

A Global “Natural” Grid Model Based on the Morse Complex

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Abstract. In the exploration and interpretation of the extensive or global natural phenomena such as environmental monitoring, climatic analysis, hydrological analysis, meteorological service, simulation of sea level rise, etc., knowledge about the shape properties of the earth surface and terrain features is urgently needed. However, traditional discrete global grids (DGG) can not directly provide it and are confronted with the challenge of the rapid data volume growth as the modern earth surveying technology develops. In this paper, a global “natural” grid (GNG) model based on the Morse complex is proposed and a relatively comprehensive and theoretical comparison with the traditional DGG models is analyzed in details as well as some issues to be resolved in the future. Finally, the experimental and analysis results indicate that this distinct GNG model built from DGG is more significant to the advance of the geospatial data acquisition technology and to the interpretation of those extensive or global natural phenomena.

1. Introduction

Recently, with the advancement in geospatial data acquisition technology, spatial data at high spatiotemporal resolution about the earth surface are being collected. However, the large data volume posts a great challenge to the traditional methods of terrain modelling and analysis, especially at the global scale. On the other hand, knowledge about the shape properties of the earth surface and terrain features is urgently needed in the exploration and interpretation of some extensive or global natural phenomena such as environmental monitoring, climatic analysis, hydrological analysis, meteorological service, simulation of sea level rise and so on. For instance, the distribution and orientation of mountains are the critical factors in the field of global climatic analysis. Additionally, river basins over the world play an important role on the global water pollution monitoring and control [1].

Over the years, many discrete grid models for the global surface have been proposed, called discrete global grid (DGG), such as Quadrangle Grid [2], Quaternary Triangular Mesh (QTM) [3], or Hexagon Grid [4] and Voronoi/TIN grid [5], as shown in Figure 1. As a result, they are qualified to simulate the global surface [6], but the DEM constructed by them can neither directly reveal the terrain natural features nor provide the knowledge about the shape properties of the earth surface. Actually, they are mainly applied to the organization or index of the global spatial data, thematic map production and 3D visualization [7]. On the contrary, the Global “Natural” Grid (GNG) model partitions the earth surface into “natural” cells by using the natural terrain feature elements, such as feature points (pits, peaks and saddles) and feature lines (such as ridges and ravines), which possibly



provide a practicable data model and analysis method to explore and understand those natural phenomena mentioned above.

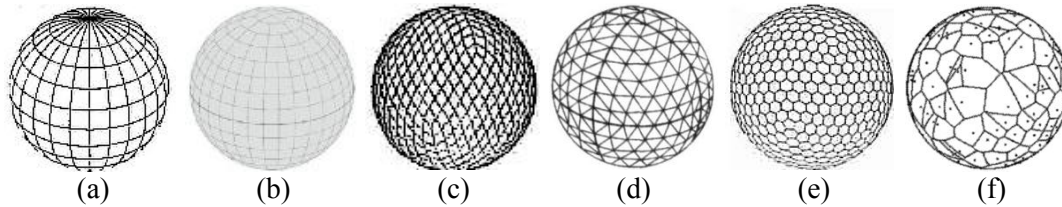


Figure 1. Traditional discrete global grids (DGG): (a) latitude and longitude grid, (b) DGG [8], (c) diamond grid [9], (d) QTM; (e) hexagon grid, (f) Voronoi/TIN grid

The topological method based on Morse theory [10,11] has been recognized as an important tool for shape analysis and understanding in many applications, including geography, physics, chemistry and medicine [12], because of the ability of feature extraction, multi-scale analysis and forming an abstract or compact description of a surface shape. Over the years, based on the Morse complex [13,14], lots of algorithms for terrain morphology have been presented (see [12,15] for more details). However, most of them and the related applications are limited to local terrain datasets homeomorphic to a 2-disk. In this paper, a GNG model based on the Morse complex is proposed for the whole earth surface which is topological equivalent to an open 2-ball.

The remainder of this paper is organized as follows. In Section 2, some basic notions and the development of Morse Theory and Morse complex are introduced briefly. In Section 3, a modeling approach for the GNG is proposed. After that, an experiment is designed to validate this method and a relatively comprehensive and theoretical comparison with the traditional DGG is further analyzed in details. Following, some issues or challenges for the GNG are discussed. Finally, some concluding remarks are drawn.

