Non-Archimedean mathematical analysis methods in description of deformation of structurally inhomogeneous geomaterials

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Abstract. Under analysis is an approach to mathematical modeling of structurally inhomogeneous rocks considering structural hierarchy and internal self-balanced stresses. The fields of stresses and strains at various scale levels of rock mass medium are characterized using the non-Archimedean analysis methods. It is shown that such model describes accumulationtion of elastic energy in the form of internal self-balanced stresses on a micro-scale. The finite element algorithm and a computer program are developed to solve plane boundary-value problems. The calculated data on compression of a rock specimen are reported. The paper shows that the behavior of plastic strain zones largley depends on the pre-set initital micro-stresses.

1. Introduction

One of the key problems of geomechanics is calculation of stresses and strains in a unit element of rock mass. The body forces and the boundary conditions are pre-set. In the meanwhile some part of internal stresses in a self-stressing rock mass can be mutually balanced. And boundary conditions offer no information on them in this case. So, it is required to know the body forces, the boundary conditions and the conditions of how the rock mass was formed. Some information on self-balancing stresses is obtained experimentally, using special testing procedures [2, 3]. These data are useful in selecting parameters for a mathematical model of self-balancing stresses. One of the modeling methods introduces hypotheses on microdeformation of a medium, then uses averaging, and for the averaged values the continuum equations are derived. This approach assumes that the unit volume of the medium is subject to uniform stresses and strains [4].

An alternative approach is based on using non-Archimedean variables. The model avoids the assumption of the uniform state of the unit volume. Equations of equilibrium and motion are formulated specifically for structural elements. The size of the elements is set in proportion to the value of $1/n$, where n is a sufficiently high natural number. Then, a discrete model is constructed. In the classical analysis, the next step is the analysis of the limit when $n \to \infty$ and, accordingly, $1/n \to 0$. As a result, the finite difference equations transform into the differential equations. In the non-Archimedean analysis, the number n tends not to infinity but to an actual infinitely high number ω. The size of a unit element tends to E. The value of E is smaller than any positive rational number but $E > 0$. For this reason, the finite difference equations transform not in the differential equation but in the actual infinitesimal difference equations. They are analyzable at a varying degree of accuracy, e.g. assuming that $E = 0$, $E^2 = 0$ or $E > 0$, $E^2 = 0$, etc. Sometimes, the replacement of

$1/n$ by E is of consequence. In the other situation, this replacement may of critical value. For instance, when the problem source parameters already include the values E, ω, or when new measurements of space and time on a macrolevel are included. Here we mean measurements included in the structure of a real number as a real number is not only a multiscale straight line segment of the length E but an infinite entity attributed with a diameter of the order of E (conclusion of Cantor's concept of number [5]).

2. Mathematical model

This paper describes the analysis of the model of self-stressing rock mass (Figure 1, plane strain) [6, 7]. Initially, elastic particles are at the points of a square lattice. Sliding between the particles is possible in accordance with the laws of plasticity, dry or viscous friction, or their combinations. Along the diagonal, the particles are connected by elastic elements (a porous medium is ether elastic or of Vickers type). Forces applied to the elastic elements can be balanced by the forces applied to contacts, consequently, internal self-balancing stresses can be considerable in case that external stresses are absent.

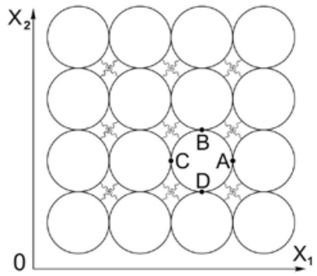

Figure 1. Model of a structurally nonuniform medium capable to accumulate elastic energy.

Any mathematical model should satisfy certain consistency constraints, e.g. to convert into model of the linear elasticity theory in a special case. This offers a way for further generalizations. Let $2L$ be the size of a domain subject to deformation and $2L/n$ –be the size of a structural element of the domain (particle). It is assumed that particles are linearly elastic, and there are no pores and sliding at contacts of the particles is absent. Furthermore, it is assumed that point moments cannot propagate through the contacts. In this case, there are four force vectors $\overline{F}$ and four displacement vectors $\overline{u}$ for each particle. One vector conforms with two scalar components (plane strain). Temperature and other parameters can also be added. The total amount of the scalar unknowns is: $4n(n-1)$—displacements at contacts, $4n(n-1)$—forces at contacts, $8n$—displacement at boundaries, $8n$—forces at boundaries. Thus, we have $8n^2+8n$ variables.

