

Numerical solution for option pricing with stochastic volatility model

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Abstract. The option pricing equations derived from stochastic volatility models in finance are often cast in the form of nonlinear partial differential equations. To solve the equations, we used the upwind finite difference scheme for the spatial discretisation and a fully implicit time-stepping scheme. The result of this scheme is a matrix system in the form of an M-Matrix and we proof that the approximate solution converges to the viscosity solution to the equation by showing that the scheme is monotone, consistent and stable. Numerical experiments are implemented to show that the behavior and the order of convergence of upwind finite difference method.

Keywords: stochastic volatility model, option pricing, upwind finite difference method, convergence, nonlinear partial differential equation

1. Introduction

Stock option is a contract that give its holder the right but not the obligation to buy (*call option*) or sell (*put option*) stocks at a specified price [1]. The time when the contract ends is known as the expiration date or the maturity date of the option. There are two types of options, the European option that can be exercised at maturity date and the American options that can be exercised on or before the maturity date. Fisher Black and Myron Scholes [2] showed that the price of the option is the solution of partial differential equations (PDE) called the standar Black-Scholes model [2].

In the Black-Scholes model assumes that the volatility is constant. This assumption is incompatible with the market price movement, where the volatility has a tendency to go down and will go up again at some point [3]. The volatility in the Black-Scholes can depend on the time, the price of the underlying stock, and/or derivative of the option price. In this case, the standard Black-Scholes equation becomes the following Nonlinear Black-Scholes equation:

$$V_t = \frac{\sigma^2(t, S, V_S, V_{SS})}{2} S^2 V_{SS} + rSV_S - rV \quad (1)$$

where σ^2 is a modified volatility as a function of t, S, V_S , and V_{SS}

Some models assume volatility surface across underlying asset prices and time [4, 5], the model assumes that the volatility follows the random process [6], and stochastic volatility models developed by Avellaneda, Levy and Paras [7]. [8], formulated partial differential model for option pricing with stochastic volatility is given by



$$V_\tau = \frac{\sigma^2(\Gamma)}{2} S^2 V_{SS} + rSV_S - rV \quad (2)$$

where $\Gamma = V_{SS}$ is the second partial derivative of V to S , $\sigma^2(\Gamma)$ is the volatility as a function of Γ . Volatility is assumed to be in the interval

$$\sigma_{min} \leq \sigma(\Gamma) \leq \sigma_{max}.$$

With such a range of possible volatility value, the equation (2) is nonlinear and does not have a unique solution. This values are obtained by either maximizing or minimizing equation (2) by selecting $\sigma^2(\Gamma)$ according to the value of $\Gamma = V_{SS}$. Specifically, if we consider the worst case for an investors with a long position in the option, then the value of $\sigma^2(\Gamma)$ is:

$$\sigma^2(\Gamma) = \begin{cases} \sigma_{max}^2 & \text{if } \Gamma \leq 0 \\ \sigma_{min}^2 & \text{if } \Gamma > 0. \end{cases} \quad (3)$$

On the other hand, the best case for an investor with a long position is determined by:

$$\sigma^2(\Gamma) = \begin{cases} \sigma_{max}^2 & \text{if } \Gamma > 0 \\ \sigma_{min}^2 & \text{if } \Gamma \leq 0. \end{cases} \quad (4)$$

Prices for the investors with short positions are given by the negative of the solutions when applying equations (3) and (4). With a transformation $\tau = T - t$, equation (2) can be written as

$$U_\tau = \frac{\sigma^2(\Gamma)}{2} S^2 U_{SS} + rSU_S - rU \quad (5)$$

for $S > 0$ and $0 < \tau \leq T$, where $\sigma^2(\Gamma)$ followed equations (3) and (4).

This model involves nonlinear partial differential equations that required a numerical method to determine the option price solution. Several numerical approach can be carried out to solve it, including finite difference method, upwind finite difference method and finite volume methods. Upwind finite difference method and finite volume method proved to be a consistent, stable and monotonous [9, 10]. Finite difference method with fully implicit discretization method is monotone and converges to the viscosity solution, while the Crank-Nicolson is only monotone conditionally [8]. In this work, we develop the upwind finite difference method for pricing the stochastic volatility model. We show theoretically that the upwind finite difference scheme in space and the implicit scheme in time is consistent, stable, and monotone, so it converges to the viscosity solution. We also derive conditions which ensure that the discrete method is monotone.

