Analytical solutions to assess the stability of rock slopes subject to cracks via limit analysis

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Abstract. Based on the kinematic approach of limit analysis, a full set of upper bound solutions for the stability of homogeneous rock slopes subjected to cracks are obtained. The generalized Hoek-Brown failure criterion is adopted to describe the non-linear strength envelope of rocks. In this paper, critical failure mechanisms are determined for cracks of known depth but unspecified location, cracks of known location but unknown depth, and cracks of unspecified location and depth. It is shown that there is a nearly up to 50% drop in terms of the stability factors for rock slopes intersected by cracks compared with intact ones. Tables and charts of solutions in dimensionless forms are presented for ease of use by practitioners.

1. Introduction

A substantial range of different numerical tools exist today to investigate slope stability. Traditionally stability analyses of rock slopes have been based on limit equilibrium methods [1, 2]. In the last four decades, numerical methods for continuum mechanics such as the finite element method with strength reduction technique [3, 4] and finite element limit analysis [5] have provided the capability to reliably detect the onset of failure in rock slopes where the presence of joints is accounted for by a so called smear approach. However, in case of moderately fractured rock the smear approach no longer works since the onset of instability is ruled by the behavior of single fractures. In this case, the Discrete Element Method can nowadays be employed for 3D analyses of jointed rock slopes [6]. The development of these analyses within computationally affordable runtimes has been made possible by recent algorithmic advances in terms of contact detection algorithms [7, 8, 9] and rock joint dataset representation [9].

Rock slopes considered in this paper are intact or heavily fractured slopes subject to cracks that can be attributed to tension. These types of cracks tend to be vertical and/or subvertical. A continuum mechanics approach that explicitly considers the presence of cracks is adopted in this paper. Utili [10] and Michalowski [11] investigated the stability of cohesive frictional slopes subject to cracks via the limit analysis upper bound method. The advantage of this method is in providing an analytical solution of general validity that can be used to assess slopes of any geometry and for any value of c, ϕ. But, unlike cohesive frictional materials, rock yield is non-linear so that the Hoek-Brown (H-B) failure criterion [12], is almost universally accepted as the best criterion to express rock yielding. Collins et al. [13] developed a "tangent line" technique to extend the application of the limit analysis upper

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bound method to the stability of rock slopes. Yang et al. [14] improved the analysis of Collins et al. [13] by modifying the generalized Hoek-Brown failure criterion [15]. However, neither Collins et al. [13] nor Yang et al. [14] considered the presence of cracks in rock slopes.

In this paper, an analytical formulation will be provided for the first time to investigate the stability of rock slopes subjected to cracks via the limit analysis upper bound method assuming that the rock slope is uniform and isotropic. Then, preliminary results of the analytical investigation are provided.

2. Hoek-Brown failure criterion

Based on the result of a series of field investigations and triaxial tests on rocks, Hoek & Brown [12] and Hoek et al. [15] introduced the well-known Hoek-Brown failure criterion which has become the most adopted method for the characterization of rock strength by far. The original H-B failure criterion can be expressed as

$$\sigma_1 = \sigma_3 + \sqrt{m\sigma_3\sigma_{ci} + s\sigma_{ci}^2} \tag{1}$$

with σ_1, σ_3 the major and minor principle stress respectively, σ_{ci} uniaxial compression strength for intact rock, m a parameter related to the rock type and s degree of fracturing of the rock mass.

The original Hoek-Brown failure criterion was designed for intact rock with high cohesion. In Hoek et al. [15], a new parameter n is introduced to extend the applicability of the criterion to loose and broken rocks. Thus, the generalized Hoek-Brown failure criterion is written as

$$\sigma_1 = \sigma_3 + \sigma_{ci}\left(m\frac{\sigma_3}{\sigma_{ci}} + s\right)^n \tag{2}$$

with n a parameter accounting for degree of imperfection of the rock mass. The failure envelop for the H-B failure criterion in terms of major and minor principle stresses is shown in Figure 1.

