The Discrete Fourier Transform on hexagonal remote sensing image

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Abstract: Global discrete grid system will subdivide the earth recursively to form a multi-resolution grid hierarchy with no Overlap and seamless which help build global uniform spatial reference datum and multi-source data processing mode which takes the position as the object and in the aspect of data structure supports the organization, process and analysis of the remote sensing big data. This paper adopts the base transform to realize the mutual transformation of square pixel and hexagonal pixel. This paper designs the corresponding discrete Fourier transform algorithm for any lattice. Finally, the paper   show the result of the DFT of the remote sensing image of the hexagonal pixel.

1. Introduction

With the rapid development of earth observation technology, Human obtain remote sensing information of big data characteristics, so remote data mining and application   becomes the focus of attention. Global discrete grid system will subdivide the earth recursively to form a multi-resolution grid hierarchy with no overlap and seamless which help build global uniform spatial reference datum and multi-source data processing mode which takes the position as the object and in the aspect of data structure supports the organization, process and analysis of the remote sensing big data. Only three basic polygons-triangles, rectangles and hexagons can seamlessly tile plane[2]. Since the adjacent triangles differ in directions, the triangle is not the optimal scheme for plane partition. So far the rectangular grid is the most commonly used . Because square pixel image is symmetry, a small amount of calculation ,easy to store and implement and many existing image processing algorithm. However, the image of square pixels also have many disadvantages, such as aliasing effects, quantization errors, inconsistency in adjacent cells and less relevance with visual process and so on. In the image processing , compared with square pixel ,hexagonal pixel have a higher degree of circular symmetry ,consistent contiguity, greater angular resolution, and require less storage space and computing[1,5]. Above all, this article will research remote-sensing image of hexagonal pixels. Since the Fourier transform has been widely used in feature extraction, spatial frequency and

filtering, image restoration and texture analysis. This article firstly describes a clear and simple DFT suitable for n-dimensional lattice L .The space domain of the DFT is a system of coset representatives of the quotient group generated by the lattice L and the sublattice of L. Then we apply this result on the image of hexagonal pixel.

2. The DFT of any lattice

This paper will introduce DFT which is suitable for any lattice. Mathematical set which can

undertake DFT   is a N order finite Abelian group D. Abelian group is D additive group.

$\widehat{D}$ denote the character group $\mathrm{Hom}(D, I_N)$ of group D. D denote space domain and $\widehat{D}$

denote frequency domain. Fourier transform is a powerful tool for spatial domain to frequency domain.

A lattice in n dimensional European space is a linear combination of linearly independent vector. Take these n vectors as base vector of the lattice. The row vectors of base matrix are base vectors of the lattice L.denote that L_0 is a n dimensional sublattice of the lattice L. Because the lattice L and the lattice L_0 are both Abelian group, The quotient group L/L_0 is also a finite Abelian group[3].

Given $L_0 < L$ and $u \in L$, $\overline{u}$ denote the coset of u in L/L_0.Similarly given $v \in L_0{}^*$, $\overline{v}$ denote the coset of v in $L_0{}^*/L^*$. Spatial domain is $D = L/L_0$ and frequency domain is $D^* = L_0{}^*/L^*$,and order $N = |G| = |G^*|$. the DFT of spatial domain into frequency domain is a linear transformation[3,4,6] , as shown in the following type:

$$(\mathcal{F}a)(\overline{v}) = \frac{1}{\sqrt{N}} \sum_{\overline{u} \in D} a(\overline{u}) e^{-2\pi i \langle u, v \rangle} \tag{1}$$

The inverse Fourier transformation:

$$(\mathcal{F}a^*)(\overline{u}) = \frac{1}{\sqrt{N}} \sum_{\overline{v} \in D^*} a^*(\overline{v}) e^{2\pi i \langle u, v \rangle} \tag{2}$$