Regarding the equations, two equations of equilibrium and one equation of moments are fulfilled for each particle in the set n^2, i.e. $3n^2$ equations all in all. Then, at each of $4n$ boundary contacts, two conditions should be set (either forces or displacements, or combination), which gives $8n$ conditions altogether. A key point is the constitutive relations. The four points A, B, C, D agree with the four vectors of displacement—eight degrees of freedom. Constitutive relations may only include such combinations of degrees of freedom, which are independent of translation and rotation of a particle, which means that three degrees of freedom are to be withdrawn. Consequently, there are five invariant combinations of displacements. The forces also have eight degrees of freedom. The vector of sum of forces and the moment are to be zero. Thus, constitutive relations connect five invariant combinations of displacements and five force characteristics. There should be five constitutive relations for one particle and $5n^2$ constitutive relations for n^2 particles. This totals $8n^2+8n$ equations. Balance has converged: the system is closed and reduces to $8n^2+8n$ algebraic equations in terms of $8n^2+8n$ unknowns.

Now, a unit volume needs five constitutive relations. On the other hand, there are merely three constitutive relations for an elastic body. This means that the classical theory contains assumptions that match two equations, moreover, these equations are of the same significance as the equations involved in the Hooke law. Let us formulate them explicitly.

First, we select five invariant combinations of displacements for the constitutive relations. The number of decision is unlimited. We chose the closest variant to the linear elasticity

$$\begin{aligned} e_{11} &= \frac{\delta u_1}{\delta x_1} = \frac{u_1(A) - u_1(C)}{2l}, \\ e_{22} &= \frac{\delta u_2}{\delta x_2} = \frac{u_2(D) - u_2(B)}{2l}, \\ e_{12} &= \frac{1}{2}\left(\frac{\delta u_1}{\delta x_2} + \frac{\delta u_2}{\delta x_1}\right) = \frac{u_1(D) - u_1(B)}{4l} + \frac{u_2(A) - u_2(C)}{4l}, \end{aligned} \tag{1}$$

$$\begin{aligned} \kappa_1 &= \frac{u_1(C) + u_1(A)}{4l} - \frac{u_1(D) + u_1(B)}{4l}, \\ \kappa_2 &= \frac{u_2(D) + u_2(B)}{4l} - \frac{u_2(C) + u_2(A)}{4l}. \end{aligned} \tag{2}$$

Here, δ denotes increment within a particle. Now, it is possible to re-write the equations of the classical elasticity theory as

$$\begin{aligned} &e_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}), \quad e_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}), \\ &e_{12} = \frac{1+\nu}{E}\sigma_{12}, \quad \kappa_1 = 0, \quad \kappa_2 = 0, \end{aligned} \tag{3}$$

where E, ν–const; $\sigma_{ij} = F_{ij}/2l$, $\sigma_{12} = \sigma_{21}$, $i, j = 1, 2$. The first three equations are the Hooke law. The last two equations are not formulated explicitly in the classical theory, although it contains some information about them. Where? The classical elasticity theory postulates diffeomorphism [8]. In other words, it is assumed that all functions are sufficiently smooth. This means that any local functions can be represented by a linear function, for instance

$$\begin{aligned} u_1 &= a_{11}x_1 + a_{12}x_2, \\ u_2 &= a_{12}x_1 + a_{22}x_2, \end{aligned} \tag{4}$$

where $a_{11}, ..., a_{22}$ are constants. Placing (4) in (2) yields $\kappa_1 \equiv 0$, $\kappa_2 \equiv 0$. The converse is valid, too. Consequently, the wanted two equations from (3) are “hidden” in the postulate of diffeomorphism. Technically, this allows transition from “derivatives” within the limits of a particle, $\delta / \delta x_i$, to the ordinary derivatives $\partial / \partial x_i$, i.e. to classical theory. Accordingly, the consistency condition for the models under analysis is satisfied.

In case of no diffeomorphism, it seems relevant to use the non-Archimedean analysis representation

$$\begin{aligned} u_1 &= u_1(x_1, x_2, \xi_1, \xi_2), \\ u_2 &= u_2(x_1, x_2, \xi_1, \xi_2). \end{aligned} \tag{5}$$

Here, the fixed values x_1, x_2 are the centers of particles, and ξ_1, ξ_2 are the coordinates within a fixed particle. The nonlinearity of the functions with respect to the coordinates ξ_1, ξ_2 agrees with the case when $\kappa_1 \neq 0$, $\kappa_2 \neq 0$. In a general case of allowable sliding at contacts and eigen stresses, the situation will be similar. From the viewpoint of mechanics, the model in Figure 1 conforms with the parallel connection of elastic and plastic elements in Figure 2.

