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Numerical solutions for BVPs governed by a Helmholtz equation of anisotropic FGM

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Abstract. A Boundary Element Method (BEM) is used for obtaining solutions to anisotropic functionally graded media (FGM) boundary value problems (BVPs) governed by a Helmholtz type equation. A technique of transforming the variable coefficient governing equation to a constant coefficient equation is utilized for deriving a boundary integral equation. Some particular problems are considered to illustrate the application of the BEM. The results show the convergence, consistency, and accuracy of the BEM solutions.

1. Introduction

The BEM has been successfully used for solving many types of problems of homogeneous media. Some works on homogeneous media problems have been recently done by Azis et. al [1, 2] and Haddade et. al [3] in which the authors considered pollutant transport problems governed by 2D diffusion-convection, and also Azis [4] in which numerical solutions for the Helmholtz boundary value problems were obtained.

However, this is generally not the case for FGM. There are two techniques usually used to deal with problems of FGM. The first one uses a technique of deriving a relevant Green function or fundamental solution to the FGM problem. Cheng [5] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For examples, Clements and Azis [6] considered the case for isotropic FGM. For anisotropic FGM some works have been studied by Azis and Clements [7, 8, 9] and Azis et. al [10]. In addition to this, recently Salam et. al [11] have been working on a class of elliptic problems for anisotropic FGM.

This paper discusses derivation of a boundary integral equation for numerically solving 2D boundary value problems governed by the (dimensionless) Helmholtz type equation of the form

$$\frac{\partial}{\partial x_i}\left[\lambda_{ij}\left(x_1,x_2\right)\frac{\partial\phi\left(x_1,x_2\right)}{\partial x_j}\right]+\beta^2\left(x_1,x_2\right)\phi\left(x_1,x_2\right)=0 \tag{1}$$

where the coefficients λ_{ij} and β^2 depend on x_1 and x_2 and the repeated summation convention (summing from 1 to 2) is employed.

A variety of problems of both isotropic and anisotropic inhomogeneous media are usually modeled with equation (1). Acoustic problems (when $\beta^2 > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta^2 = 0$) are the areas for which the

IOP Conf. Series: Earth and Environmental Science **279** (2019) 012008 doi:10.1088/1755-1315/279/1/012008

governing equation is of the type (1). Recently some works on the Helmholtz equation for isotropic and/or homogeneous media have been done. For example, Barucq et. al. in [12], Loeffler et. al. in [13] considered Helmholtz equation as the governing equation for isotropic and/or homogeneous media which is a special case of the equation (1).

The technique of transforming (1) to constant coefficient equations will be used for obtaining a boundary integral equation for the solution of (1). It is necessary to place some constraint on the class of coefficients λ_{ij} and β for which the solution obtained is valid. The analysis of this paper is purely formal; the main aim being to construct effective BEM for class of equations which falls within the type (1).

2. The boundary value problem

Referred to a Cartesian frame Ox_1x_2 a solution to (1) is sought which is valid in a region Ω in R^2 with boundary $\partial\Omega$ which consists of a finite number of piecewise smooth closed curves. On $\partial\Omega_1$ the dependent variable $\phi(\mathbf{x})$ ($\mathbf{x} = (x_1, x_2)$) is specified and on $\partial\Omega_2$

$$P(\mathbf{x}) = \lambda_{ij} \left(\partial\phi/\partial x_j\right) n_i \tag{2}$$

is specified where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$.

For all points in Ω the matrix of coefficients $[\lambda_{ij}]$ is a real symmetric positive definite matrix so that throughout Ω equation (1) is a second order elliptic partial differential equation. Therefore equation (1) may be written explicitly as

$$\frac{\partial}{\partial x_1}\left(\lambda_{11}\frac{\partial\phi}{\partial x_1}\right) + 2\frac{\partial}{\partial x_1}\left(\lambda_{12}\frac{\partial\phi}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\lambda_{22}\frac{\partial\phi}{\partial x_2}\right) + \beta^2\phi = 0$$

Further, the coefficients λ_{ij} and β are required to be twice differentiable functions of the two independent variables x_1 and x_2.