Theorem If the quotient group L/L_0 have elementary divisor $N_1, N_2, \cdots, N_d$, DFT and inverse DFT can be reduced to computing multiple one dimensional DFT. The running time needed for this result is:

$$O \left(N \sum_{i=1}^{n} \frac{\log N_i}{N_1 N_2 \cdots N_{i-1}} \right)$$

Proof: Proof the result that DFT can be reduced to computing multiple one dimensional DFT, the proof of inverse DFT is similar. Given a n dimensional lattice L and a sublattice L_0 ,the base vector of L with respect to L_0 compose the set $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ .Let $M = \{i_1 \vec{v}_1 + i_2 \vec{v}_2 + \cdots + i_n \vec{v}_n : 0 \le i_j < N_j, 0 \le j \le n\}$,the $N_1, N_2, \cdots, N_n$ are elementary divisors of L/L_0.Hence M is a system of coset representatives of the quotient group L/L_0. The base vectors of L with respect to L_0 compose the base matrix W_L.The base vectors of L compose the base matrix W. The diagonal elements of the diagonal matrix B are composed of the corresponding L/L_0 elementary divisors. As the base vector of L_0 is integer linear combination of the base vector of L, there is an integer matrix T to make TW a base matrix of L_0. $P_1, P_2, \cdots, P_a$ and $Q_1, Q_2, \cdots, Q_b$ is the elementary matrix used to diagonalize matrix T. So you can get the diagonal matrix B:

$$B = P_a \cdots P_2 P_1 T Q_1 Q_2 \cdots Q_b$$

With the relationship between matrix W_L and the matrix W:

$$W_L = (Q_1 Q_2 \cdots Q_a)^{-1} W$$

Let $Q = Q_1 Q_2 \cdots Q_b$ and $P = P_a \cdots P_2 P_1$. As the row vector of TW is the base vector of L_0 and V is a unit matrix, the row vector of the matrix $W_{L_0} = P(TW)$ is also the base vector of L_0, similarly the row vector of the matrix $W_L = Q^{-1}W$ is also the base vector L. Therefore

$$BW_L = (PAQ)(Q^{-1}W) = P(TW) = W_{L_0} \tag{3}$$

The row vector of the matrix of W_L is the base vector of L with respect to L_0, and the corresponding divisors are on the diagonal of the matrix B.

The dual lattice of the lattice L is:

$$L^* = \{ \mathbf{y} \in R^n : \langle \mathbf{x}, \mathbf{y} \rangle \text{ is an integer for all } \mathbf{x} \in L \}$$

$\langle \mathbf{x}, \mathbf{y} \rangle$ is the inner product of a standard European space. The following three properties is:

- $L^* < L_0{}^*$
- $\widehat{L/L_0} \cong L_0{}^* / L^*$
- If W is the base matrix of L, W^{-T} is the base matrix of L^*.

As the quotient group of L/L_0 is the space domain of DFT, according to the above second properties, the quotient group $L_0{}^* / L^*$ is the frequency domain.

The base vector $\{ \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \}$ of L with respect to L_0 is the row vector of the base matrix W. According to the above third properties, we get The base vector of L_0^* with respect to L^* is the row vector $\{ \vec{v}_1^*, \vec{v}_2^*, \cdots, \vec{v}_n^* \}$ of the base matrix $(BW)^{-T}$. According to

$$(\langle \vec{v}_j, \vec{v}_k^* \rangle) = (BW)^{-T} W^T = B^{-1} \tag{4}$$

Therefore

$$\langle \vec{v}_j, \vec{v}_k^* \rangle = \begin{cases} \frac{1}{N_j} & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

So

$$\left\langle \sum_{i=1}^{n} j_i v_i, \sum_{i=1}^{n} k_i v_i^* \right\rangle = \sum_{i=1}^{n} \frac{1}{N_i} j_i k_i \tag{5}$$