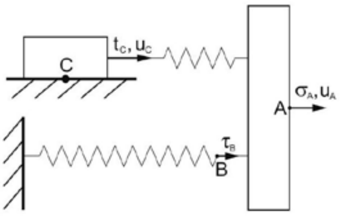

Figure 2. Mechanical model of a medium having structure.

The macrostress σ_A matches the stress τ_B in a porous medium and the stress t_C at the particle contacts. Depending on $t_C - u_C$ curve and on the values of the elastic constants, the $\sigma_A - u_A$ curve may have either ascending or descending branches. Of special interest is the case when $t_C = p$, $\tau_B = -p$, and, thus, $\sigma_A = 0$: the element is subject to self-stressing. Depending on the value of the eigen stresses p, the $\sigma_A - u_A$ curve behaves differently: it is possible that $\Delta\sigma_A > 0$, $\Delta u_A > 0$, i.e., the curve first ascends and then descends. Under high p it can be that $\Delta u_A > 0$ while $\Delta\sigma_A < 0$. In this case, the element behaves as a source of energy.

We introduce micro-variables to describe averaged micro-strains and micro-stresses in grains, $\begin{pmatrix} \varepsilon_{11}^t \\ \varepsilon_{22}^t \\ \varepsilon_{12}^t \end{pmatrix} = T \cdot \begin{pmatrix} t_{11} \\ t_{22} \\ t_{12} \end{pmatrix}$, $\begin{pmatrix} \varepsilon_{11}^\tau \\ \varepsilon_{22}^\tau \\ \varepsilon_{12}^\tau \end{pmatrix} = T \cdot \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{pmatrix}$, in porous medium, $\begin{pmatrix} \varepsilon_{11}^p \\ \varepsilon_{22}^p \\ \varepsilon_{12}^p \end{pmatrix} = P \cdot \begin{pmatrix} p_{11} \\ p_{22} \\ p_{12} \end{pmatrix}$, at the grain contacts, $\begin{pmatrix} \varepsilon_{11}^R \\ \varepsilon_{22}^R \\ \varepsilon_{12}^R \end{pmatrix} = R \cdot \begin{pmatrix} t_{11} \\ t_{22} \\ t_{12} \end{pmatrix}$ (Figure 3), where the third order square matrixes T, P, R depend, respectively, on elastic constants of grains and porous medium, and on moduli of plastic sliding at the contacts of grains. The inter-grain sliding curve is approximated by a piecewise-linear function as is shown in Figure 4 [9, 10]. The introduced values obey the consistency conditions given by

$$\begin{pmatrix} \overset{*}{\sigma}_{11} \\ \overset{*}{\sigma}_{22} \\ \overset{*}{\sigma}_{12} \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{22} \\ t_{12} \end{pmatrix} + 2 \cdot m \cdot \begin{pmatrix} p_{11} \\ p_{22} \\ p_{12} \end{pmatrix} = \begin{pmatrix} t_{11} \\ t_{22} \\ t_{12} \end{pmatrix} + 2 \cdot (1-m) \cdot \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{pmatrix}, \quad (6)$$

$$\begin{pmatrix} \overset{*}{\varepsilon}_{11} \\ \overset{*}{\varepsilon}_{22} \\ \overset{*}{\varepsilon}_{12} \end{pmatrix} = \begin{pmatrix} \varepsilon_{11}^t \\ \varepsilon_{22}^t \\ \varepsilon_{12}^t \end{pmatrix} + \begin{pmatrix} \varepsilon_{11}^R \\ \varepsilon_{22}^R \\ \varepsilon_{12}^R \end{pmatrix} = (1-m) \cdot \begin{pmatrix} \varepsilon_{11}^\tau \\ \varepsilon_{22}^\tau \\ \varepsilon_{12}^\tau \end{pmatrix} + m \cdot \begin{pmatrix} \varepsilon_{11}^p \\ \varepsilon_{22}^p \\ \varepsilon_{12}^p \end{pmatrix}. \quad (7)$$

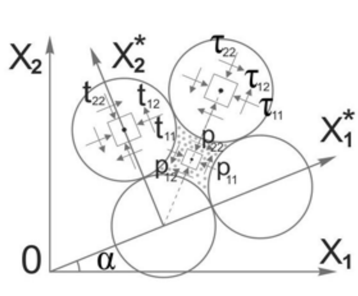

Figure 3. Internal structure of the medium (orientations of effective forces).