2. Initial and Boundary Condition

The nonlinear Black-Scholes model has a domain $S \in (0, \infty)$, but for computational purpose, it is necessary to truncate it into $S \in (0, S_{max})$, where S_{max} denotes a sufficiently large positive number to ensure the accuracy of the solution. Then the initial and boundary conditions for nonlinear Black-Scholes equations as follows:

$$U(S, 0) = u_0(S), \quad S \in (0, S_{max}), \quad (6)$$

$$U(0, \tau) = g_1(\tau), \quad \tau \in (0, T], \quad (7)$$

$$U(S_{max}, \tau) = g_2(\tau), \quad \tau \in (0, T], \quad (8)$$

with u_0 , g_1 , and g_2 are given functions $u_0(0) = g_1(0)$ and $u_0(S_{max}) = g_2(0)$. The choices of u_0 , g_1 , and g_2 depend on type of option. Popular European options are vanilla, butterfly, dan cash or nothing (CoN) with the initial and boundary conditions are given by:

$$\begin{aligned}
 u_0 &= \begin{cases} \max(S - K, 0) & \text{Vanilla call} \\ \max(K - S, 0) & \text{Vanilla put} \\ \max(S - K_1, 0) - 2 \max(S - K_2, 0) + \max(S - K_3, 0) & \text{butterfly spread} \\ B \times \mathcal{H}(S - K) & \text{CoN} \end{cases} \\
 g_1 &= \begin{cases} 0 & \text{Vanilla call} \\ Ke^{-r\tau} & \text{Vanilla put} \\ 0 & \text{butterfly spread} \\ 0 & \text{CoN} \end{cases} \\
 g_2 &= \begin{cases} S_{max} - Ke^{-r\tau} & \text{Vanilla call} \\ 0 & \text{Vanilla put} \\ 0 & \text{butterfly spread} \\ Be^{-r\tau} & \text{CoN} \end{cases}
 \end{aligned}$$

with \mathcal{H} is a Heaviside functions, B is constanta, K , K_1 , K_2 , and K_3 is a strike price.

3. Discretization

Price discretization, let $I = (0, S_{max})$ divided to sub-interval M , where $I_i = (S_i, S_{i+1})$, $i = 0, 1, \dots, M-1$ with $0 = S_0 < S_1 < \dots < S_M = S_{max}$, and for each $i = 0, 1, \dots, M-1$ let $h_i = h = S_{i+1} - S_i$. For time discretization, let $\tau = (0, T)$ divided to sub-interval N , where $\tau_n = (\tau_n, \tau_{n+1})$, $n = 0, 1, \dots, N-1$ with $0 = \tau_0 < \tau_1 < \dots < \tau_N = T$ and for each $n = 0, 1, \dots, N-1$ being let $\Delta\tau_n = \tau_{n+1} - \tau_n$. For any $W^n = (W_0^n, W_1^n, \dots, W_M^n)^T$ and $W_i = (W_i^0, W_i^1, \dots, W_i^N)^T$ with $i = 0, 1, \dots, M$ and $n = 0, 1, \dots, N$, the first and second derivatives followed finite difference operator:

$$(\delta_\tau W_i)(n) = \frac{W_i^{n+1} - W_i^n}{\Delta\tau_n}, (\delta_S^+ W^n)(i) = \frac{W_{i+1}^n - W_i^n}{h}, (\delta_S^- W^n)(i) = \frac{W_i^n - W_{i-1}^n}{h} \quad (9)$$

$$(\delta_{SS} W^n)(i) = \frac{W_{i-1}^n - 2W_i^n + W_{i+1}^n}{h^2} \quad (10)$$

By using operators (9-10), nonlinear Black-Scholes model (5) was approximated to the upwind finite difference method:

$$\delta_\tau U_i(n) - \frac{1}{2} \sigma^2 ((\delta_{SS} U^{n+1})(i) S_i^2 (\delta_{SS} U^{n+1})(i) - r S_i (\delta_S^+ U^{n+1})(i) + r U_i^{n+1}) = 0, \quad (11)$$