The method of solution will be to obtain boundary integral equations from which numerical values of the dependent variables ϕ and P may be obtained for all points in Ω. The analysis here is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$ and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. The coefficients λ_{ij} and β are required to take the form

$$\lambda_{ij}(\mathbf{x}) = \overline{\lambda}_{ij} g(\mathbf{x}) \tag{3}$$

$$\beta^2(\mathbf{x}) = \overline{\beta}^2 g(\mathbf{x}) \tag{4}$$

where the $\overline{\lambda}_{ij}$ and $\overline{\beta}$ are constants and g is a differentiable function of $\mathbf{x}$. Use of (3) and (4) and in (1) yields

$$\overline{\lambda}_{ij}\frac{\partial}{\partial x_i}\left(g\frac{\partial\phi}{\partial x_j}\right) + \overline{\beta}^2 g\phi = 0 \tag{5}$$

Let

$$\phi(\mathbf{x}) = g^{-1/2}(\mathbf{x})\,\psi(\mathbf{x}) \tag{6}$$

so that (5) may be written in the form

$$\overline{\lambda}_{ij}\frac{\partial}{\partial x_i}\left[g\frac{\partial\left(g^{-1/2}\psi\right)}{\partial x_j}\right] + \overline{\beta}^2 g^{1/2}\psi = 0$$

That is

$$\overline{\lambda}_{ij}\left[\left(\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}-\frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}\right)\psi+g^{1/2}\frac{\partial^2 \psi}{\partial x_i \partial x_j}\right]+\overline{\beta}^2 g^{1/2}\psi=0 \tag{7}$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j}=-\frac{1}{4}g^{-3/2}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j}+\frac{1}{2}g^{-1/2}\frac{\partial^2 g}{\partial x_i \partial x_j}$$

permits (7) to be written in the form

$$g^{1/2}\overline{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j}-\psi\overline{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j}+\overline{\beta}^2 g^{1/2}\psi=0$$

It follows that if g is such that

$$\overline{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j}=0 \tag{8}$$

then the transformation (6) carries the variable coefficients equation (5) to the constant coefficients equation

$$\overline{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j}+\overline{\beta}^2\psi=0 \tag{9}$$

And if g is such that

$$\overline{\lambda}_{ij}\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j}-\overline{\beta}^2 g^{1/2}=0 \tag{10}$$

then

$$\overline{\lambda}_{ij}\frac{\partial^2 \psi}{\partial x_i \partial x_j}=0 \tag{11}$$

Also, substitution of (3) and (6) into (2) gives

$$P=-P_g\psi+P_\psi g^{1/2} \tag{12}$$

where

$$P_g(\mathbf{x})=\overline{\lambda}_{ij}\frac{\partial g^{1/2}}{\partial x_j}n_i \qquad P_\psi(\mathbf{x})=\overline{\lambda}_{ij}\frac{\partial \psi}{\partial x_j}n_i$$

A boundary integral equation for the solution of (9) and (11) is given in the form

$$\eta(\mathbf{x}_0)\,\psi(\mathbf{x}_0)=\int_{\partial\Omega}\left[\Gamma(\mathbf{x},\mathbf{x}_0)\,\psi(\mathbf{x})-\Phi(\mathbf{x},\mathbf{x}_0)\,P_\psi(\mathbf{x})\right]ds(\mathbf{x}) \tag{13}$$

where $\mathbf{x}_0=(a,b)$, $\eta=0$ if $(a,b)\notin\Omega\cup\partial\Omega$, $\eta=1$ if $(a,b)\in\Omega$, $\eta=\frac{1}{2}$ if $(a,b)\in\partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (a,b).

The so called fundamental solution Φ in (13) is any solution of the equation

$$\overline{\lambda}_{ij}\frac{\partial^2 \Phi}{\partial x_i \partial x_j}+\overline{\beta}^2\Phi=\delta(\mathbf{x}-\mathbf{x}_0)$$

and the Γ is given by

$$\Gamma(\mathbf{x},\mathbf{x}_0)=\overline{\lambda}_{ij}\frac{\partial\Phi(\mathbf{x},\mathbf{x}_0)}{\partial x_j}n_i$$

where δ is the Dirac delta function. Following Azis [14], for two-dimensional problems Φ and Γ are given by