2. Morse theory and Morse complex

Morse theory is a powerful tool to capture the morphology of manifold on which a scalar function f is defined. Let f be a C^2 -differentiable real-valued function defined over a domain D in the 2D space. A point p of D is a critical point of f if and only if the gradient of f vanishes on p . Function f is said to be a Morse function when all its critical points are non-degenerate (i.e., if and only if the Hessian matrix $\text{Hess}_p f$ of the second derivatives of f at p is non-singular: the determinant of $\text{Hess}_p f$ is not zero). This implies that the critical points of f are isolated. The number of negative eigenvalues of $\text{Hess}_p f$ is called the index of a critical point p . There are three types of non-degenerate critical point p : a minimum (pit), a saddle (pass), or a maximum (peak) if and only if p has index 0, 1 or 2, respectively. Points which are not critical are said to be regular.

An integral line of f , going through a point p , is a maximal path which is everywhere tangent to the gradient of f . Integral lines that converge to (originate from) a critical point p of index i form an i -cell (($2-i$)-cell) called the descending (ascending) cell of p . The collection of all descending cells form a Euclidean cell complex, called descending Morse complex (Figure 2a), and the collection of all ascending cells form also a Euclidean cell complex, called ascending Morse complex (Figure 2b), which is dual with respect to the descending Morse complex. A Morse function f satisfying the Morse-Smale condition, that is, descending and ascending Morse complexes intersect only transversally, is called a Morse-Smale function. This means that the intersection of the edges of the descending and ascending cells is a saddle point. In this case, we can define a complex, called the Morse-Smale complex, whose cells are the connected components of the intersections of the descending and ascending cells (Figure 2c). Integral lines connecting saddle points to other critical points are called separatrices. All separatrices forms a network, called the critical net. It is the 1-skeleton of the Morse-Smale complex.

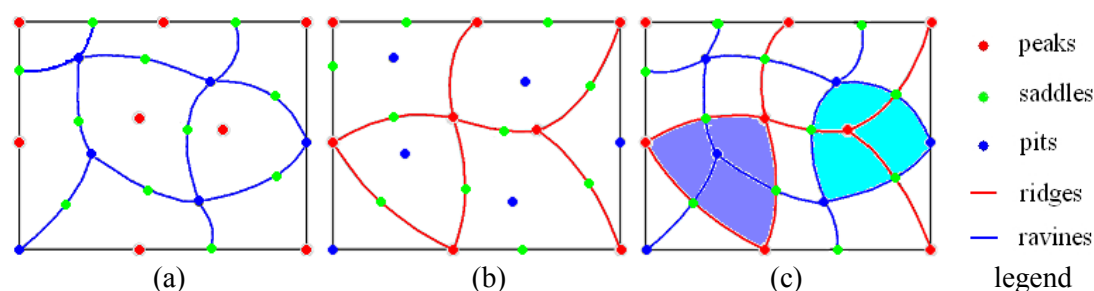


Figure 2. Morse and Morse-Smale complex: (a) ascending Morse complex, (b) descending Morse complex, (c) Morse-Smale complex

As mentioned above, the fundamental restrictive condition with the Morse theory is the assumption that surface is twice differentiable and that the Hessian matrix of partial derivatives is everywhere non-singular. However, real surfaces are seldom sufficiently smooth enough, especially in the discrete case (eg. the DEM). As a result, to harmonize the contradiction between the theory and application, main two extensions of the classic Morse theory are proposed, that is, piece-wise linear Morse theory [16] and discrete Morse theory [17]. Figure 3 indicates the development of Morse theory and Morse complex.