We get:

$$\begin{aligned}
 U_{i-1}^{n+1} \left(-\frac{1}{2h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2 \right) + U_i^{n+1} \left(\frac{1}{\Delta\tau_n} + \frac{1}{h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2 + \frac{r S_i}{h} + r \right) + \\
 U_{i+1}^{n+1} \left(-\frac{1}{2h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2 - \frac{r S_i}{h} \right) = \frac{U_i^n}{\Delta\tau_n}.
 \end{aligned} \quad (12)$$

For simplicity, equation (12) can be written into the following form:

$$\alpha_i^{n+1} (U^{n+1}) U_{i-1}^{n+1} + \beta_i^{n+1} (U^{n+1}) U_i^{n+1} + \gamma_i^{n+1} (U^{n+1}) U_{i+1}^{n+1} = \frac{1}{\Delta\tau_n} U_i^n, \quad (13)$$

for $i = 1, \dots, M-1$ and $n = 1, \dots, N-1$, where:

$$\alpha_i^{n+1} (U^{n+1}) = -\frac{1}{2h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2, \quad (14)$$

$$\beta_i^{n+1} (U^{n+1}) = \frac{1}{\Delta\tau_n} + \frac{1}{h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2 + \frac{r S_i}{h} + r, \quad (15)$$

$$\gamma_i^{n+1} (U^{n+1}) = -\frac{1}{2h^2} \sigma^2 (\delta_{SS} U^{n+1})(i) S_i^2 - \frac{r S_i}{h}. \quad (16)$$

Using (6-8), the initial and boundary conditions for equation (13) is defined as follows

$$U_i^0 = g_1(S_i) \quad U_0^n = g_2(\tau_n) \quad U_M^n = g_3(\tau_n) \quad (17)$$

for $i = 1, 2, \dots, M-1$ and $n = 1, \dots, N$, the equation of (13) can be written into the following matrix form

$$A^{n+1}(U^{n+1}) \hat{U}^{n+1} = \frac{1}{\Delta\tau_n} \hat{U}^n + B^{n+1}, \quad (18)$$

for $n = 1, \dots, N-1$, where

$$A^{n+1}(U^{n+1}) = \begin{bmatrix} \beta_1^{n+1} & \gamma_1^{n+1} & 0 & \dots & 0 & 0 & 0 \\ \alpha_2^{n+1} & \beta_2^{n+1} & \gamma_2^{n+1} & \dots & 0 & 0 & 0 \\ 0 & \alpha_3^{n+1} & \beta_3^{n+1} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{M-3}^{n+1} & \gamma_{M-3}^{n+1} & 0 \\ 0 & 0 & 0 & \dots & \alpha_{M-2}^{n+1} & \beta_{M-2}^{n+1} & \gamma_{M-2}^{n+1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_{M-1}^{n+1} & \beta_{M-1}^{n+1} \end{bmatrix}$$

$$\hat{U}^k = (U_1^k, U_2^k, \dots, U_{M-1}^k)' \text{ for } k = n, n+1$$

$$B^{n+1} = (-\alpha_1^{n+1}U_0^{n+1}, 0, \dots, 0, -\gamma_{M-1}^{n+1}U_M^{n+1})'$$

$A^{n+1}(U^{n+1})$ is a M matrix for any given U^{n+1} .

3.1. Theorem 1. M-Matrix

For any $n = 0, 1, \dots, N$, $A^n = (A_{ij}^n)$ is a M matrix for any U^n that being given.

Proof:

To prove Theorem 1, it must be shown that for $i = 1, 2, \dots, M-1$:

$$\alpha_i^n < 0, \quad \beta_i^n > 0, \quad \gamma_i^n < 0 \quad (19)$$

$$\beta_i^n \geq |\alpha_i^n| + |\gamma_i^n| \quad (20)$$

For Matrix A^{n+1} , from models (14-16) it can be seen that the condition (19) is obviously true. Further conditions (20), because $r \geq 0$ and $\frac{1}{\Delta\tau_n} \geq 0$ so:

$$\begin{aligned} \beta_i^{n+1} &\geq |\alpha_i^{n+1}| + |\gamma_i^{n+1}| + r + \frac{1}{\Delta\tau_n} \\ &\geq |\alpha_i^{n+1}| + |\gamma_i^{n+1}| \end{aligned} \quad (21)$$

From definition of $A^n = (A_{ij}^n)$ and depend on (21), we got:

$$A_{ij}^n \leq 0, \quad i \neq j, \quad A_{ii}^n > 0, \quad A_{ii}^n > \sum_{j=1}^{M-1} |A_{ij}^n|$$

It is obvious that A^n is irreducible. Thus, A^n is a M-matrix for any given U^n .