$$\begin{aligned}
\Phi(\mathbf{x},\mathbf{x}_0) &= \begin{cases} \frac{K}{2\pi}\ln R & \text{if } \overline{\beta}^2 = 0 \\ \frac{\imath K}{4} H_0^{(2)}(\omega R) & \text{if } \overline{\beta}^2 > 0 \end{cases} \\
\Gamma(\mathbf{x},\mathbf{x}_0) &= \begin{cases} \frac{K}{2\pi}\frac{1}{R}\overline{\lambda}_{ij}\frac{\partial R}{\partial x_j} n_i & \text{if } \overline{\beta}^2 = 0 \\ \frac{-\imath K\omega}{4} H_1^{(2)}(\omega R)\,\overline{\lambda}_{ij}\frac{\partial R}{\partial x_j} n_i & \text{if } \overline{\beta}^2 > 0 \end{cases}
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
K &= \ddot{\tau}/\zeta \\
\omega &= \sqrt{|\overline{\beta}^2/\zeta|} \\
\zeta &= \left[\overline{\lambda}_{11} + 2\overline{\lambda}_{12}\dot{\tau} + \overline{\lambda}_{22}\left(\dot{\tau}^2 + \ddot{\tau}^2\right)\right]/2 \\
R &= \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \\
\dot{x}_1 &= x_1 + \dot{\tau}x_2 \\
\dot{a} &= a + \dot{\tau}b \\
\dot{x}_2 &= \ddot{\tau}x_2 \\
\dot{b} &= \ddot{\tau}b
\end{aligned}$$

where $\dot{\tau}$ and $\ddot{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic

$$\overline{\lambda}_{11} + 2\overline{\lambda}_{12}\tau + \overline{\lambda}_{22}\tau^2 = 0$$

and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively and $\imath$ represents the square root of minus one. The derivatives $\partial R/\partial x_j$ needed for the calculation of the Γ in (14) are given by

$$\begin{aligned}
\frac{\partial R}{\partial x_1} &= \frac{1}{R}(\dot{x}_1 - \dot{a}) \\
\frac{\partial R}{\partial x_2} &= \dot{\tau}\left[\frac{1}{R}(\dot{x}_1 - \dot{a})\right] + \ddot{\tau}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right]
\end{aligned}$$

Use of (6) and (12) in (13) yields

$$\begin{aligned}
\eta(\mathbf{x}_0)\, g^{1/2}(\mathbf{x}_0)\,\phi(\mathbf{x}_0) &= \int_{\partial\Omega}\left\{\left[g^{1/2}(\mathbf{x})\,\Gamma(\mathbf{x},\mathbf{x}_0) - P_g(\mathbf{x})\,\Phi(\mathbf{x},\mathbf{x}_0)\right]\phi(\mathbf{x})\right. \\
&\quad \left. - \left[g^{-1/2}(\mathbf{x})\,\Phi(\mathbf{x},\mathbf{x}_0)\right] P(\mathbf{x})\right\} ds(\mathbf{x})
\end{aligned} \tag{15}$$

This equation provides a boundary integral equation for determining ϕ and P at all points of Ω.

4. Numerical examples

Some particular boundary value problems will be solved numerically by employing the integral equation (15). Standard boundary element method is employed to obtain numerical results. The integrals in equation (15) are evaluated numerically using the Bode's quadrature (see Abramowitz and Stegun [15]). The main aim is to show the validity of the analysis used above for deriving the boundary integral equation (15) and the appropriateness of the BEM in solving the problems through the derived boundary integral equation (15).

IOP Conf. Series: Earth and Environmental Science **279** (2019) 012008 doi:10.1088/1755-1315/279/1/012008

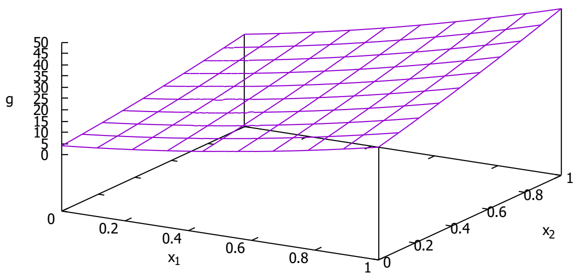

Figure 1. A quadratic inhomogeneity function $g(\mathbf{x}) = [2\,(1 + 1.5x_1 + x_2)]^2$