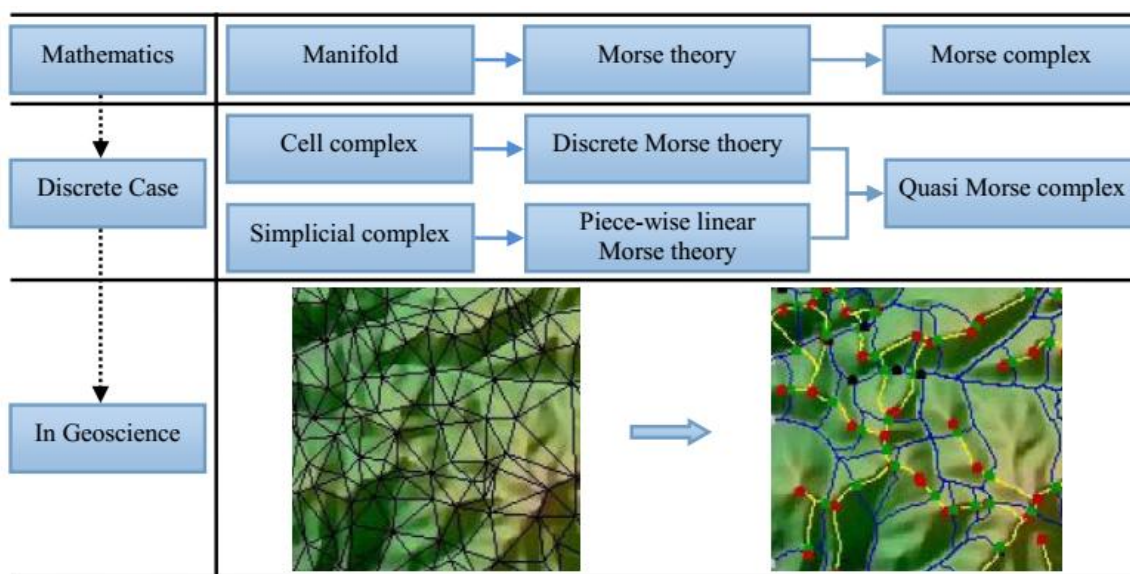


Figure 3. The development of Morse theory and Morse complex

In terrain analysis, the separatrix starting from saddle to peak is called the ridge line (or ridge), while the one starting from saddle to pit is called the ravine line (or ravine). Theoretically, it is impossible that valley and ridge line intersect at regular points and that the isolated critical points occur. Meanwhile, the earth surface is just a 2-manifold, where Morse theory can be used directly. However, as shown in Figure 4, when applying the existing algorithms for local terrain to the global DEM, some abnormal situations will occur, such as the various local intersections between a ridge and a ravine and some ridges passing across the ocean without the marine identification.

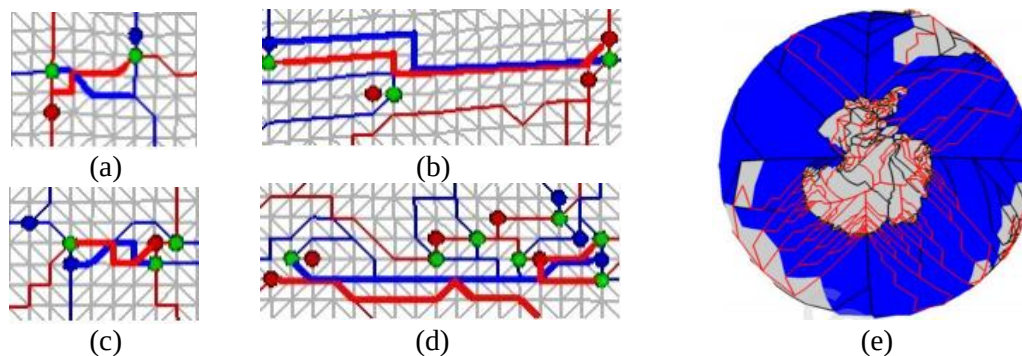


Figure 4. The abnormal when applying the existing construction algorithms of Morse complex to the global DEM: (a) single-point intersection, (b) intersections forming a line, (c) multiple intersections, (d) multi-intersections between a ravine and some ridges, (e) some ridges passing across the ocean

3. A Modeling Approach of GNG

To effectively avoid those defaults discussed above (in Figure 4), a GNG modelling method is proposed in this section. Generally speaking, this method mainly consists of three steps: Firstly, based on the restricted quadtree [18], a variable-resolution geometric representation is constructed to simulate the global surface; Secondly, the feature elements are extracted and an original GNG model is constructed; Thirdly, the topological simplification or refinement operators [13,14] are directly applied to the original GNG model recursively to establish a multi-resolution representation. Here, some details about the geometrical simulation and the original GNG construction are our focus.

3.1. A Variable-resolution Geometrical Simulation of the Earth Surface

Since the data volume of the global DEMs is huge, a variable-resolution representation of the earth surface is more appropriate to simulate the earth surface with fine expression in land area while rough expression in marine region (about 71% of the whole earth surface). However, some cracks or T-junctions may occur, which is adverse to the extraction of the feature elements. As a result, in view of these two aspects, a triangulated global diamond grid based on the restricted quadtree [18] is adopted to construct a seamless expression of the global DEMs.