$J = (j_1, j_2, \cdots, j_n)$ denote $j_1 \vec{v}_1 + j_2 \vec{v}_2 + \cdots + j_n \vec{v}_n$ and $K = (k_1, k_2, \cdots, k_n)$ denote $k_1 \vec{v}_1^* + k_2 \vec{v}_2^* + \cdots + k_n \vec{v}_n^*$. The formula (5) in the above formula (1):

$$
\begin{aligned}
(\mathcal{F}a)(K) &= \frac{1}{\sqrt{N}} \sum a(J) e^{-2\pi i \frac{j_1 k_1}{N_1}} e^{-2\pi i \frac{j_2 k_2}{N_2}} \cdots e^{-2\pi i \frac{j_n k_n}{N_n}} \\
&= \frac{1}{\sqrt{N}} \sum_{j_n = 0}^{N_n - 1} e^{-2\pi i \frac{j_n k_n}{N_n}} \\
&\quad \times \left(\cdots \left(\sum_{j_2 = 0}^{N_2 - 1} e^{-2\pi i \frac{j_2 k_2}{N_2}} \left(\sum_{j_1 = 0}^{N_1 - 1} a(J) e^{-2\pi i \frac{j_1 k_1}{N_1}} \right) \right) \right)
\end{aligned} \tag{6}
$$

$0 \leq j_1 < N_1, \cdots, 0 \leq j_n < N_n$.

Let

$$b_1(k_1, j_2, \cdots, j_n) = \sum_{j_1 = 0}^{N_1 - 1} a(j_1, j_2, \cdots, j_n) e^{-2\pi i \frac{j_1 k_1}{N_1}},$$

$$b_2(k_1, k_2, j_3, \cdots, j_n) = \sum_{j_2=0}^{N_2-1} b_1(k_1, j_2, \cdots, j_n) e^{-2\pi i \frac{j_2 k_2}{N_2}},$$

$$\cdots$$

$$b_n(k_1, k_2, \cdots, k_n) = \sum_{j_n=0}^{N_n-1} b_{n-1}(k_1, k_2, \cdots, k_{n-1}, j_n) e^{-2\pi i \frac{j_n k_n}{N_n}}$$

So

$$
\begin{aligned}
(Fa)(k) &= b_n(k_1, k_2, \cdots, k_n) \\
&= \sum_{j_n=0}^{N_n-1} b_{n-1}(k_1, k_2, \cdots, k_{n-1}, j_n) e^{-2\pi i \frac{j_n k_n}{N_n}}
\end{aligned}
\tag{7}
$$

Because the run time of one dimension DFT of size N_i is $O(N_i \log N_i)$. So the complexity of the above formular is:

$$O\left(N \sum_{i=1}^{n} \frac{\log Ni}{N_1 N_2 \cdots N_{i-1}} \right)$$

3. pixel conversion

Current image acquisition and display devices are all based on the square grid structure. In order to do discrete Fourier transform on the hexagonal pixel remote sensing image, firstly it needs to transform the square pixel image into a hexagonal pixel image.

The base vectors of traditional square pixel image are $\vec{a}_1 = (1, 0)$ and $\vec{a}_2 = (0, 1)$. The base vectors of hexagonal are $\vec{v}_1 = (1, 0)$ and $\vec{v}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$. The matrix C is the transition matrix which can transform square pixel into hexagonal pixel. The formular of the base transformation is as shown as below :

$$(\vec{v}_1, \vec{v}_2) = (\vec{a}_1, \vec{a}_2)C \tag{8}$$

Hence the transition matrix C is:

$$C = \begin{pmatrix} 1 & \frac{-1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Let the vector $\boldsymbol{x} = m_1 a_1 + m_2 a_2 = n_1 v_1 + n_2 v_2$, Therefore

$$
\begin{aligned}
\boldsymbol{x} &= (\vec{a}_1, \vec{a}_2) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = (\vec{v}_1, \vec{v}_2) \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
&= (\vec{a}_1, \vec{a}_2) C \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}
\end{aligned}
$$

So

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = C \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \tag{9}$$