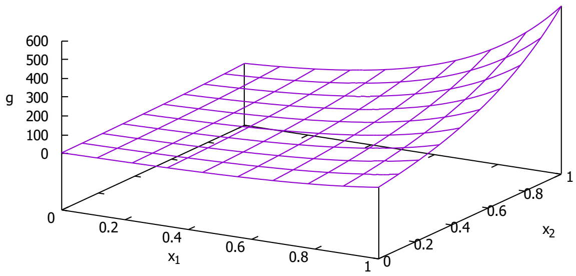

Figure 2. An exponential inhomogeneity function $g(\mathbf{x}) = [2\exp(1.5x_1 + x_2)]^2$

4.1. Examples with analytical solutions

In order to see the convergence and accuracy of the BEM we will consider some examples of problems with analytical solutions. Two possible multiparameter forms of the inhomogeneity function $g(\mathbf{x})$ satisfying (8) and (10) are quadratical and exponential functions respectively. For all problems considered in this section we take quadratical and exponential functions $g(\mathbf{x})$ respectively as

$$g(\mathbf{x}) = [A\,(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]^2 \tag{16}$$

$$g(\mathbf{x}) = [A\exp(\alpha_1 x_1 + \alpha_2 x_2)]^2 \quad \overline{\beta}^2 = \overline{\lambda}_{11}\alpha_1^2 + 2\overline{\lambda}_{12}\alpha_1\alpha_2 + \overline{\lambda}_{22}\alpha_2^2 \tag{17}$$

with $A = 2, \alpha_0 = 1, \alpha_1 = 1.5, \alpha_2 = 1$. Plots of $g(\mathbf{x})$ are shown in Figures 1–2. The geometry of the region Ω and the boundary conditions are as depicted in Figure 3. The values of the constant coefficients $\overline{\lambda}_{ij}$ for the governing equation (1) are

$$\overline{\lambda}_{11} = 1, \overline{\lambda}_{12} = 0.5, \overline{\lambda}_{22} = 2$$

For the case when $g(\mathbf{x})$ satisfies equation (8), $\psi(\mathbf{x})$ must satisfy (9). For the analytical solution ϕ we will take

$$\psi(\mathbf{x}) = B\left[\cos(\gamma_1 x_1 + \gamma_2 x_2) + \sin(\gamma_1 x_1 + \gamma_2 x_2)\right] \quad \overline{\beta}^2 = \overline{\lambda}_{11}\gamma_1^2 + 2\overline{\lambda}_{12}\gamma_1\gamma_2 + \overline{\lambda}_{22}\gamma_2^2$$

$$\psi(\mathbf{x}) = B\,(\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2) \quad \overline{\beta}^2 = 0$$

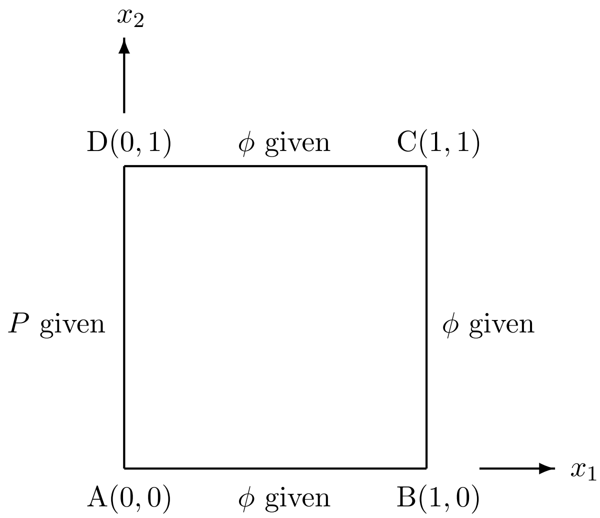

Figure 3. The geometry of all problems in Section 4.1

Whereas if $g(\mathbf{x})$ satisfies equation (10) then $\psi(\mathbf{x})$ satisfies (11). And we will take

$$\psi(\mathbf{x}) = B\left(\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2\right)$$