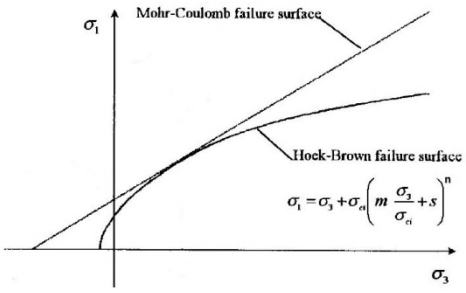


Figure 1. Hoek-Brown failure criterion.

3. Rock slope stability analysis

Although there is no lower/upper bound solution available for the stability of rock slopes with cracks, it was still expected by Li et al. [5] that the rigorous limit analysis results were found to bracket the true slope stability number to within ±9% or better for the intact case. In this paper, the upper bound solutions following the Mohr-Coulomb failure criterion are provided in the first place. The solutions are not directly applicable to materials obeying the H-B criterion. However, in principle, limit analysis can still be adopted as long as the material considered obeys the normality rule. A detailed upper bound solution of a rock slope with a vertical tension crack will be given in this section.

3.1. Limit analysis for slopes following Mohr-Coulomb failure criterion

For illustrative purposes, only a horizontal upper slope with failure line passing through the slope toe is examined (see Figure 2). For more general cases like failure line passing below the slope toe and non-horizontal upper slope are illustrated in Appendix. The slope has a height of H and an inclination of β, with a region of rock $\widehat{EDCB}$ rigidly rotating away about a centre of rotation P. The remaining part is bound by a crack B-C and logarithmic spiral $\widehat{DC}$ with an equation written in polar coordinates with reference to P,

$$r = r_0 \exp\left[\tan\phi\left(\theta - \theta_0\right)\right]$$

with r the distance of a generic point of the spiral to its centre P, θ the angle formed by r with a reference horizontal axis (see Figure 2), and θ_0 and γ_0 identifying the angle and distance of a particular point of the spiral F to its centre. According to normality rule [16], the angle between the sliding rate $\dot{u}$ of the rock mass and the failure line $\widehat{DC}$ must always equal to ϕ. However, it is not the case for the angle φ between $\dot{u}$ and the crack B-C, which can be different from ϕ. [10]

In Figure 2, the following geometrical relationships are obtained

$$r_\varsigma = r_\chi \exp\left[\tan\phi\left(\zeta - \chi\right)\right] \tag{3}$$

and

$$r_\upsilon = r_\chi \exp\left[\tan\phi\left(\nu - \chi\right)\right] \tag{4}$$

with r_ζ and r_υ the radii of the spiral at the ζ and υ angles respectively, and

$$H = r_\chi \left\{\exp\left[\tan\phi\left(\nu - \chi\right)\right]\sin\nu - \sin\chi\right\} \tag{5}$$

$$\delta = r_\chi \left\{\exp\left[\tan\phi\left(\zeta - \chi\right)\right]\sin\zeta - \sin\chi\right\} \tag{6}$$

$$L_1 = r_\chi \left\{\frac{\sin\left(\chi + \beta\right)}{\sin\beta} - \exp\left[\tan\phi\left(\nu - \chi\right)\right]\frac{\sin\left(\nu + \beta\right)}{\sin\beta}\right\} \tag{7}$$

$$L_2 = r_\chi \left\{\cos\chi - \exp\left[\tan\phi\left(\zeta - \chi\right)\right]\cos\zeta\right\} \tag{8}$$

with L_1 and L_2 horizontal lengths as indicated in Figure 2, and δ being the crack depth.