According to the above formulas of the base transformation and coordinated transformation ,it can resample square pixel image to the desired hexagonal pixel remote sensing images. This paper uses bilinear interpolation to resample . The hexagonal images resampled are shown as below:
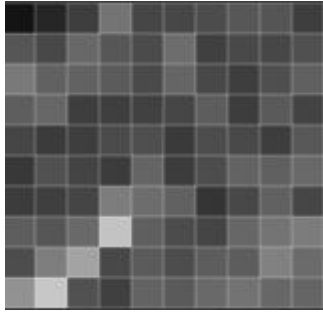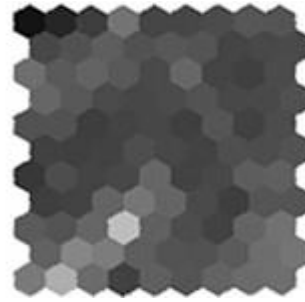
Figure 1.square pixel image

Figure 2 hexagonal pixel image

4. The DFT of the Hexagonal lattice

Remote sensing image is a function which is defined on a finite set A of the two dimensional lattice H.The set A must be nature Abelian group.H_0 is a sublattice of H.the set A is a system of coset representatives of the quotient group H/H_0. the quotient group H/H_0 is the space domain of DFT. The vector $\vec{v_1} = (1,0)$ and $\vec{v_2} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ are base vectors of hexagonal lattice H with respect to H_0 and the corresponding divisors are N_1 and N_2.So the corresponding formular of DFT of hexagonal lattice.

$$(\mathcal{F}a)(K) = \frac{1}{\sqrt{N}} \sum a(J) e^{-2\pi i \frac{j_1 k_1}{N_1}} e^{-2\pi i \frac{j_2 k_2}{N_2}}$$

$$= \frac{1}{\sqrt{N}} \sum_{j_2=0}^{N_2-1} e^{-2\pi i \frac{j_2 k_2}{N_2}} \left(\sum_{j_1=0}^{N_1-1} a(J) e^{-2\pi i \frac{j_1 k_1}{N_1}} \right)$$

$0 \leq j_1 < N_1, 0 \leq j_2 < N_2$.

Let

$$b_1(k_1, j_2) = \sum_{j_1=0}^{N_1-1} a(j_1, j_2) e^{-2\pi i \frac{j_1 k_1}{N_1}},$$

$$b_2(k_1, k_2) = \sum_{j_2=0}^{N_2-1} b_1(k_1, j_2) e^{-2\pi i \frac{j_2 k_2}{N_2}}$$

So

$$(Fa)(k) = b_2(k_1, k_2)$$

$$= \sum_{j_2=0}^{N_2-1} b_1(k_1, j_2) e^{-2\pi i \frac{j_2 k_2}{N_2}}$$

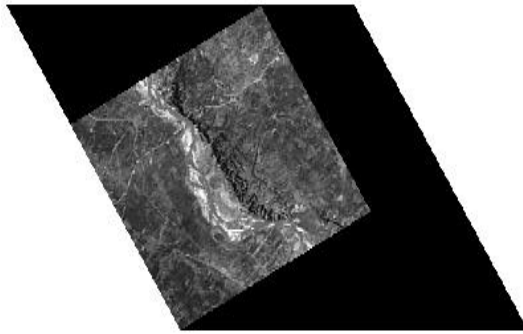The experimental result is shown as below:

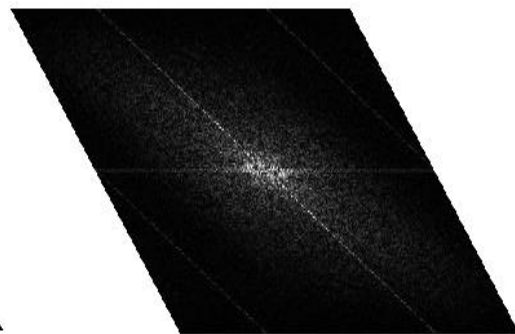Figure 3.The source hexagonal pixel image   Figure 4.The result of DFT

5. Conclusion

This paper firstly describes DFT suitable for n-dimensional lattice and then introduces base the DFT on a particular subdivision structure of hexagonal pixel remote sensing image .

6. References

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