Then the analytical solution ϕ can be obtained from equation (6). The parameters for ψ are taken to be

$$B = 2.5, \gamma_0 = 1, \gamma_1 = 1, \gamma_2 = 1.5$$

4.1.1. Quadratically graded media: $\boldsymbol{g}(\mathbf{x})$ *is of the form* (16)

Problem 4.1.1.1: $\overline{\boldsymbol{\beta}}^{\mathbf{2}} > \mathbf{0}$ *in equation* (9). The analytical solution is

$$\phi\left(\mathbf{x}\right) = \frac{B\left[\cos\left(\gamma_1 x_1 + \gamma_2 x_2\right) + \sin\left(\gamma_1 x_1 + \gamma_2 x_2\right)\right]}{A\left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2\right)}$$

so that $\overline{\beta}^2 = \overline{\lambda}_{ij}\gamma_i\gamma_j = 7$. The results are shown in Table 1. The BEM solution converges to the analytical solution as the number of elements increases.

Problem 4.1.1.2: $\overline{\boldsymbol{\beta}}^{\mathbf{2}} = \mathbf{0}$ *in equation* (9). Now we choose analytical solution

$$\phi\left(\mathbf{x}\right) = \frac{B\left(\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2\right)}{A\left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2\right)}$$

The results are shown in Table 2. Again, the BEM solution converges to the analytical solution as the number of elements increases.

4.1.2. Exponentially graded media: $\boldsymbol{g}(\mathbf{x})$ *is of the form* (17)

Problem 4.1.2. We take the analytical solution

$$\phi\left(\mathbf{x}\right) = \frac{B\left(\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2\right)}{A\exp\left(\alpha_1 x_1 + \alpha_2 x_2\right)}$$

Table 3 shows the results of the analytical and BEM solutions with 160, 320 and 640 elements of equal length. The BEM solution converges to the analytical solution as the number of elements increases.

IOP Conf. Series: Earth and Environmental Science **279** (2019) 012008 doi:10.1088/1755-1315/279/1/012008

Table 1. BEM and analytical solutions for Problem 4.1.1.1

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 160 elements			BEM 320 elements		
(0.1,0.5)	1.0670	-1.0382	-0.7508	1.0681	-1.0399	-0.7511
(0.3,0.5)	0.8734	-0.9078	-0.8044	0.8742	-0.9090	-0.8043
(0.5,0.5)	0.7012	-0.8184	-0.8408	0.7018	-0.8193	-0.8405
(0.7,0.5)	0.5448	-0.7469	-0.8561	0.5453	-0.7477	-0.8557
(0.9,0.5)	0.4020	-0.6823	-0.8495	0.4023	-0.6831	-0.8494
(0.5,0.1)	0.9463	-0.6379	-0.3210	0.9465	-0.6379	-0.3194
(0.5,0.3)	0.8497	-0.7617	-0.6285	0.8502	-0.7627	-0.6276
(0.5,0.7)	0.5195	-0.8168	-0.9614	0.5201	-0.8176	-0.9616
(0.5,0.9)	0.3223	-0.7655	-0.9966	0.3229	-0.7663	-0.9966
	BEM 640 elements			Analytical		
(0.1,0.5)	1.0686	-1.0405	-0.7514	1.0691	-1.0411	-0.7517
(0.3,0.5)	0.8746	-0.9096	-0.8043	0.8750	-0.9102	-0.8043
(0.5,0.5)	0.7021	-0.8198	-0.8403	0.7024	-0.8203	-0.8402
(0.7,0.5)	0.5455	-0.7481	-0.8555	0.5457	-0.7486	-0.8553
(0.9,0.5)	0.4024	-0.6834	-0.8493	0.4025	-0.6838	-0.8491
(0.5,0.1)	0.9466	-0.6382	-0.3189	0.9468	-0.6387	-0.3183
(0.5,0.3)	0.8504	-0.7632	-0.6272	0.8507	-0.7637	-0.6269
(0.5,0.7)	0.5204	-0.8180	-0.9616	0.5207	-0.8183	-0.9618
(0.5,0.9)	0.3232	-0.7664	-0.9969	0.3234	-0.7665	-0.9972