4. Convergence of The Numerical Scheme

[11] showed that a discretisation scheme is convergence to the viscosity solution if it is a stable, consistent, and monotone discretization. For $1 \leq i \leq M - 1$ and $0 \leq n \leq N - 1$, a functional F_i^{n+1} defined by:

$$F_i^{n+1} = \left(-\frac{rS_i}{h}\right) U_{i+1}^{n+1} + \left(\frac{1}{\Delta\tau_n} + \frac{rS_i}{h} + r\right) U_i^{n+1} - \frac{U_i^n}{\Delta\tau_n} - \frac{1}{h^2} \sigma^2((\Gamma^{n+1})(i))(\Gamma^{n+1})(i). \quad (22)$$

Then, equation (22) becomes

$$F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1}, U_{i-1}^{n+1}, U_i^n) = 0.$$

4.1. Lemma 1. Monotonicity

Discretization scheme (13) is monotone, i.e., for any $\varepsilon > 0$ and $1 \leq i \leq M - 1$,

$$F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1} + \varepsilon, U_{i-1}^{n+1} + \varepsilon, U_i^n + \varepsilon) \leq F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1}, U_{i-1}^{n+1}, U_i^n) \quad (23)$$

And

$$F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1} + \varepsilon, U_{i-1}^{n+1} + \varepsilon, U_i^n + \varepsilon) \leq F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1}, U_{i-1}^{n+1}, U_i^n). \quad (24)$$

Proof

Since $-\frac{rS_i}{h} \leq 0$, $\frac{1}{\Delta\tau_n} > 0$ and $\frac{1}{\Delta\tau_n} + \frac{rS_i}{h} + r > 0$, the first three (linear) terms on the right hand side of equation (22) are respectively non-increase in U_{i+1}^{n+1} , increasing in U_i^{n+1} and decreasing in U_i^n . Let $E_k = \left(0, 0, \dots, \underset{k}{1}, 0, \dots, 0\right)'$ be the $(M - 1) \times 1$ column vector. From the definition δ_{SS} (11), we have

$$\begin{aligned} \delta_{SS}(U^{n+1} + \varepsilon E_{i-1} + \varepsilon E_{i+1})(i) &= \frac{(U_{i-1}^{n+1} + \varepsilon) - (2U_i^{n+1}) + (U_{i+1}^{n+1} + \varepsilon)}{h^2} \\ &= (\Gamma^{n+1})(i) + \frac{\varepsilon}{h^2} \end{aligned}$$

and

$$\begin{aligned} \delta_{SS}(U^{n+1} + \varepsilon E_i)(i) &= \frac{(U_{i-1}^{n+1}) - 2(U_i^{n+1} + \varepsilon) + (U_{i+1}^{n+1})}{h^2} \\ &= (\Gamma^{n+1})(i) - \frac{2\varepsilon}{h^2} \end{aligned}$$

For the nonlinear term on the right hand side $-\frac{1}{h^2} \sigma^2((\Gamma^{n+1})(i))(\Gamma^{n+1})(i)$, we consider case 1 and case 2 are the worst case for long position in option, case 3 and case 4 are the best case for long position.

Case 1

For $\Gamma_i^{n+1} \leq 0$, implies that $\sigma^2(\Gamma_i^{n+1}) = \sigma_{max}^2 > 0$

Then we have;

$$-\frac{\sigma^2}{h^2}(\Gamma^{n+1}(i))(\Gamma^{n+1}(i)) = -\frac{\sigma_{max}^2}{h^2}(\Gamma^{n+1}(i))$$

$$\sigma^2(S, \Gamma) \cdot \Gamma = \sigma^2(S, Z) \cdot Z$$

For any S, Z_1 and Z_2 , which Z_1 and $Z_2 \leq 0$,

$$-\frac{1}{h^2}((\sigma_{max}^2)Z_1 - (\sigma_{max}^2)Z_2) = \begin{cases} -\frac{1}{h^2}\sigma_{max}^2(Z_1 - Z_2) \geq 0; & \text{if } Z_1 < Z_2 \quad Z_1, Z_2 \leq 0 \\ -\frac{1}{h^2}\sigma_{max}^2(Z_1 - Z_2) \leq 0; & \text{if } Z_1 > Z_2 \quad Z_1, Z_2 \leq 0 \end{cases}$$