Three different types of mechanisms will be analysed:

(a) slopes with a crack of known depth but unknown location
(b) slopes with a crack of known location but unknown depth
(c) slopes with cracks of unknown location and depth

In conditions (a) and (b), certain geometric constraints are introduced for the selections of χ, υ and ζ. For cracks of known depth, the following constraint is found:

$$\exp\left(\tan\phi\cdot\zeta\right)\sin\zeta = \exp\left(\tan\phi\cdot\chi\right)\sin\chi\left(1 - \frac{\delta}{H}\right) + \frac{\delta}{H}\exp\left(\tan\phi\cdot\nu\right)\sin\nu \tag{9}$$

Similarly, for cracks of known location, the following constraint is found:

$$\exp\left(\tan\phi\cdot\chi\right)\sin\chi=\exp\left(\tan\phi\cdot\nu\right)\sin\nu+\frac{\exp\left(\tan\phi\cdot\nu\right)\cos\nu-\exp\left(\tan\phi\cdot\zeta\right)\cos\zeta}{x/H} \tag{10}$$

Then, the internally dissipated energy rate $\dot{W}_d$ along the failure line $\widehat{DC}$ and the external work rate $\dot{W}_\gamma$ for the slope are to be detailed respectively.

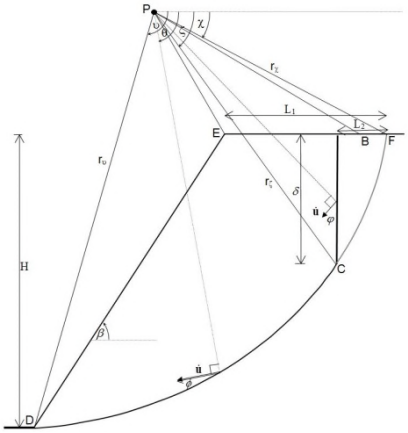


Figure 2. Failure mechanism

According to upper bound theorem and the assumed kinematic mechanism, energy is dissipated only along the logarithmic spiral shaped failure line $\widehat{DC}$, which is indicated as Γ in the integral

$$\dot{W}_d=\int_\Gamma c\dot{u}\cos\phi\frac{rd\theta}{\cos\phi}=c\omega\int_\Gamma r^2d\theta=c\omega r_\zeta^2\int_\nu^\zeta\exp\left[2\tan\phi\left(\theta-\zeta\right)\right]d\theta \tag{11}$$

from which the following expression is obtained

$$\dot{W}_d=c\omega r_\chi^2\exp\left[2\tan\phi\left(\zeta-\chi\right)\right]\frac{\exp\left[2\tan\phi\left(\nu-\zeta\right)\right]-1}{2\tan\phi}=c\omega r_\chi^2\,f_d\left(\chi,\nu,\zeta,\phi\right) \tag{12}$$

As shown in (Utili and Nova [17]; Utili and Crosta [18,19]), the rate of external work due to the rock weight of region $\widehat{EDCB}$ is computed as the work done by region $\widehat{EFD}$ minus the work of region $\widehat{BFC}$. The rate of external work for region $\widehat{EFD}$ is the result of the work done by region $\widehat{PFD}$ minus P-F-E and P-E-D. Likewise, work done by region $\widehat{BFC}$ can be expressed by the summation of the work done by region $\widehat{PFC}$ minus P-F-B and P-B-C. $\dot{W}_1$, $\dot{W}_2$, $\dot{W}_3$, $\dot{W}_4$, $\dot{W}_5$ and $\dot{W}_6$ indicate the work done by $\widehat{PFD}$, P-F-E, P-E-D, $\widehat{PFC}$, P-F-B and P-B-C respectively. Therefore, the total rate of external work due to the rock weight is given by:

$$\dot{W}_\gamma=\dot{W}_1-\dot{W}_2-\dot{W}_3-\left(\dot{W}_4-\dot{W}_5-\dot{W}_6\right)=\dot{W}_1-\dot{W}_2-\dot{W}_3-\dot{W}_4+\dot{W}_5+\dot{W}_6=\omega\gamma r_\chi^3\left(f_1-f_2-f_3-f_4+f_5+f_6\right) \tag{13}$$

