$d_k(u,v) = \sum_{\Omega_j \in \Gamma_k} \int_{\Omega_j} \varepsilon(u) \cdot D[\varepsilon(v)] d\Omega_j,$$

$$b_{2,k}(\tau,v) = \sum_{\Omega_j \in \Gamma_k} \int_{\Omega_j} \tau \cdot \varepsilon(v) d\Omega_j,$$

$$b_{1,k}(\tau,v) = \sum_{\Omega_j \in \Gamma_k} \oint_{\partial\Omega_j} (\tau \cdot n) \cdot v ds,$$

$\forall v \in U_k$, $T_c(v) := v_c$, v_c is the conforming part of v and n represents the unit outer normal to $\partial\Omega_j$.

Concretely, Wilson's interpolation space in three-dimensional case is adopted as U_k.

In order to achieve the high performance of the above hybrid scheme, the stress space is subjected to the constraint of energy-compatible condition [9], i. e. $\forall \tau \in V_k$, $v \in U_k$,

$b_{1,\Omega_j}(\tau, v - T_c(v)) := \oint_{\partial\Omega_j} \tau \cdot n \cdot (v - T_c(v)) ds = 0$.

Hence, the following two discrete spaces $H_{0-1,k}$ and $H_{0-1+,k}$ are given:

$$H_{0-1,k} := \{\tau \in H_{1,k} \big| b_{1,\Omega_j}(\tau, v - T_c(v)) = 0\}, \forall v \in U_k,$$

where

$$H_{1,k} := \{\tau, \tau\big|_{\Omega_j} = \hat{\tau} \circ F_j^{-1}, \forall \Omega_j \in \Gamma_k\},$$

$$\hat{\tau} = [I_6 \; I_6\xi \; I_6\eta \; I_6\zeta] := \hat{N}_\sigma q_\tau, q_\tau \in R^{24},$$

$$H_{0-1+,k} := \{\tau \in H_{1+,k} \big| b_{1,\Omega_j}(\tau, v - T_c(v)) = 0\}, \forall v \in U_k,$$

where

$$H_{1+,k} := \{\tau, \tau\big|_{\Omega_j} = \hat{\tau} \circ F_j^{-1}, \forall \Omega_j \in \Gamma_k\},$$

$$\hat{\tau} = [I_6 \; I_6\xi \; I_6\eta \; I_6\zeta \; I_6\xi\eta \; I_6\eta\zeta \; I_6\zeta\xi] := \hat{N}_\sigma q_\tau, q_\tau \in R^{42},$$

I_6 is the unit matrix and F_j is the isoparameteric mapping from the referential square $[-1,1]\times[-1,1]\times[-1,1]$ to Ω_j.

The combinations $U_k \times H_{0-1,k}$ and $U_k \times H_{0-1+,k}$ correspond to the two 8-node hexahedron combined hybrid elements CHH(0−1) and CHH(0−1)+. Meanwhile (1) can be simplified as follows: to find $(\sigma_k, u_k) \in V_k \times U_k$ such that

$$\begin{cases} \alpha s_k(\sigma_k,\tau) - \alpha b_{2,k}(\tau,u_k) = 0, \forall \tau \in V_k, \\ \alpha b_{2,k}(\sigma_k,v) + (1-\alpha) d_k(u_k,v) = (f,v), \forall v \in U_k. \end{cases} \quad (2)$$

$s_k(\tau,\tau)$ is positive definite, so the stress can be expressed linearly by the displacement on each element. In consequence, by eliminating the stress parameters, a final linear system containing the displacement variables only can be generated: Find $u_k \in U_k$, such that

$$A_k u_k = f_k \quad (3)$$

Obviously, the scheme (3) is equivalent to (2).

3 Multigrid preconditioned conjugate gradient method

In this section, we give the multigrid preconditioned conjugate gradient method which is a preconditioned conjugate gradient method with multigrid method as a preconditioner. First of all, multigrid method is described.