Specifically, according to the principle of the global diamond subdivision [19], the earth surface is uniformly partitioned to a certain level at first. After that, the maximum elevation difference of each diamond vertexes is calculated and compared with the given threshold, and then taking advantage of the algorithm of restricted quadtree tiling [18], the diamond node is determined to be recursively refined, until the maximum elevation difference does not exceed the threshold value any more. As a result, the representation level of the flat ocean region is progressively shallower than that of the fluctuant land area, where the level between two adjacent diamonds exceeds one at most. Finally, leaf node is triangulated according to the adjacencies of the quadtree nodes.

3.2. A Modified Algorithm of the original GNG

To the various local intersections, the duality of the Morse Complex is considered here. The main principle is as follows:

From the dual structure of the ascending and descending Morse complex (in Figure 2), it can be observed that the boundary of an ascending cell is consisted of ravines and saddles, and that the ridges starting from the boundary saddles to the central peak route within the cell. Dually, the boundary of a descending cell is consisted of ridges and saddles, the ravines starting from the boundary saddles to the central pit route within the cell.

By this principle, a modified algorithm for the original GNG is designed, which mainly consists of three steps. Firstly, one of Morse complex, that is to say, descending or ascending Morse complex is

constructed by the existing algorithm (for example, the simple STD algorithm [20] without floating-point computations), after the topology relationship of each two adjacent vertexes and two adjacent triangles as well as vertex and triangle is established in the triangulated global diamond grid. Here, all the marine areas are identified first of all and each is regarded as a pit region according to the common sense. Secondly, ridges (ravines), that is, the separatrix of the dual Morse complex is extracted in every constructed cell to prevent those intersections. The saddles belonging to a cell are classified into two groups: the boundary ones or the internal ones. So, a cell saddle whether located on the boundary or not need to be determined. What's more, there may be some macro-saddles, that is, one feature line passes two saddles and the path between the two saddles (called macro-saddle line) plays the dual role of both ravine line and ridge line. Therefore, all the macro-saddles are identified and any of the macro-saddle lines is extracted according to the corresponding path established in the first stage. Furthermore, it should be noticed that the ravines starting from the saddles located on the boundary of the marine regions are not extracted and the end point of each ravine is a pit or a macro-saddle or a point of some marine area. Finally, regarding the feature lines extracted in the second stage as the boundary, the dual complex of the former is constructed and the topological representation of the earth surface, i.e. the original GNG is constructed. Figure 5 shows the pseudo codes of this algorithm.

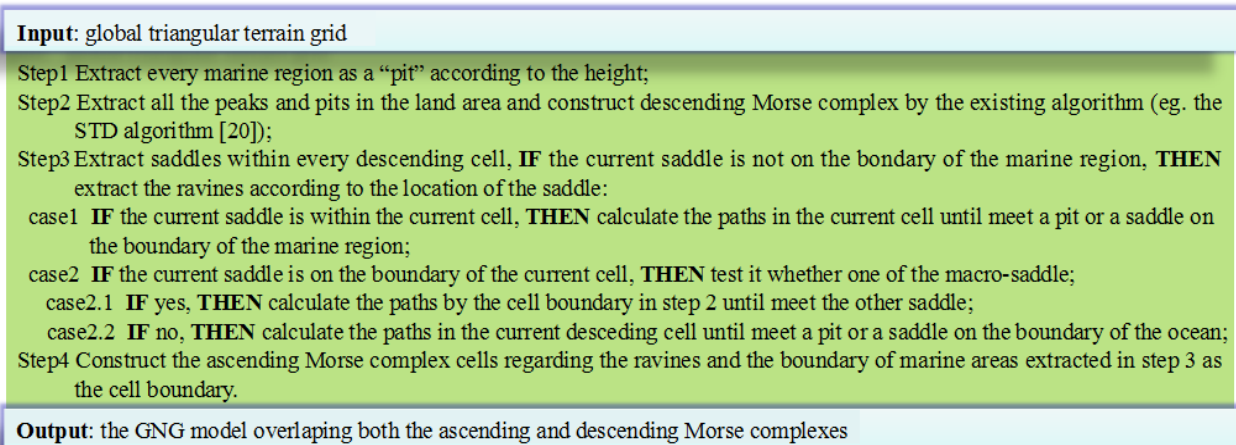


Figure 5. The pseudo codes of this modified algorithm of the original GNG

4. Experiments and Comparison Analysis

In this section, an experiment system with C++ and OSG (Open Scene Graph) developed by using GTOPO30 data is designed to validate the feasibility and correctness of our approach for the GNG model.