The calculation of the work rates $\dot{f}_1$, $\dot{f}_2$, $\dot{f}_3$, $\dot{f}_4$, $\dot{f}_5$ and $\dot{f}_6$ can be found in Utili [10]. Equating the rate of external work $\dot{W}_\gamma$ to the rate of internal energy dissipation $\dot{W}_d$ gives

$$\dot{W}_\gamma=\dot{W}_d \tag{14}$$

$$\omega\gamma r_\chi^3\left(f_1-f_2-f_3-f_4+f_5+f_6\right)=c\omega r_\chi^2 f_d \tag{15}$$

Dividing by ω and r_χ^2 and rearranging, the stability factor $N_{M-C}=\frac{\gamma H}{c}$ for Mohr-Coulomb failure criterion is obtained as

$$N_{M-C}=\frac{\gamma H}{c}=g\left(\chi,\zeta,\nu\right)=\frac{f_d\left\{\exp\left[\tan\phi\left(\nu-\chi\right)\right]\sin\nu-\sin\chi\right\}}{f_1-f_2-f_3-f_4+f_5+f_6} \tag{16}$$

3.2. Limit analysis for slopes following Hoek-Brown failure criterion

Drescher & Christopoulos [20] first proposed a linear failure surface which is a tangent to the actual non-linear failure surface to get an upper bound solution. Meanwhile, the same "tangent line method" was adopted by Collins et al. [13] to linearize the Hoek-Brown failure criterion. Yang et al. [14] introduced a similar method as Collins et al. [13] but more complicated conditions such as sophisticated geometry (e.g., inclined upper slope), different strength parameters (e.g., n for the generalized H-B failure criterion) and pore pressure distribution are considered.

Considering the generalized Hoek-Brown failure criterion, Yang et al. [14] proposed a revised stability factor $N_{H-B}=\frac{\gamma H}{s^n\sigma_c}$, with $s^n\sigma_c$ the uniaxial compressive strength of the rock to be used in the definition of the stability factor N_{H-B} instead of c in equation (16).

The tangential line to the H-B failure criterion is expressed as

$$\tau=\sigma\tan\varphi_t+c_t \tag{17}$$

where τ and σ are the shear and normal stress, c_t is the intercept of the tangential line to τ axes in the (σ,τ) stress space, and φ_t is the angle of the tangential line at the point considered M to the σ axis (see Figure 3.).

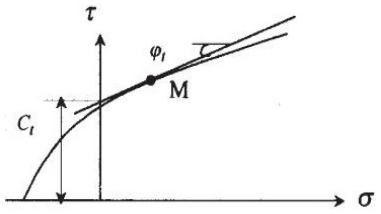


Figure 3. A tangential line for a non-linear failure criterion

According to Yang et al. [14], τ and σ along the failure envelope are determined by the following two equations

$$\frac{\tau}{\sigma_c}=\frac{\cos\varphi_t}{2}\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{n/(1-n)} \tag{18}$$

$$\frac{\sigma}{\sigma_c}=\left(\frac{1}{m}+\frac{\sin\varphi_t}{mn}\right)\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{1/(1-n)}-\frac{s}{m} \tag{19}$$

From equation (18) and equation (19), c_t is found by

$$\frac{c_t}{\sigma_c}=\frac{\cos\varphi_t}{2}\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{n/(1-n)}-\frac{\tan\varphi_t}{m}\left(1+\frac{\sin\varphi_t}{n}\right)\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{1/(1-n)}+\frac{s}{m}\tan\varphi_t \tag{20}$$

For the material following the original H-B failure criterion when $n=0.5$, equation (20) can be simplified in the form

$$\frac{c_t}{\sigma_c}=\frac{m\left(1-\sin\varphi_t\right)^2}{16\sin\varphi_t\cos\varphi_t}+\frac{s}{m}\tan\varphi_t \tag{21}$$