3.1 Multigrid method

Multigrid method, as a very effective iterative solver method, is used for solving a linear equation system

$$A_k v = b. \quad (4)$$

The multigrid cycle (iteration) looks like:

1. Pre-smoothing: The original vector is v_0. Iterate on (4) to reach v_1 by $m(k)$ traditional iteration methods.
2. Compute the residual: $r_k = b - A_k v_1$.
3. Restriction: restrict the residual to the coarser grid by $r_{k-1} = R r_k$, where R is the restriction matrix from the fine grid to the coarser grid.
4. Solve the coarse grid error equation

$$A_{k-1} e = r_{k-1}. \quad (5)$$

If $k-1$ is the coarsest level, (5) is solved by the direct method. Otherwise, give the initial value $e_0 = 0$ and obtain a multigrid approximation solution e_1 by γ recursive calls of multigrid method on $k-1$ level. If $\gamma = 1$, it is V-cycle multigrid method. If $\gamma = 2$, it is W-cycle multigrid method.

5. Prolongation: The coarser grid error e_1 is interpolated to the fine grid by $v_2 = P e_1$, where P is the interpolation matrix.
6. Correction: the vector v_1 obtained in the pre-smoothing process is corrected as: $v_3 = v_1 + v_2$.
7. Post-smoothing: perform $m(k)$ iterations with the initial value v_3 on the fine grid, then a multigrid approximation v_4 is generated.

Remark If the iteration number on the smoothing step increases exponentially with the decrease of the grid number in V cycle, it is the variable V-cycle multigrid method.

3.2 Multigrid preconditioned conjugate gradient method

Multigrid preconditioned conjugate gradient method uses the multigrid method as a preconditioner for conjugate gradient method.

Here $Au = f$ is concerned. Multigrid preconditioned conjugate gradient method is explained in the following:

Assume $M = LL^T$, where L is a nonsingular matrix. Let $\tilde{A} = L^{-1}AL^{-T}$, $\tilde{u} = L^T u$ and $\tilde{f} = L^{-1} f$. Then solve $\tilde{A}\tilde{u} = \tilde{f}$ using the plain conjugate gradient method:

u_0 is an initial value, then an initial residual r_0 is $r_0 = f - Au_0$. $Ms_0 = r_0$ is relaxed by the multigrid method and an initial direction vector is $p_0 := s_0$.

$i = 0$
While (! Convergence) {
$\alpha_i = (s_i, r_i)/(p_i, Ap_i)$,
$x_{i+1} = x_i + \alpha_i p_i$,
$r_{i+1} = r_i - \alpha_i Ap_i$,
Convergence test,
Relax $Ms_{i+1} = r_{i+1}$ using the multigrid method,
$\beta_i = (s_{i+1}, r_{i+1})/(s_i, r_i)$
$p_{i+1} = s_{i+1} + \beta_i p_i$,
$i++$.
}

4 Numerical example

In this section, based on the trilinear interpolate operator (a generalization of the bilinear interpolation operator to three-dimension), we will consider multigrid preconditioner for the linear systems resulting from combined hybrid hexahedron elements discretization of the following cantilever beam problem.

Example The cantilever beam problem [4].

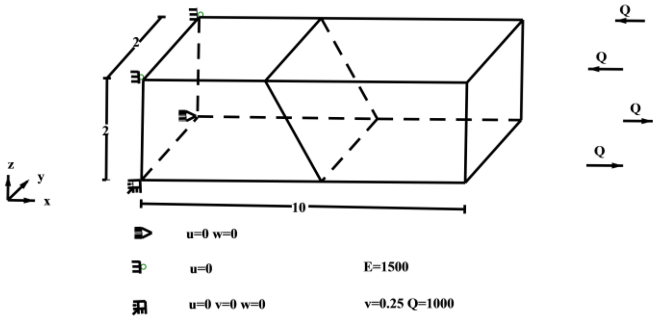

Figure 1: the cantilever beam problem

Firstly, the equation is discretized by CHH(0−1) and CHH(0−1)$^+$. Then the preconditioned conjugate gradient (PCG) method is applied to the discrete systems. Here V-cycle, W-cycle and the variable V-cycle multigrid methods are adopted as the preconditioners respectively. We take symmetric Gauss-Seidel iterations for the smoother, the zero vector for the initial value and trilinear interpolation operator for the prolongation operator in the multigrid method. The smoothing number is 2^{J-k} on grid level k $(1 \le k \le J)$ in variable V-cycle multigrid method. In the following tables, the iterations number and the CPU time (in seconds) needed are listed as the relative error of the residual is reduced by a factor of 10^{-5}.

