Analytical solution for the vertical structure eigenfunction in diagnosis of the low frequency wave-type perturbations in primitive dynamics equation

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S E Rainchik[1]*

[1] Institute of Information Systems and Geotechnologies, Russian State Hydrometeorological University, 79 Voronezhskaya str., St.-Petersburg 192007 Russia

E-mail: srainchik@rshu.ru

Abstract. This study presents an analytical solution to the problem of determining the vertical modal structure of low-frequency atmospheric oscillations. This analytical solution is critically compared with the numerical solution of this problem known in the literature, and good agreement for vertical normal modes with the wave number less than 9 is found. Applicability of the analytical solution for diagnosing the atmospheric climate variability and the quality of its simulation by contemporary GCM is briefly discussed.

1. Introduction

Using a classical approach, a special diagnostic technique to analyze the long-wave-type eigen-solutions of governing primitive equations, which characterize free, unforced, temporal changes in the atmospheric dynamics, simulated by GCM. These results have demonstrated that at special conditions, the generated eigen-oscillations influence the climate statistics by changing the time-averaged basic state of the atmospheric circulation [3].

Within this general eigenvalue problem, the problems of determination of the horizontal normal modes and of the vertical normal modes, respectively, are intrinsically coupled. So, in general case, this problem does not allow for separation of variables and has to be studied numerically. The horizontal normal modes can be specified as a sum of associated Legendre polynomials or, what sometimes is preferred, as a sum of Hough functions being the eigenfunctions of Laplace's tidal equations on a sphere. However, in this short note we would like to remind the reader on existence of an exact analytical solution by Monin and Gavrilin to a very similar problem of determination of vertical normal modes in the quasi-geostrophic atmospheric dynamics. This analytical approach has a potential of testing and improving the diagnostic technique applied to atmospheric data and GCM outputs.

2. Diagnostic Problem Formulation

We use the primitive equations written in the log-pressure coordinates. These equations are linearized with respect to the perturbations superimposed on a basic state. The basic state corresponds to a real horizontal & vertical distribution of meteorological parameters, averaged over time. To study the eigensolutions of the primitive equations and to recognize the individual atmospheric wave modes, by following the approach of Tanaka, this system was re-formulated in terms of the Galerkin projection

onto orthogonal basis functions. As the result, the original set of governing equations is split into two sub-systems, corresponding to the horizontal normal modes and the vertical normal modes, respectively.

3. Analytical Solution for the Vertical Normal Mode Structure

The thermodynamic energy balance equation, taken under adiabatic approximation, gives an ordinary differential equation describing the vertical structure of oscillation [1].

$$\frac{d}{dp}\left(\frac{p^2}{R\gamma}\frac{dG_m}{dp}\right)+\frac{G_m}{h_m}=0 \tag{3.1}$$

with p as the pressure, G_m is the vertical structure function depending on height (pressure) and on the vertical modal index m; γ is the static stability parameter variable with height as

$$\gamma=\frac{RT_0}{c_p}-p\frac{dT_0}{dp} \tag{3.2}$$

where $T_0(p)$ is the basic state temperature. The boundary condition follows from the energy balance at the top and the bottom (Earth surface) levels in the atmosphere, as in [1].

$$\frac{dG_m}{dp}+\frac{gG_m}{pT_0}=0 \tag{3.3}$$

It is taken both at the surface level $p = p_0$ and at the top level $p = p_H$.

Equation (3.1) together with the boundary conditions (3.3), constitutes the classical Sturm-Liouville problem.

To solve this problem, we apply an exact analytical solution of by *Monin* and *Gavrilin* for a similar differential operator and similar boundary conditions. In terms of (3.1), (3.3), the exact analytical solution

$$G_m(p)=\pm\sqrt{\frac{p_0}{p}}\left[\sqrt{2}\nu_m\cos\left(\nu_m\ln\frac{p_0}{p}\right)+\left(1-2\alpha_0^2\right)\sin\left(\nu_m\ln\frac{p_0}{p}\right)\right] \tag{3.4}$$

The surface pressure p_0 is taken equal to 1000hPa; the eigenvalues are given by the formula

$$\nu_m=\pi m/\ln\left(p_0/p_H\right) \tag{3.5}$$

where m is the vertical wave number (changing from 0 to 12); $p_H=50\,hPa$;

$$\alpha_0^2=\gamma/T_0 \tag{3.6}$$

is the non-dimensional static stability parameter. It should be noted that (3.4) describes both the barotropic and baroclinic components, which can be compared with numerical results [1] (Table 1). The input $T_0(p)$ and $\gamma(p)$ data at pressure levels similar as in *Tanaka and Kung* [1] have been used from the paper just cited; see 1st-3rd columns of Table 1.

Table 1. Basic state temperature T_0, stability parameter γ, and the equivalent height h_m; from *Tanaka and Kung* [1986].

p(hPa)	To(K)	γ(K)	m	h_m(m)
50	208.22	53.18	0	9564.8
70	210.42	53.44	1	1629.1
100	212.75	53.98	2	338.6

150	215.52	49.54	3	163.6
200	220.50	39.53	4	92.5
250	227.10	31.19	5	64.6
300	233.95	25.94	6	47.8
400	247.54	22.14	7	28.1
500	253.44	24.76	8	21.9
700	274.23	32.45	9	14.4
850	282.80	35.13	10	10.6
1000	290.40	32.28	11	8.6

4. Results and Analysis

The analytical solution (3.4) is in good agreement with the numerical results (Figures 1 to 4). Note, that no indication on the tropopause is apparent in the global mean state shown in Table 1, because of insufficient model resolution in the stratosphere. The wavenumber $m = 0$ corresponds to the barotropic mode, whereas the modes with $m \geq 1$ are regarded as the baroclinic modes, with the number of nodes given by m. In numerical computations, the vertical structure of the higher baroclinic modes may depend on the selection of vertical levels. Since these modes have an aliasing problem in the stratosphere, these issues are addressed in the subsequent computations.