Compared with equation (16), the revised stability factor for H-B failure criterion is defined as

$$N_{H-B}=\frac{\gamma H}{s^n\sigma_c}=\frac{\gamma H}{c_t}\cdot\frac{c_t}{s^n\sigma_c}=N_{M-C}\frac{c_t}{s^n\sigma_c}=\frac{f_d}{f_1-f_2-f_3-f_4+f_5+f_6+p_w}\times \left\{\begin{array}{l}\dfrac{\cos\varphi_t}{2}\left[\dfrac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{n/(1-n)}-\dfrac{\tan\varphi_t}{m}\left(1+\dfrac{\sin\varphi_t}{n}\right)\left[\dfrac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{1/(1-n)}\\ +\dfrac{s}{m}\tan\varphi_t\end{array}\right\}\times\frac{\exp\left[\tan\phi_t\left(\nu-\chi\right)\right]\sin\nu-\sin\chi}{s^n} \tag{22}$$

The minimum value of the function $N_{H-B}(\chi,\upsilon,\zeta,\varphi_t)$ was found by evaluating repeatedly the function over the four variables χ,υ,ζ and φ_t in the range of values of engineering interest.

Results

In most cases, no information is available about the depth and the location of cracks, therefore the most critical scenario for the stability of the slope is assumed. This scenario entails the existence of the most critical crack in terms of depth and position for the slope. Obviously this is an unrealistic assumption since in general cracks other than the most critical one will be present. However, this assumption has the advantage of being on the conservative side. In the absence of data a conservative assumption has to be preferred. To search for the upper bound corresponding to the most critical scenario, equation (22) is minimized over all possible χ,υ,ζ and φ_t. In Figure 4, it is quite apparent that with the increase of the slope inclination, the values of the stability factors N_{H-B} drop significantly.

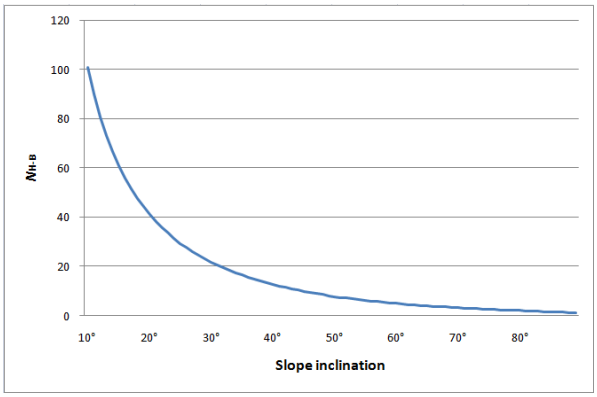


Figure 4. Stability factor N_{H-B} against slope inclination, $m=7.3$: limestone, $s=1$.

In practice, the generalized H-B failure criterion accounts for the variability of natural rock mass and the difference of fractured degree with n as a parameter in equation (2). The exponent n varies from 0.5 to 0.7. Sometimes the upper slope has an angle $\alpha \neq 0$ (see Figure 6 in Appendix), which complicates the solutions of the stability factors. Table 1 provides the most critical values of the stability factors $N_{critical}$ for cracked slopes with the slope inclination β varying from 30° to 90°, and α being equal to 5°, 10° and 15°.

From Table I, it is found that the parameter α has little influence on the stability factors. Figure 5 shows the influence of the exponent n (ranging from 0.5 to 0.7) on the stability factors for limestone slopes with $m = 7.3$, $s = 1$ and $\alpha = 5°$. The stability factors $N_{critical}$ increase with n.

Table 1. The stability factors $N_{critical}$ for limestone rock ($m = 7.3$, $s = 1$) with various values of n, α and β.