Similarity, we can prove that it also holds for:

Case 2

For $\Gamma_i^{n+1} > 0$, implies that $\sigma^2(\Gamma_i^{n+1}) = \sigma_{min}^2 > 0$

Case 3

For $\Gamma_i^{n+1} > 0$, implies that $\sigma^2(\Gamma_i^{n+1}) = \sigma_{max}^2 > 0$

Case 4

For $\Gamma_i^{n+1} \leq 0$, implies that $\sigma^2(\Gamma_i^{n+1}) = \sigma_{min}^2 > 0$

Thus, for any $\varepsilon > 0$ and $1 \leq i \leq M - 1$ we have

$$F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1} + \varepsilon, U_{i-1}^{n+1} + \varepsilon, U_i^n + \varepsilon)$$

$$= \left(-\frac{rS_i}{h}\right)(U_{i+1}^{n+1} + \varepsilon) + \left(\frac{1}{\Delta\tau_n} + \frac{rS_i}{h} + r\right)U_i^{n+1} - \frac{1}{\Delta\tau_n}(U_i^n + \varepsilon)$$

$$- \frac{1}{h^2}S_i^2[\sigma^2(u_i^{n+1} + \varepsilon)](u_i^{n+1} + \varepsilon)$$

$$\leq F_i^{n+1}(U_i^{n+1}, U_{i+1}^{n+1}, U_{i-1}^{n+1}, U_i^n)$$

This is (23), Similarly, it is easy to show that (24) also hold true. So, the discretization scheme is monotone.

4.2. Lemma 2. Stability

For each $0 \leq n \leq N - 1$ let $U^{n+1} = (U_0^{n+1}, (\hat{U}^{n+1})^T, U_M^{n+1})^T$ let \hat{U}^{n+1} is a solution (13). Then, U^{n+1} satisfies

$$\|U^{n+1}\|_{\infty} \leq \max\{\|u_0\|_{\infty}, \|g_1\|_{\infty}, \|g_2\|_{\infty}\}$$

where u_0, g_1, g_2 is initial and boundary conditions (6–8), and $\|\cdot\|_{\infty}$ is norm l_{∞} .

Proof

For any $0 \leq n \leq N - 1$, (13) can be written as:

$$\beta_i^{n+1} U_i^{n+1} = -\alpha_i^{n+1} U_{i-1}^{n+1} - \gamma_i^{n+1} U_{i+1}^{n+1} + \frac{1}{\Delta \tau_n} U_i^n$$

For each $1 \leq i \leq M-1$. Remember that $\alpha_i^{n+1} < 0$, $\gamma_i^{n+1} < 0$ and $\beta_i^{n+1} > 0$.

If $\|U^{n+1}\|_\infty = |U_k^{n+1}|$ for $k \in \{1, 2, \dots, M-1\}$, then the equation

$$\beta_i^{n+1} \|U_i^{n+1}\|_\infty \leq -\alpha_i^{n+1} \|U_{i-1}^{n+1}\|_\infty - \gamma_i^{n+1} \|U_{i+1}^{n+1}\|_\infty + \frac{1}{\Delta \tau_n} \|U_i^n\|_\infty$$

with $i = k$ becomes $(\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1}) \|U^{n+1}\|_\infty \leq \frac{1}{\Delta \tau_n} \|U^n\|_\infty$

Thus, since $\alpha_i^{n+1} < 0$ and $\gamma_i^{n+1} < 0$ then we obtain

$$\begin{aligned} \|U^{n+1}\|_\infty &\leq \frac{1/\Delta \tau_n}{(\alpha_i^{n+1} + \beta_i^{n+1} + \gamma_i^{n+1})} \|U^n\|_\infty \leq \|U^n\|_\infty \\ &\leq \|U^{n-1}\|_\infty \leq \dots \leq \|U^0\|_\infty \leq \|g_1\|_\infty. \end{aligned} \quad (25)$$