Table 2. BEM and analytical solutions for Problem 4.1.1.2

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 160 elements			BEM 320 elements		
(0.1,0.5)	1.4017	-0.5173	0.2893	1.4016	-0.5170	0.2881
(0.3,0.5)	1.3142	-0.3701	0.2896	1.3141	-0.3700	0.2886
(0.5,0.5)	1.2501	-0.2778	0.2794	1.2500	-0.2778	0.2786
(0.7,0.5)	1.2011	-0.2160	0.2655	1.2010	-0.2161	0.2649
(0.9,0.5)	1.1625	-0.1723	0.2506	1.1624	-0.1727	0.2503
(0.5,0.1)	1.1141	-0.2272	0.4136	1.1145	-0.2281	0.4122
(0.5,0.3)	1.1887	-0.2599	0.3366	1.1889	-0.2601	0.3356
(0.5,0.7)	1.3014	-0.2866	0.2356	1.3012	-0.2865	0.2349
(0.5,0.9)	1.3449	-0.2897	0.2016	1.3446	-0.2894	0.2008
	BEM 640 elements			Analytical		
(0.1,0.5)	1.4016	-0.5168	0.2875	1.4015	-0.5165	0.2870
(0.3,0.5)	1.3141	-0.3699	0.2881	1.3141	-0.3698	0.2876
(0.5,0.5)	1.2500	-0.2778	0.2782	1.2500	-0.2778	0.2778
(0.7,0.5)	1.2010	-0.2162	0.2646	1.2010	-0.2163	0.2643
(0.9,0.5)	1.1623	-0.1729	0.2502	1.1623	-0.1731	0.2501
(0.5,0.1)	1.1147	-0.2281	0.4116	1.1149	-0.2283	0.4109
(0.5,0.3)	1.1889	-0.2602	0.3351	1.1890	-0.2603	0.3346
(0.5,0.7)	1.3011	-0.2864	0.2346	1.3010	-0.2863	0.2343
(0.5,0.9)	1.3445	-0.2893	0.2005	1.3443	-0.2892	0.2002

Table 3. BEM and analytical solutions for Problem 4.1.2

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 160 elements			BEM 320 elements		
(0.1,0.5)	1.2074	-1.1589	-0.2267	1.2073	-1.1586	-0.2275
(0.3,0.5)	0.9911	-1.0035	-0.2645	0.9910	-1.0032	-0.2652
(0.5,0.5)	0.8058	-0.8507	-0.2677	0.8058	-0.8506	-0.2681
(0.7,0.5)	0.6501	-0.7095	-0.2515	0.6500	-0.7097	-0.2517
(0.9,0.5)	0.5209	-0.5853	-0.2249	0.5209	-0.5844	-0.2260
(0.5,0.1)	0.8810	-0.7844	-0.0791	0.8812	-0.7872	-0.0789
(0.5,0.3)	0.8528	-0.8417	-0.1954	0.8529	-0.8418	-0.1961
(0.5,0.7)	0.7479	-0.8288	-0.3073	0.7478	-0.8285	-0.3076
(0.5,0.9)	0.6844	-0.7868	-0.3247	0.6843	-0.7860	-0.3239
	BEM 640 elements			Analytical		
(0.1,0.5)	1.2073	-1.1585	-0.2280	1.2072	-1.1583	-0.2284
(0.3,0.5)	0.9910	-1.0032	-0.2655	0.9910	-1.0031	-0.2659
(0.5,0.5)	0.8058	-0.8506	-0.2683	0.8058	-0.8506	-0.2686
(0.7,0.5)	0.6500	-0.7097	-0.2519	0.6500	-0.7097	-0.2520
(0.9,0.5)	0.5209	-0.5846	-0.2260	0.5208	-0.5847	-0.2260
(0.5,0.1)	0.8814	-0.7877	-0.0794	0.8815	-0.7880	-0.0801
(0.5,0.3)	0.8529	-0.8419	-0.1964	0.8530	-0.8420	-0.1968
(0.5,0.7)	0.7477	-0.8284	-0.3077	0.7477	-0.8283	-0.3079
(0.5,0.9)	0.6842	-0.7862	-0.3240	0.6842	-0.7862	-0.3241

4.2. Examples without analytical solutions

In this section we will consider some examples of problems without simple analytical solutions. We setup some problems for a homogeneous isotropic material by taking $g(\mathbf{x})$ constant, $\overline{\lambda}_{11} = \overline{\lambda}_{22} = 1, \overline{\lambda}_{12} = 0$ and with symmetrical boundary conditions. This function $g(\mathbf{x})$ satisfies equation (8) and so the corresponding $\psi(\mathbf{x})$ should satisfy (9). The aim is to see the consistency of the BEM of whether it produces symmetrical solutions.