		Slope inclination β°						
n	α	30°	40°	50°	60°	70°	80°	90°
0.50	5	21.89	13.41	9.65	7.39	5.74	4.33	2.88
	10	21.71	13.35	9.62	7.36	5.70	4.28	2.83
	15	21.02	13.50	9.96	7.71	6.01	4.52	3.00
0.55	5	31.66	17.61	12.15	9.08	6.94	5.17	3.40
	10	31.49	17.58	12.13	9.06	6.90	5.11	3.34
	15	30.75	17.95	12.70	9.60	7.35	5.46	3.57
0.60	5	49.42	24.34	16.00	11.64	8.74	6.43	4.18
	10	49.22	24.33	15.98	11.62	8.69	6.35	4.09
	15	48.35	24.14	15.89	11.54	8.60	6.25	4.01
0.65	5	85.89	38.17	24.42	17.52	13.02	9.51	6.15
	10	85.63	38.17	24.40	17.48	12.96	9.40	6.02
	15	84.61	37.92	24.28	17.37	12.83	9.26	5.90
0.70	5	175.25	64.28	38.66	26.90	19.61	14.12	8.65
	10	175.20	64.34	38.67	26.87	19.52	13.95	8.83
	15	174.29	64.02	38.53	26.73	19.35	13.75	8.65

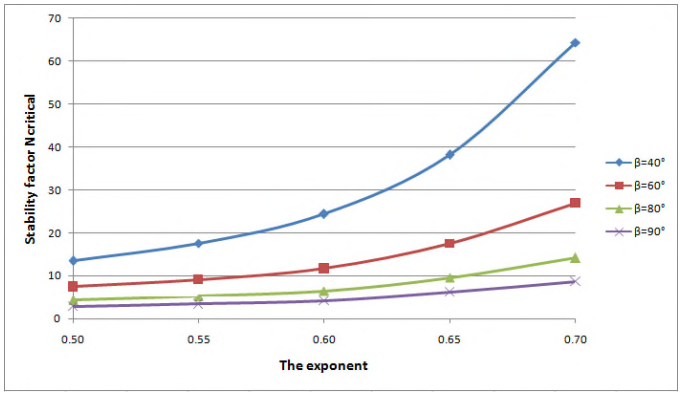


Figure 5. Influence of exponent n on the stability factor for $m = 7.3$, $s = 1$ and $\alpha = 5°$.

4. Conclusions

The kinematic approach of limit analysis and the tangent technique were applied to investigate the stability of rock slopes subjected to tension cracks with rocks obeying the generalized Hoek-Brown failure criterion. From the performed analyses it emerges that rock slopes subject to cracks can suffer a nearly up to 50% drop in stability factor in comparison with the case of intact rock slopes.

Appendix

Case of a non-horizontal upper slope ($\alpha \neq 0$)

The case of $\alpha \neq 0$ is illustrated in Figure 6. In the following, only the equations that assume a different expression from the equations illustrated in the paper for a horizontal upper slope ($\left(1-\frac{N_{k\neq 0}}{N_{k=0}}\right)\times 100\%$) are shown.

Equation (5) becomes

$$H = r_\chi \left\{ \exp\left[\tan\phi\left(\upsilon - \chi\right)\right] \sin\upsilon - \sin\chi \right\} \tag{14}$$

Equation (6) becomes

$$\delta = r_\chi \left\{ \exp\left[\tan\phi\left(\zeta - \chi\right)\right] \sin\zeta - \sin\chi \right\} \tag{15}$$

Equation (7) becomes

$$L_1 = r_\chi \left\{ \frac{\sin\left(\upsilon-\chi\right)}{\sin\left(\upsilon+\alpha\right)} - \exp\left[\tan\phi\left(\upsilon-\chi\right)\right] \times \sin\left(\upsilon+\alpha\right) \frac{\sin\left(\upsilon+\beta\right)}{\sin\left(\upsilon+\alpha\right)\sin\left(\beta-\alpha\right)} - \sin\left(\chi+\alpha\right) \right\} \tag{16}$$

Equation (8) becomes

$$L_2 = r_\chi \left\{ \cos\chi - \exp\left[\tan\phi\left(\zeta - \chi\right)\right]\cos\zeta \right\} \tag{17}$$