If $\|U^{n+1}\|_\infty = |U_0^{n+1}|$ or $\|U^{n+1}\|_\infty = |U_M^{n+1}|$, from equations (6), (7) and (8) we can see that

$$\|U^{n+1}\|_\infty \leq \max\{|U_0^{n+1}|, |U_M^{n+1}|\} \leq \max\{\|g_1\|_\infty, \|g_2\|_\infty\} \quad (26)$$

Combining equations (25) and (26), gives

$$\begin{aligned} \|U^{n+1}\|_\infty &\leq \max\{\|U^n\|_\infty, |U_0^{n+1}|, |U_M^{n+1}|\} \\ &\leq \max\{\|u_0\|_\infty, \|g_1\|_\infty, \|g_2\|_\infty\} \end{aligned}$$

Hence, the discretization scheme (13) is stable.

4.3. Lemma 3. Consistency

Discretization scheme (13) is consistent.

Proof:

Lax equivalence theorem [12] states that the finite difference method is consistent for a given initial value problem.

4.4. Theorema 2. Convergence

Discretization scheme (13) is convergent to solutions (5) with boundary conditions (6-8) as $(h, \Delta \tau) \rightarrow 0$, where $h = \max_{0 \leq i \leq M-1} h_i$ and $\Delta \tau = \max_{0 \leq n \leq N-1} \Delta \tau_n$.

Proof:

[11], proved that if a discretization of the nonlinear 2nd order PDE is consistent, stable and monotone, then the solution of the fully discretized system converges to the viscosity solution to the PDE. Since discretization (13) proved to be a consistent, stable and monotone, then discretizing (13) converges. Theorem 4 is just a consequence of Lemma 1, 2 and 3.

5. Numerical Solution

The calculation of the option value by using the parameter $r = 0.1$, $T = 1$, $K = 40$, $S_{max} = 80$, $\sigma_{max}^2 = 0.0625$, $S_{min}^2 = 0.0225$, with a uniform mesh $h = 2$, $M = 40$ and $\Delta\tau = 0.05$ $N = 20$. Comparison of the option price on the best case and worst case for a position as an option buyer (long position) can be seen in Figure 1. Since the exact solution is unknown, we use the numerical solution on the uniform mesh with $h = 0.03125$, $M = 2560$ and $\Delta\tau = 0.00078125$, $N = 1280$ as the “exact” solution, V_{exact} .

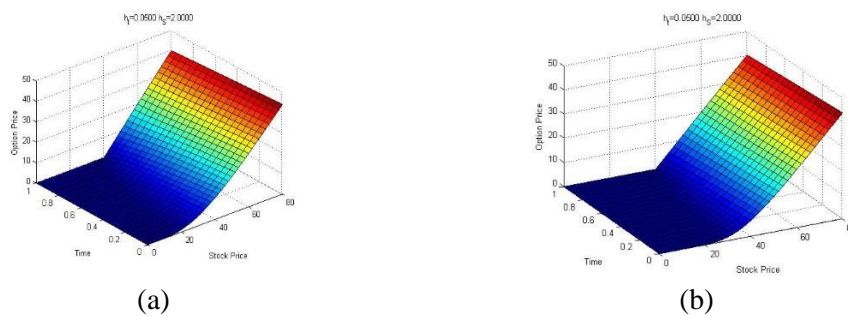


Figure 1 The price of the European call option for a position as a buyer for option with, (a) the best case and (b) the worst

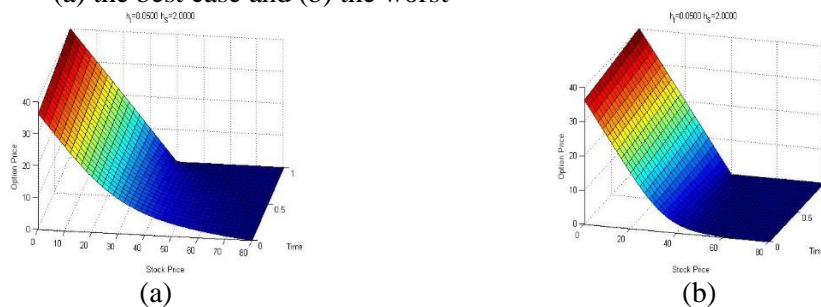


Figure 2 The price of the European put option for a position as an option to, (a) the best case and (b) the worst case