In particular, the baroclinic mode with $m = 2$ has a single maximum at 150 hPa. For $m = 3$ this maximum is shifting down to 180 hPa, but a secondary maximum appears at 100 hpa.

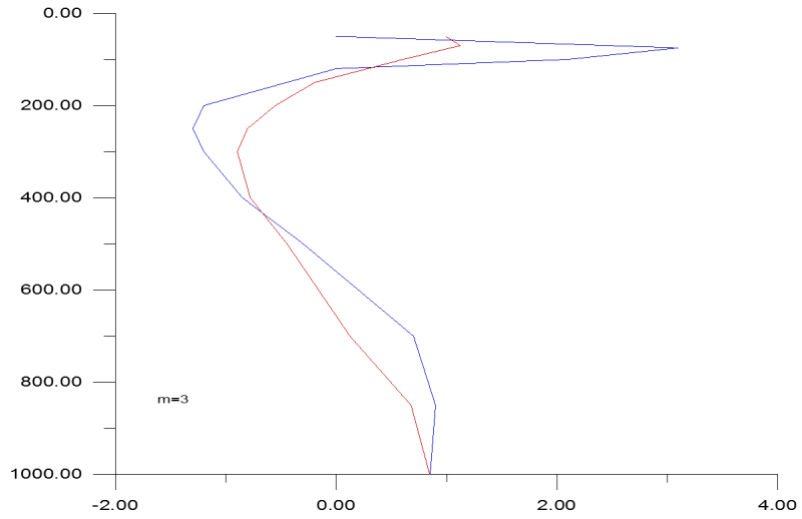

Figure 1. Vertical eigenvectors for vertical mode for $m = 3$ (comparison for analytical and numerical solutions).

The Baroclinic mode shows a maximum at 150 and 70 hPa. For the baroclinic mode has three maxima, at 230, 90, and 50 hPa.

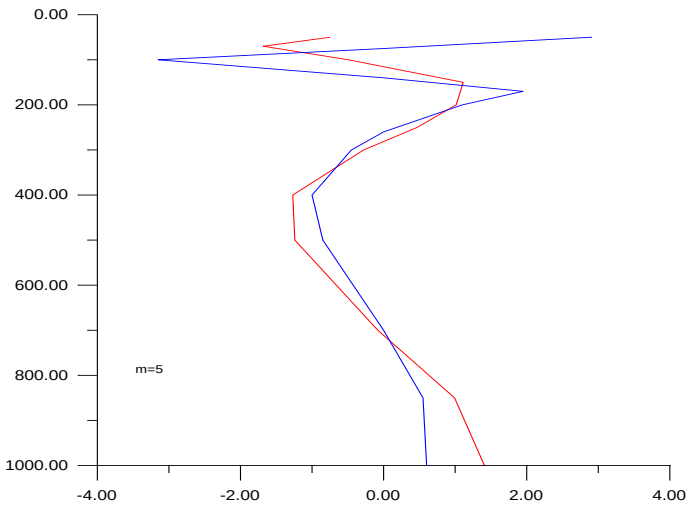

Figure 2. Vertical eigenvectors for vertical mode for m = 5 (comparison for analytical and numerical solutions).

Modes with m = 6 have maximum at 250 hPa, waves with m = 6 show maxima at 300, 200, and 150 hPa. Wave number 8 is associated with the baroclinic mode having maxima at 700, 450, 270, and200 hPa. For m = 9 maxima occur at 500, 400, and 250 hPa.

The comparison of the analytical solution with the numerical eigenfunction profiles shows good agreement for m = 9, but for higher m-values the analytical solution does not match the numerical computations. However, depending on the considered problem the analytical solution can be easily used in various circumstances for the purpose of comparison with numerical calculations of the baroclinic modes.

5. Summary and Conclusuions

In this work, we deal with an analytical solution to the problem of the vertical eigenmodes determination. In distinction to previously obtained numerical solutions, our approach is based on the exact analytical solution by Monin [2] which greatly facilitates analysis and allows for easier diagnostics of different dynamical mechanisms, which can govern the waves vertical structure and their propagation with altitude. Comparison of our analytical results with the previously obtained numerical profiles show good correspondence and confirm the applicability and usefulness of the analytical solution.

Following the approach proposed by H. Tanaka, the present investigation deals with developing of diagnostic techniques to be applied to analysis of the wave-type modal structures of the atmospheric general circulation, by using the primitive governing equations. This study might be useful to illuminate such an important modern question as to what extent can the atmospheric dynamics be responsible for the low frequency atmospheric climate variability? Additionally, this diagnostic could assist in finding some deficiencies in the numerical solutions to the climate models, as far as it concerns the adequate reproduction of the low-frequency climate variability and climate statistics.

Finally, our approach opens an easier way to future diagnostics of nonlinear interactions between modes, of mechanisms responsible for the upscale energy fluxes, which contribute to the atmospheric mean climate state and its variations.

References

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