4.2.1. Problem 4.2.1: $\overline{\boldsymbol{\beta}}^{\mathbf{2}} > \mathbf{0}$ *in equation* (9). We take $g(\mathbf{x}) = 4, \overline{\beta}^2 = 1$ and boundary conditions are as shown in Figure 4. Table 4 shows the results of the BEM solution using 80, 160, 320 and 640 elements of equal length. As expected, the results converge as the number of elements increases and also they are symmetrical about the axes $x_2 = 0.5$.

4.2.2. Problem 4.2.2: $\overline{\boldsymbol{\beta}}^{\mathbf{2}} = \mathbf{0}$ *in equation* (9). We consider a problem with $g(\mathbf{x}) = 9, \overline{\beta}^2 = 0$ and the boundary conditions are as shown in Figure 5. Table 5 shows the results of the BEM solution using 80, 160, 320 and 640 elements of equal length. The results converge as the number of elements increases and also they are symmetrical about the axes $x_1 = 0.5$.

5. Conclusion

The Helmholtz type governing equation (1) is sometimes used for modeling physical problems such as acoustic problems (when $\beta^2 > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta^2 = 0$). The boundary integral equation (15) is derived from this governing equation (1) and then from (15) a BEM is constructed for calculation of numerical solutions to the problems for anisotropic functionally graded media including quadratically and exponentially graded media. The results show that the BEM solution gives a convergence, consistency, and accuracy. Therefore the results also prove that the analysis used for deriving

Table 4. BEM solution for Problem 4.2.1

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 80 elements			BEM 160 elements		
(0.1,0.5)	0.3612	-0.2875	-0.0000	0.3619	-0.2876	-0.0000
(0.3,0.5)	0.2969	-0.3534	-0.0000	0.2975	-0.3537	-0.0000
(0.5,0.5)	0.2207	-0.4053	-0.0000	0.2213	-0.4058	-0.0000
(0.7,0.5)	0.1358	-0.4410	-0.0000	0.1363	-0.4416	-0.0000
(0.9,0.5)	0.0455	-0.4590	-0.0000	0.0459	-0.4598	-0.0000
(0.5,0.1)	0.2207	-0.4056	0.0000	0.2213	-0.4059	-0.0000
(0.5,0.3)	0.2207	-0.4054	0.0001	0.2213	-0.4058	0.0000
(0.5,0.7)	0.2207	-0.4054	-0.0001	0.2213	-0.4058	-0.0000
(0.5,0.9)	0.2207	-0.4056	-0.0000	0.2213	-0.4059	0.0000
	BEM 320 elements			BEM 640 elements		
(0.1,0.5)	0.3622	-0.2876	0.0000	0.3623	-0.2876	0.0000
(0.3,0.5)	0.2978	-0.3538	0.0000	0.2980	-0.3539	0.0000
(0.5,0.5)	0.2216	-0.4059	0.0000	0.2217	-0.4060	0.0000
(0.7,0.5)	0.1365	-0.4419	0.0000	0.1366	-0.4420	0.0000
(0.9,0.5)	0.0460	-0.4601	0.0000	0.0461	-0.4603	0.0000
(0.5,0.1)	0.2216	-0.4060	-0.0000	0.2217	-0.4060	-0.0000
(0.5,0.3)	0.2216	-0.4059	0.0000	0.2217	-0.4060	-0.0000
(0.5,0.7)	0.2216	-0.4059	-0.0000	0.2217	-0.4060	0.0000
(0.5,0.9)	0.2216	-0.4060	0.0000	0.2217	-0.4060	0.0000