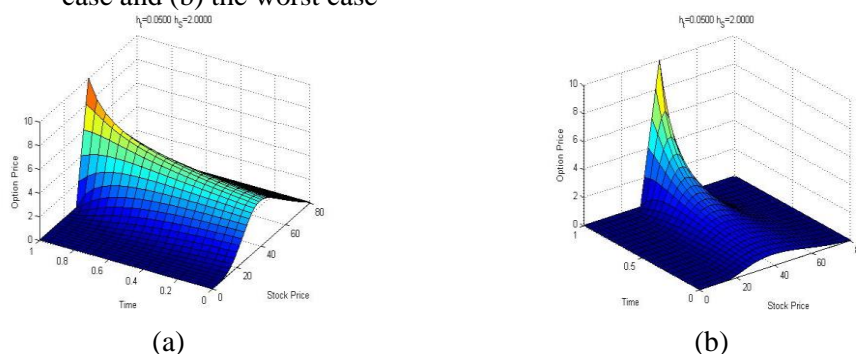


Figure 3 The price of the European butterfly option for a position as an option to, (a) the best case and (b) the worst case

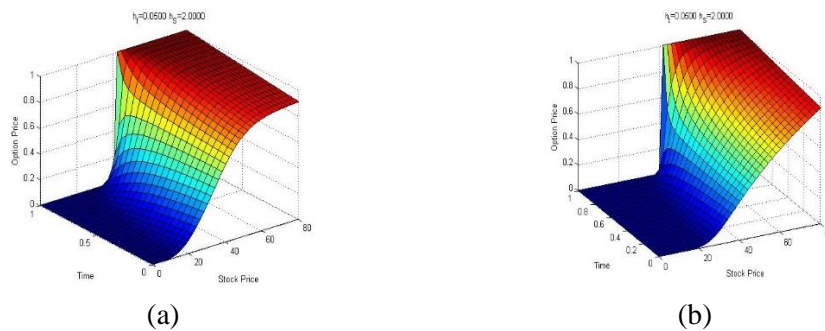


Figure 4 The price of the European cash or nothing option for a position as a buyer option to, (a) the best case and (b) the worst case

Table 1 Computed rated of convegence for the *call option*

M	N	The Best Case		The Worst Case	
		$\ \cdot\ _{h,\infty}$	Ratio	$\ \cdot\ _{h,\infty}$	Ratio
41	21	2.1583e-01		1.7887e-01	
81	41	1.2705e-01	1.70	9.9884e-02	1.79
161	81	7.7695e-02	1.64	5.6592e-02	1.76
321	161	4.7794e-02	1.63	3.2573e-02	1.74
641	321	2.7615e-02	1.73	1.8008e-02	1.81
1281	641	1.2437e-02	2.22	7.9266e-03	2.27

Using the “exact” solution, we can calculate the ratio of the numerical solution from the mesh with

$$\text{Ratio} = \frac{\|V_h^{\Delta\tau} - V_{exact}\|_{h,\infty}}{\|V_{h/2}^{\Delta\tau/2} - V_{exact}\|_{h,\infty}}$$

Where $V_h^{\Delta\tau}$ denotes a solution on the mesh with h is mesh size of stock dan $\Delta\tau$ mesh size of time, and

$$\|V_h^{\Delta\tau} - V_{exact}\|_{h,\infty} := \max_{1 \leq i \leq M; 1 \leq n \leq N} |V_i^n - V_{exact}(S_i, \tau_n)|.$$

Ratio calculation results in Table 1 shows the convergence order of upwind method of call option with the best case and worst case are about 1.6 and 1.7 respectively.

Futhermore, calculating rates of convergence from butterfly option, put option, and cash-or-nothing (CoN) option are the same as calculating the rates of convergence of call option. Therefore, we get that the rates of convergence of put option with the best case and the worst case are about 1.6 and 1.7 respectively, butterfly option with the best case and the worst case are about 1.6 and 1.7 respectively, and cash or nothing (CoN) option with the best case and the worst case is about 1.3 and 1.3 respectively.

6. Conclusions

This study used upwind finite difference method for the discretization of space and implicit method for time discretization of nonlinear PDE in option pricing with stochastic volatility model, where this discretization scheme proved monotone, consistent and stable. Based on the results of numerical simulations, it showed that the order of convergence for upwind finite difference method with stochastic volatility models is about 1.6 to 1.7 for the worst case and best case, with a position as a buyer of the option (long position).

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