Figure 6a shows a triangulated variable-resolution global diamond grid is constructed based on the restricted quadtree, where the resolution of the land areas is higher than that of the marine regions and there are no cracks and T-junctions as well as thin and long triangles. Here, the earth surface is uniformly partition into the diamond grid at level 3 first of all, and then each diamond node is refined recursively to the max level 7, if the maximum elevation difference of the diamond vertexes exceeds the given threshold 10.0 m.

The saperatrix of original GNG by our modified algorithm from that variable-resolution global triangulated terrain grid is shown in the Figure 6g. From the view of structure, it can be regarded as the overlap of the global descending Morse complex (in Figure 6d) and the global ascending Morse complex (in Figure 6e), while Figure 6b and 6c are the GNG cells. In hydrology, each descending cell is a catchment area and all the rivers converge into the sea. What's more, Figure 6f indicates that some islands occur in the marine areas which represent as isolated peaks, that is, no saddles and ravines connect them. Meanwhile, there are also some saddles to the marine minimum but no paths, locating at the boundary of some marine region. Besides, from the Figure 6g, 6h and 6i, we can see that a

multi-resolution GNG is achieved by applying 2000-step and 6000-step topological simplification operators respectively to the original model, while the inverse of this simplification is the refinement.

Further, according to this modelling method and the experimental results, we can see some relationship between the GNG and traditional DGG models (eg. the Quaternary Triangular Mesh [3]), that is, traditional DGG provides a fundamental geometrical model by the simulation of the earth surface because the separatrices of the GNG are extracted from the global DEM by the DGG. As a result, the veracity and accuracy of the separatrix extraction depend on the simulated global DEM. Meanwhile, the GNG is the resegmentation and classification of the cells of traditional DGG models, according to the hydrological characteristics of the global terrain surface.

On the other hand, there are also some distinguishing characteristics of the GNG comparing with the traditional DGG, which is discussed and analyzed as follows:

1) From the perspective of the theoretical basis, Morse theory provides a strict mathematical foundation for the GNG model and supports a knowledge-based approach to analyze, understand and visualize the behaviour in space and time of the scalar field [21]. In fact, GNG can capture the topological structure information related to the level sets of the height function defined on the earth surface and provides a knowledge-based approach to analyze, visualize and understand the earth surface behaviour through a global perspective.

2) To the way of surface description, GNG can describe the shape of the earth surface by those fundamental feature elements (pits, peaks, ravines and ridges), which effectively act as landmarks and can be used to get a representative coverage of the entire earth surface. Therefore, it is an abstract and compact topological description rather than the geometrical formulation by the traditional DGG, which usually contains data redundancy and cannot directly reveal the nature feature elements. Moreover, nature features at the different levels can be analyzed and the main terrain morphological structure can be visualized through the multi-resolution representation of GNG by the topological simplification or refinement, which we can see from the Figure 6g, 6h and 6i.

3) In terms of the complex cell, as shown in Figure 6b and 6c, GNG decomposes the earth surface into irregular “natural” cells rather than regular “data” cells by the traditional DGG models. Since the cells of GNG or Morse complex are the segmentation of the terrain into regions of influence of minima and maxima or into regions of uniform gradient field, they relates closely to the hydrological characteristics of the earth surface. For example, the descending Morse complex of a minimum corresponds to a basin in hydrology [22]. However, the adjacency relationship among the cells of GNG, coded by Morse Incidence Graph [23], seems to be more complex than the one among the cells of the DGG models.

4) As for the aspect of the practical application, the background and purpose of the traditional DGG models is to overcome some shortcomings such as data broken, geometric distortion, when the plain grid models are applied to the organization and visualization of the global multi-scale spatial data [24,25]. However, since the separatrices of the GNG model are consisted of terrain feature lines and points, it provides the terrain morphological information or knowledge about the shape properties of the earth surface which is necessary to the exploration and interpretation of those natural phenomena mentioned in Section 1. In addition, as any descending Morse complex corresponds to a basin, a topological simplification operation [26] of a descending Morse complex can be considered as the simulation of the phenomenon that water overflows along the lowest saddle when the corresponding basin is flooded. As a result, when applied in the field of the rise-or-fall simulation and analysis of global sea levels, GNG is a better option than the traditional DGG models (eg. the QTM [27]), because of the grid cell, granularity and calculation method.