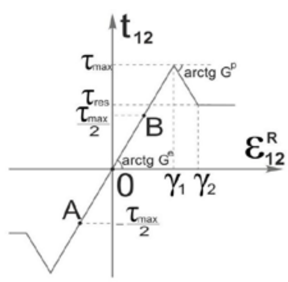

Figure 4. Inter-grain sliding curve.

Here, $\overset{*}{\varepsilon}_{ij}, \overset{*}{\sigma}_{ij}$ are the components of the averaged macro-strains and macro-stresses in the local coordinates connected with the orientation of the grain matrix, $0 < m < 1$ is understood as a characteristic of clear opening. After elimination of the micro-variables, the constitutive relations will be given by

$$\begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{pmatrix} = W \cdot \left((T+R)^{-1} + 2 \cdot (T+P)^{-1} \right)^{-1} \cdot W^{-1} \cdot \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad (8)$$

where $\varepsilon_{ij}, \sigma_{ij}$ are the components of the averaged macro-strains and macro-stresses in arbitrary rectangular coordinates; the matrix W is reflective of the orientation of the grain matrix relative to the rectangular coordinates and depends on α (refer to Figure 3).

Based on the described approach and the mathematical model of a structurally nonuniform rock mass (8), the finite element algorithm and software program are developed to solve 2D boundary value problems, considering internal self-balancing stresses.

3. Numerical model

This section exemplifies calculation of a geomaterial compression. Let a rectangular specimen (see Figure 5) be compressed between two parallel plates displaced vertically. The boundary conditions are given by

$$\begin{aligned} &\Delta u_2\big|_{\Gamma_1^+} = \Delta d, \quad \Delta u_2\big|_{\Gamma_1^-} = -\Delta d, \\ &\Delta u_1\big|_{\Gamma_1^+} = \Delta u_1\big|_{\Gamma_1^-} = \Delta\sigma_{11}\big|_{\Gamma_2} = \Delta\sigma_{12}\big|_{\Gamma_2} = 0, \end{aligned} \quad (9)$$

where Δu_i, $\Delta\sigma_{ij}$ are the increments in the displacements and stresses at the certain segment of the boundary; Δd is the increment of the loading parameter (due to nonlinearity, the problem is solved in the quasi-static formulation, by loading steps). Let the parameter of the orientation of the grain matrix be set as $\alpha = \pi / 4$ and the parameter of the clear opening—as $m = 0.5$. All stresses are referred to the value of a maximum shear stress at the inter-grain contacts, $\tau_{\max}$ (see Figure 4): $\tau_{\max} = 1$, $\tau_{res} = 0.4$. It is noteworthy that with such orientation of the grain structure, the influence of self-balancing shear stresses in the medium is the most pronounced.

The aim of the numerical modeling is to assess influence of the self-balancing (eigen) stresses. To this effect, three calculation series are performed for three different scenarios of setting the self-balancing stresses. In all series, the calculations end upon reaching one and the same value of the loading parameter, namely, $dH / H = -0.012$.

The first calculations series is performed for a geomaterial specimen without eigen self-balancing stresses, i.e. all components $t_{ij} = p_{ij} = 0$ at the start time and, consequently, $\sigma_{ij} = 0$. In this case, each point of the medium is in the position of the point O in the plot of the contact interaction of grains in Figure 4. Figure 5a illustrates the calculation results. Hereinafter, the unshaded areas show the state of strengthening at the contacts: $0 < t_{12} < 1$, $\varepsilon_{12}^R < \gamma_1$; the light-grey color marks the zones of weakening between grains: $0.4 < t_{12} < 1$, $\gamma_1 < \varepsilon_{12}^R < \gamma_2$; the dark-grey color depicts the regions of the residual shear strength: $t_{12} = 0.4$, $\varepsilon_{12}^R > \gamma_2$.

