

PAPER • OPEN ACCESS

About Interval Type Walras Dynamic Model for Single Product Market

To cite this article: V Sokolov and M Sevodin 2019 *IOP Conf. Ser.: Earth Environ. Sci.* **272** 032221

View the [article online](#) for updates and enhancements.

About Interval Type Walras Dynamic Model for Single Product Market

V Sokolov¹, M Sevodin¹

¹Faculty of Applied Mathematics and Mechanics, Perm National Research Polytechnic University, Perm, Russia

E-mail: m.sevodin@mail.ru

Abstract. The article is devoted to the application of the theory of fuzzy sets to the problem of determining certain characteristics of economic subjects of the economy. On the basis of fuzzy estimates of the function of excess demand, a modification of the Walrasian dynamic model of the single commodity market is presented. Estimates of the interval type of the equilibrium position of the model are obtained. The conditions for its stability are found, which represent a system of inequalities for the parameters of the model. The boundary value problem, equivalent in some sense to Walras's usual dynamic model of the single commodity market with constant coefficients for adjusting the supply price, as well as the demand price, is studied. An approximate solution of this boundary-value problem is constructed with a guaranteed degree of accuracy. These results are carried over to the case of the considered modification of the interval type Walras model. The problem was set and solved to change the price of the commodity to the final moment of time.

1. Introduction

Thanks to its application orientedness, the theory of differential equations with deviating arguments stands out among other mathematical theories. It describes a large number of application-related situations [1]. In economics, we see examples of such equations when modeling the economy of production, pricing, interaction among planned market mechanisms, etc. Especially interesting is the study of linear differential equations with lagging arguments using methods of functional analysis [1,2]. This approach allowed us to explore the stability of various economic model modifications and establish some of those models' parameters.

At the same time, any economic process is subject to uncertainty due to the large number of factors involved. [3], for example, states that uncertainty and incompleteness are fundamental properties of economic processes. It is therefore important to extend the above-mentioned methods to the case of economic models with various kinds of uncertainties.

This article discusses the Walrasian model of stabilization of single-product market prices, which, if formulated deterministically, can be considered a classical problem [4-6]. Beyond the traditional theory, we see new aspects in interpreting it and defining an equilibrium associated with the fuzzy, interval-type uncertainty of the coefficients and functions involved in the model. The intervality here arises from our incomplete knowledge about the object, modeling errors, model parameter calculation errors, and/or the consequences of supply and demand linearization.



The multitude of possible implementations of model parameters within the problem's intervals is what distinguishes interval problems from deterministic ones. At the same time, the little amount of information we have about the actual values of the coefficients does not allow us to consider it a stochastic problem and use conventional filtering methods [7].

At the moment there is a methodology for solving similar tasks that involves methods of the fuzzy set theory. The mathematical foundations of fuzzy research methods were mainly laid in the second half of the 20th century [8-10].

Interval analysis is one of the more effective research tools in the study of various mathematical models characterized by high uncertainty. Originally created for the needs of computational mathematics, it has been increasingly used in control theory [11-14], operations research [15, 16], and game theory [17,18].

In economics, methods of the fuzzy set theory have been used since the late 1970s. For example, [19] considers a system of differential equations with fuzzy parameters and introduces Leontief's input-output matrix, whose elements are triangular fuzzy numbers. Also notable here is the monograph [20], which considers a wide range of possible applications of the fuzzy set theory, from evaluating investment efficiency to managing personnel and equipment replacements.

We want to specifically highlight the use of triangular fuzzy numbers and refer to the previous research in the field of investments [19-24]. The theory of triangular numbers allows us to reduce qualitative, expert-based assessments to ones that are quantitative and numerical, even if fuzzy. On the other hand, fuzzy sets provide an expert with greater flexibility. For example, when assessing a given economic indicator, the expert can consider a pessimistic, optimistic and most likely scenarios and combine the resulting information in a triangular fuzzy number. This is precisely the approach we follow herein.

This article focuses on applying triangular fuzzy numbers to the problem of determining some characteristics of economic objects. Based on fuzzy estimates of the excess demand function, we come up with a modification of the dynamic Walrasian model for a single-product market. We also give interval-type estimates of the model's equilibrium position and its stability conditions. We study a boundary value problem that is in a sense equivalent to the conventional dynamic Walrasian model for a single-product market with constant coefficients for adjusting the offer price and the demand price. We give, with a guaranteed degree of accuracy, an approximate solution of said boundary value problem. We then apply these results to the case of the interval-type Walrasian model in question. Finally, we formulate and solve the problem of an n -fold increase in the price of good by a certain point in time.

2. Fuzzy Walrasian model

Consider a modified linear Walrasian model [25] of a single-product market with a piecewise constant lag of the supply price.

$$E'(t) = E(P(t), P([t/T]T), T) + \bar{\eta}(t). \quad (1)$$

Here, $E(t) = D(t) - S(t)$ is the excess demand function, $D(t) = \alpha - aP(t)$, $S(t) = -\beta + bP([t/T]T)$ are the demand and supply functions, $P(t)$ is the price of a unit of goods at time t , T is the price lag, λ is a response time coefficient, α, β are positive parameters, $[t/T]$ is the integer part of t/T , a, b are the coefficients for adjusting the supply price, $\bar{\eta}(t)$ is an uncontrollable perturbation and n is a natural number.