A brief comparison of GNG with the traditional DGG models is summarized in Table 1.

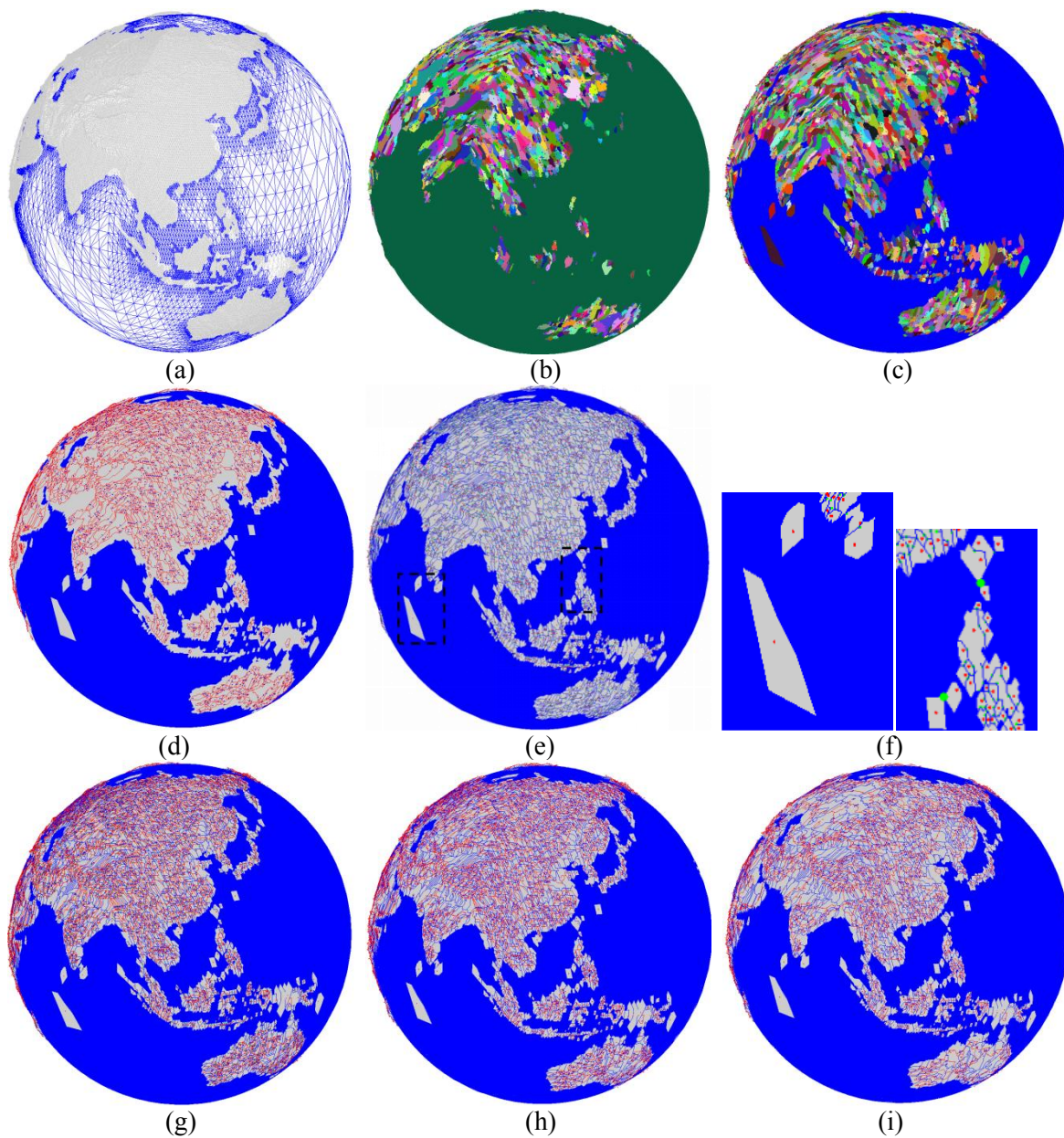


Figure 6. The original construction and multi-scale representation of GNG: (a) variable-resolution triangular mesh, (b) cells of descending Morse complex, (c) cells of ascending Morse complex, (d) sepratrix of the global descending Morse complex, (e) sepratrix of the ascending Morse complex, (f) local enlarged details of degeneration in the dotted region in figure (e), (g) the original global natural grid without simplification, (h) 2000-step simplification, (i) 6000-step simplification