Table 5. BEM solution for Problem 4.2.2

(x_1, x_2)	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$	ϕ	$\partial\phi/\partial x_1$	$\partial\phi/\partial x_2$
	BEM 80 elements			BEM 160 elements		
(0.1,0.5)	0.5221	0.2153	0.9230	0.5222	0.2155	0.9222
(0.3,0.5)	0.5576	0.1288	0.8031	0.5578	0.1288	0.8020
(0.5,0.5)	0.5708	0.0000	0.7607	0.5710	0.0000	0.7595
(0.7,0.5)	0.5576	0.1288	0.8031	0.5578	-0.1288	0.8020
(0.9,0.5)	0.5221	0.2154	0.9230	0.5222	-0.2155	0.9222
(0.5,0.1)	0.3472	0.0000	0.2866	0.3479	0.0000	0.2855
(0.5,0.3)	0.4353	0.0000	0.5773	0.4357	0.0000	0.5761
(0.5,0.7)	0.7339	0.0000	0.8588	0.7339	0.0000	0.8575
(0.5,0.9)	0.9106	0.0000	0.9005	0.9103	-0.0000	0.8990
	BEM 320 elements			BEM 640 elements		
(0.1,0.5)	0.5223	0.2156	0.9218	0.5224	0.2157	0.9217
(0.3,0.5)	0.5579	0.1289	0.8014	0.5579	0.1289	0.8011
(0.5,0.5)	0.5711	-0.0000	0.7589	0.5711	-0.0000	0.7586
(0.7,0.5)	0.5579	-0.1289	0.8014	0.5579	-0.1289	0.8011
(0.9,0.5)	0.5223	-0.2156	0.9218	0.5224	-0.2157	0.9217
(0.5,0.1)	0.3482	-0.0000	0.2851	0.3484	0.0000	0.2849
(0.5,0.3)	0.4360	-0.0000	0.5754	0.4361	0.0000	0.5751
(0.5,0.7)	0.7338	-0.0000	0.8568	0.7338	-0.0000	0.8565
(0.5,0.9)	0.9101	-0.0000	0.8983	0.9100	-0.0000	0.8979

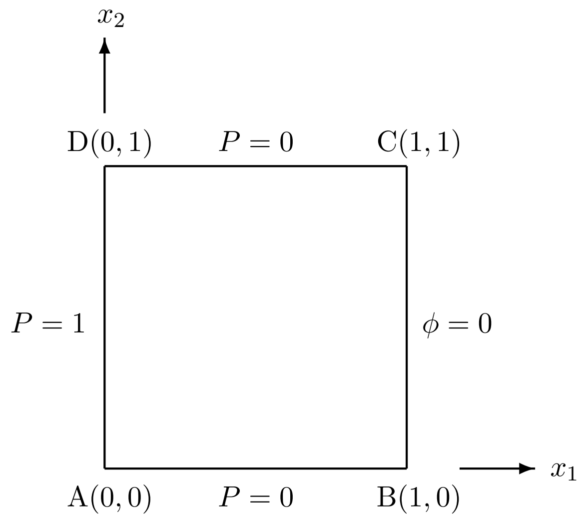

Figure 4. The geometry of Problem 4.2.1

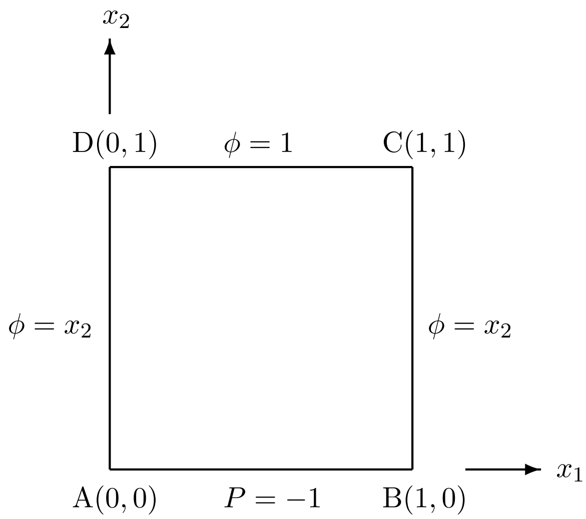

Figure 5. The geometry of Problem 4.2.2

the boundary integral equation (15) is valid. Together with its ease in implementation, it may be concluded that BEM is a useful numerical method for solving such kind of problems.

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