Table 1: variable V-cycle PCG for CHH (0−1) and CHH (0−1)+

	Grid	2×4×4	4×8×8	8×16×16	16×32×32
	Dofs	135	675	4131	28611
CHH(0-1)	Iter	18	15	17	17
	Time	0.05	0.20	4.16	115.17
$CHH(0\text{-}1)^+$	Iter	18	15	17	17
	Time	0.05	0.20	4.08	112.97

Table 2: W-cycle PCG for CHH (0−1) and CHH (0−1) + with one smoothing step

	Grid	2×4×4	4×8×8	8×16×16	16×32×32
	Dofs	135	675	4131	28611
CHH(0-1)	Iter	18	15	15	15
	Time	0.05	0.25	3.80	103.25
$CHH(0\text{-}1)^+$	Iter	18	15	16	15
	Time	0.05	0.23	4.10	107.81

Table 3: V-cycle PCG for CHH (0−1) and CHH (0−1) + with one smoothing step

	Grid	2×4×4	4×8×8	8×16×16	16×32×32
	Dofs	135	675	4131	28611
CHH(0-1)	Iter	18	18	21	23
	Time	0.05	0.27	4.63	141.92
$CHH(0\text{-}1)^+$	Iter	18	18	22	24
	Time	0.05	0.23	4.96	154.92

Table 4: V-cycle PCG for CHH (0−1) and CHH (0−1) + with two smoothing step

	Grid	2×4×4	4×8×8	8×16×16	16×32×32
	Dofs	135	675	4131	28611
CHH(0-1)	Iter	12	13	16	18
	Time	0.03	0.25	5.45	180.08
$CHH(0\text{-}1)^+$	Iter	12	13	16	18
	Time	0.03	0.27	5.76	183.73

Seen from the tables, various PCG methods show a quick convergence for CHH(0-1) and $CHH(0\text{-}1)^+$. The results in Table 1 and 2 demonstrate that W-cycle PCG need less time and iteration number than variable V-cycle PCG, in spite of the same smoothing number in the two methods. Moreover, from Table 3 and 4, it is obvious that the iteration number is dropped sharply but more CPU time is lost, with the increase of the smoothing number.

5 Conclusion

Based on the trilinear interpolate operator, we present the preconditioned conjugate gradient method with various multigrid preconditioners for the two 8-node hexahedron combined hybrid elements with high performance. The results of the numerical example exhibit that the method is convergent and efficient.

Acknowledgements

This research is supported by National Natural Science Foundation of China (Nos. 11471262, 11501450).

References

[1] J. H. Bramble, J. E. Pasciak, J. C. Xu. “The analysis of multigrid algorithms with nonnested spaces or noninherited quadratic forms”, *Math. Comput.* **56**, pp. 1-34, (1991).
[2] S. C. Brenner. “Nonconforming mixed multigrid method for the pure displacement problem in planar linear elasticity”, *Siam J. Numer. Anal.*, **30**, pp. 116-135, (1993).
[3] F. Brezzi, M. Fortin. “Mixed and hybrid finite element method”, *Springer-Verlag*, Berlin, 1991.
[4] W. J. Chen, Y. K. Cheung. “Three-dimensional 8-node and 20-node refined hybrid isoparameteric elements”, *Int. J. Numer. Meth. Eng.*, **35**, pp. 1871-1889, (1992).
[5] H. L. Wang, Y. F. Nie. “A variable V-cycle preconditioner for the combined hybrid element discrete system on quadrilateral meshes with two intergrid transfer operators”, *Unpublished results*.
[6] H. L. Wang, Y. F. Nie, J. G. Wang, Z. B. Yuan. “The multigrid method for combined hybrid quadrilateral elements of linear elasticity problem”, *Unpublished results.*
[7] T. X. Zhou. “Finite element method based on combination of saddle point variational formulation”, *Sci. China Ser. E.,* **40**, pp. 285–300, (1997).
[8] T. X. Zhou. “Stabilized hybrid finite element methods based on the combination of saddle point principles of elasticity problems”, *Math. Comput.,* 72, pp. 1655-1673, (2003).
[9] T. X. Zhou, Y. F. Nie. “Combined hybrid approach to finite element schemes of high performance”, *Int. J. Numer. Meth. Eng*, **51**, pp. 181-202, (2001).
[10] T. X. Zhou, Y. F. Nie. “8-node hexahedron combined hybrid element with high performance”, *J. Num. Method. Comp. Appl.,* **3**, pp.231-240, (2003).