5. Challenges for the GNG model

Although a distinctive GNG model is proposed in this paper, there are still some issues or challenges to be considered in the future works:

5.1. Simplification operations for local degeneration of GNG

As mentioned above, there may be some local degeneration in the GNG model. However, the connection of the feature points coded by the Morse incidence graph is the precondition and

foundation of the topological simplification and construction of the multi-scale Morse complexes [23]. As a result, the generalization and refinement operators proposed by [13,14] are no longer suitable for the global terrain morphology. So, some further extension and improvement of those operators needs to be done, in order to construct and achieve an overall and correct multi-resolution representation of the GNG model more than that of the mainland of earth surface. Here, the marine areas may be used to resolve this issue. After that, a combined geometrical and topological multi-scale representation of the GNG just can be built based on the MMT [28] or HFT [29].

5.2. Construction based on the discrete Morse theory

Here, the GNG model was built from the global triangular terrain mesh, just based on the piece-wise linear Morse theory. However, based on the discrete Morse theory, a large number of algorithms (see [12,15] for more details) have been designed and proposed from the local DEM, including both the triangular irregular network (TIN) and the regular square grid (RSG). Actually, the discrete Morse theory overcomes the inherent default of the piece-wise linear Morse theory [12] and may be more suitable for the parallel computation of the Morse complex in the case of a large data sets [30,31]. Therefore, based on the discrete Morse theory the construction and parallel computation of the GNG model from other cell types of the DGG may be one of our further work directions.

5.3. Local refinement operation for GNG

In the field of the global GIS, the whole and expansive earth surface is usually regarded as the study object and the size of data volume is very large, especially under the background of the rapid development of the modern earth observing technology. However, because of the difference of major perspective, some small objects such as house, rabbit, human body and molecule are usually considered at the most existing literature on the Morse complex in computer graphics, biology, physics, chemistry and so on. As a result, the local update operation can be achieved by the reconstruction of the Morse complex in these fields, which is impossible to the GNG model. Consequently, a scientific and effective approach of local update and refinement for the GNG model is particularly needed in global GIS, in order to meet the practical application requirements at the different scales and the development trend of the modern earth observing technology.

Additionally, there are also some other issues to be resolved such as the real-time LOD (level of details) visualization and specific application schema in case of the analysis and understanding of those related nature phenomena.

6. Concluding remarks

In this paper, a global "natural" grid (GNG) model base on the Morse complex is proposed. As a compact structure, GNG segments the earth surface by those fundamental feature elements (pits, peaks, ravines and ridges) and is consisted of the natural cells, which can directly reveal the natural features and provide the knowledge about the shape of the whole earth surface. As a result, it is of great significance to the development trend of the modern earth surveying technologies and has a closer link to the analysis and understanding of some extensive or global nature phenomena. The main work of this paper is as follows:

- 1) A global natural grid (GNG) based on the Morse complex is proposed, of which the modeling approach includes the simulation of the earth surface, original construction and multi-resolution representation.

- 2) A relatively comprehensive and theoretical comparison between the GNG and traditional DGG is analyzed in details. In brief, traditional DGG provides a geometrical foundation of the feature extraction and cell grouping for the GNG. However, there are also some individual distinguishing characters of the GNG, such as the theoretical foundation, description way, model cell, and background of application.

3) Some issues or challenges for the GNG to be considered in the future works are also discussed and given out, mainly including the multi-resolution representation, the construction and parallel computation based on the discrete Morse theory, the local update and refinement operation.

Table 1. The comparison of GNG with the traditional DGG models

Model	Relationship	Difference					
		theoretical foundation	description	cell	adjacency coding	feature revealing	applications
GNG	traditional DGG provides a foundation of cell grouping for the GNG	rigorous	compact and topological	irregular natural cells	based on Morse incidence graph (MIG)	able	environmental monitoring, climatic and hydrological analysis, meteorological service, simulation of sea level rise, etc.
Traditional DGG		relatively loose	accurate and geometrical	regular data cells	mostly based on a tree	disable	organization or index of global spatial data, thematic map production and visualization

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