Let the start-time shearing be set as $t_{12} = -0.5$ at the contacts of grains and as $p_{12} = 0.5$ in the porous medium. The other components of the micro-stress tensors of grains and pores are assumed zero. In the plot in Figure 4, the pre-set initial conditions conform with the point A. Apparently, the internal micro-stresses balance each other so that all macro-stress components $\sigma_{ij} = 0$. The eigen stresses are set so that they "resist" compression. This situation is illustrated in Figure 5b. It is evident that the pre-set self-balancing stresses greatly decelerate growth of the plastic strain zones: these zones cover smaller domain as against the case of the absent self-balancing stresses under the same external force. Figure 5c shows the calculation results with the alternate self-balancing stresses $t_{12} = 0.5$, $p_{12} = -0.5$, which fall at the point *B* in the plot in Figure 4. In this case, these stresses "contribute" to compression. From the comparison of the deformation patterns obtained under equal external loading in Figures 5a–5c, the eigen stresses exert a considerable influence on evolution of plastic strain zones, and can either decelerate (Figure 5b) or accelerate (Figure 5c) expansion of plastic deformation and failure of specimens.

(a) (b) (c)

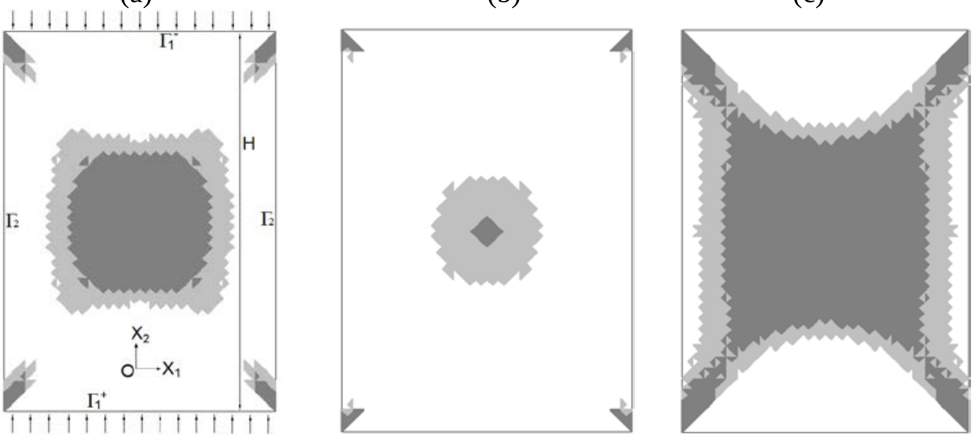

Figure 5. Plastic deformation zones: (a) without initial stresses included; (b) the initial stresses "resist" compression; (c) the initial stresses "contribute" to compression.

4. Conclusion

The proposed approach allows mathematical modeling of deformation of structurally nonuniform media, considering internal self-balancing stresses.

The internal self-balancing stresses have a considerable influence on the behavior of plastic deformation zones, and can both decelerate and accelerate failure.

5. Acknowledgements

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References

[1] Moroz AI 2004 *Self-Stress State of Rocks* Moscow: MGGU (in Russian)
[2] Birger IA 1963 *Residual Stresses* Moscow: Mashgiz (in Russian)
[3] *Organizational Standard STO 36554501-019-2009 Identification of Self-Stress State in Rocks NITs Stroitelstvo* Moscow (in Russian)
[4] Revuzhenko AF 2000 *Mechanics of Elastoplastic Media and Nonstandard Analysis* Novosibirsk: NGU (in Russian)
[5] Revuzhenko AF 2012 *Mathematical Analysis of Functions of Non-Archimedean Variable: Application-Specific Mathematical Tool for Description of Structural Levels in a Geomedium* Novosibirsk: Nauka (in Russian)
[6] Kurlenya MV, Oparin VN, Revuzhenko AF and Shemyakin EI 1987 Some feature of rock response to near-range blasting *Doklady AN SSSR* Vol 293 No 1
[7] Revuzhenko AF 1990 Rock as a medium with internal energy source and sinks. Report 1 *Journal of Mining Science* Vol 26 No 4 pp 301–308
[8] Trusov PV 2009 Some issues of the nonlinear mechanics of a deformable solid (Disputable) *Matemat. Model. Sistem Protsessov* 2009 No 17

[9] Lavrikov SV, Mikenina OA, Revuzhenko AF and Shemyakin EI 2008 Concept of non-Archimedean multiscale space and model of plastic media having structure *Fizich. Mezomekh.* Vol 11 No 3
[10] Lavrikov SV, Mikenina OA, Revuzhenko AF 2008 Rock mass deformation modeling using the non-Archimedean analysis *Journal of Mining Science* Vol 44 No 1 pp 1–14