$\dot{W}_2$ becomes

$$\dot{W}_2 = \omega\gamma r_\chi^3 \left[\frac{1}{6}\sin\chi \frac{L_1}{r_\chi}\left(2\cos\chi - \frac{L_1}{r_\chi}\right)\right] = \omega\gamma r_\chi^3 f_2\left(\chi,\upsilon,\beta,\phi\right) \tag{18}$$

$\dot{W}_3$ becomes

$$\begin{aligned} \dot{W}_3 &= \omega\gamma r_\chi^3 \left[\begin{array}{l} \frac{1}{6}\exp\left[\tan\phi\left(\upsilon - \chi\right)\right]\left[\sin\left(\upsilon - \chi\right) - \frac{L_1}{r_\chi}\sin\upsilon \right] \\ \times\left\{ \cos\chi - \frac{L_1}{r_\chi} + \cos\upsilon\exp\left[\tan\phi\left(\upsilon - \chi\right)\right]\right\} \end{array} \right] \\ &= \omega\gamma r_\chi^3 f_3\left(\chi,\upsilon,\beta,\phi\right) \end{aligned} \tag{19}$$

$\dot{W}_6$ becomes

$$\begin{aligned} \dot{W}_6 &= \omega\gamma r_\chi^3 \left(\frac{1}{3}\exp\left[2\tan\phi\left(\zeta - \chi\right)\right]\left(\cos\zeta\right)^2 \times\left\{ \exp\left[\tan\phi\left(\zeta - \chi\right)\right]\sin\zeta - \sin\chi\right\}\right) \\ &= \omega\gamma r_\chi^3 p_3\left(\chi,\zeta,\phi\right) \end{aligned} \tag{20}$$

Equation (22) becomes

$$\begin{aligned} N_{H-B} = \frac{\gamma H}{s^n\sigma_c} = \frac{\gamma H}{c_t}\cdot\frac{c_t}{s^n\sigma_c} = N_{M-C}\frac{c_t}{s^n\sigma_c} = \frac{f_d}{f_1 - f_2 - f_3 - f_4 + f_5 + f_6}\times\frac{\sin\beta}{\sin\left(\beta - \alpha\right)} \\ \left\{ \frac{\cos\varphi_t}{2}\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{n/(1-n)} - \frac{\tan\varphi_t}{m}\left(1 + \frac{\sin\varphi_t}{n}\right)\left[\frac{mn\left(1-\sin\varphi_t\right)}{2\sin\varphi_t}\right]^{1/(1-n)} + \frac{s}{m}\tan\varphi_t \right\}\times \\ \frac{\exp\left[\tan\phi_t\left(\upsilon - \chi\right)\right]\sin\left(\upsilon + \alpha\right) - \sin\left(\chi + \alpha\right)}{s^n} \end{aligned} \tag{21}$$

Equation (9) becomes

$$\exp\left(\tan\phi\cdot\zeta\right)\sin\left(\zeta+\alpha\right)=$$
$$\exp\left(\tan\phi\cdot\chi\right)\sin\left(\chi+\alpha\right)\left(1-\frac{\delta}{H}\frac{\sin\beta}{\sin\left(\beta-\alpha\right)}\right)$$
$$+\frac{\delta}{H}\frac{\sin\beta}{\sin\left(\beta-\alpha\right)}\exp\left(\tan\phi\cdot\upsilon\right)\sin\left(\upsilon+\alpha\right) \tag{22}$$

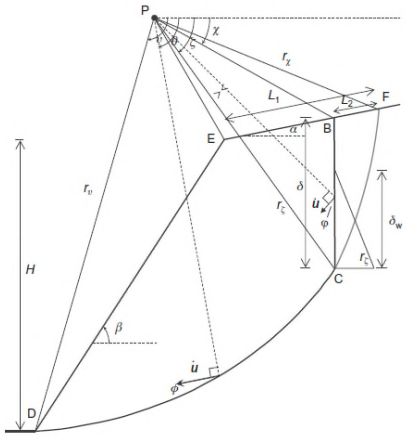


Figure 6. Inclined slope upper surface ($\alpha \neq 0$).

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