Express $E(t)$ via $P(t)$ and define $p = \lambda b$, $q = \lambda a$, $\lambda(\alpha + \beta) + \bar{\eta}(t) = f(t)$. Then (1) can be expressed as

$$P'(t) + pP([t/T]T) + qP(t) = f(t). \quad (2)$$

(1) assumes that all parameters and the excess demand function itself have exact values. In practice, however, exact values cannot be usually obtained.

Consider the parameters of (1) as triangular numbers $\tilde{c} = (c_1, c_2, c_3)$. c_1 is the least possible value of c , c_3 is the greatest possible value of c , and c_2 is the most likely value of c . Functions of (1) are also considered as triangular [7], i.e. $\tilde{f}(t) = (f_1(t), f_2(t), f_3(t))$, where $f_i(t)$ has the same meaning as c_i , $i=1,2,3$. Note that arithmetic operations and differentiation over fuzzy triangular functions are performed according to rules that can be found in e.g. in [7].

Now let the fuzzy parameters in (1) be such that we can obtain an equation similar to (2) for each component of the function $\tilde{P}(t)$:

$$P'_i(t) + p_i P_i([t/T]T) + q_i P_i(t) = f_i(t), i=1,2,3. \quad (3)$$

For each of these equations there is a point of equilibrium where the rate of change of the corresponding price is zero. These equilibrium positions at $\bar{\eta}(t) \equiv 0$ are defined by equality relations [25]: $P_i^* = (\alpha_i + \beta_i) / (a_i + b_i)$, $i=1,2,3$. It is known [25] that the obtained positions are exponentially stable. This allows us to assert price stabilization. The equilibrium price can be expressed as a triangular fuzzy number $\tilde{P}^* = (P_1^*, P_2^*, P_3^*)$. This triangular number can be considered a fuzzy equilibrium for the problem (1).

We highlight separately the case when the fuzzy equilibrium becomes a non-fuzzy number. This occurs if

$$(\alpha_1 + \beta_1) / (a_1 + b_1) = (\alpha_2 + \beta_2) / (a_2 + b_2) = (\alpha_3 + \beta_3) / (a_3 + b_3).$$

Obviously, the exponential stability holds in this case.

Now we will express the above processes in terms of solving a boundary value problem, considering its deterministic case first.

3. Boundary value problem and its solution

Take n and consider (1) within the time interval $[0, nT]$.

Formulate the problem on an ω -fold increase in the price of good by the target point in time nT :

$$P(nT) = \omega P(0) \quad (4)$$

Express the boundary condition (3) as $\ell P \equiv \Psi \cdot P(0) + \int_0^{nT} \varphi(s) P'(s) ds = \bar{\beta}$ where $\Psi = 1 - \omega$, $\varphi(s) \equiv 1$, $\bar{\beta} = 0$ ([1], page 32). Using the number Ψ and the function φ find a function $u(t)$ such that $u(0) \neq 1$, $\ell u = 1$: $u(t) = (t\omega + nT) / (nT)$

Then the system of equations

$$P'(t) + B(t)P(0) = z(t), \ell P = 0, B(t) = -u'(t)/u(0),$$

is uniquely solvable ([1], page 52) and has the solution that can be expressed as

$$P(t) = \int_0^{nT} W(t,s) z(s) ds, \quad (5)$$

where $W(t,s) = \{1 - u(t)\varphi(s), 0 \leq s \leq t \leq nT; -u(t)\varphi(s), 0 \leq t < s \leq nT\}$.

Apply a "W-bstitution" (4) to (2):

$$P'(t) + B(t)P(0) = -pP([t/T]T) + qP(t) + B(t)P(0) + f(t)$$

We get the integral equation

$$z(t) = \int_0^{nT} K(t,s) z(s) ds + f(t), \quad (6)$$

If $K(t,s) = B(t)W(0,s) - pW([t/T]T,s) - qW(t,s)$.

Replace the equation (5) with

$$\tilde{z}(t) = \int_0^{nT} \tilde{K}(t,s) \tilde{z}(s) ds + f(t). \quad (7)$$

with a degenerate kernel $\tilde{K}(t,s) = \sum_{j=0}^m a_j(t) b_j(s)$, where the functions $a_j(t), b_j(t)$ are determined by a piecewise constant approximation of the kernel $K(t,s)$, where such approximation corresponds to a uniform partition of the square $[0, nT] \times [0, nT]$ into smaller squares with side lengths meeting the precision requirement (see (11)).

Then (6) can be expressed as

$$\tilde{z}(t) = \int_0^{nT} \left(\sum_{j=0}^m a_j(t) b_j(s) \right) \tilde{z}(s) ds + f(t). \quad (8)$$

Multiply both parts of (7) by $b_i(t)$ and integrate it member-wise from 0 to nT for all $i = \overline{0, m}$.

Then the system (8) can be expressed as

$$P_i = \sum_{j=0}^m \alpha_{ij} P_j + c_i, \quad i = \overline{0, m},$$

$$\text{where } P_i = \int_0^{nT} b_i(t) z(t) dt, \alpha_{ij} = \int_0^{nT} b_i(t) a_j(t) dt, c_i = \int_0^{nT} b_i(t) f(t) dt, i = \overline{0, m}.$$

If the matrix

$$A = \{\gamma_{ij}\}, \quad \gamma_{ij} = e_{ij} - \alpha_{ij}, \quad \text{where } e_{ij} = \{1, i = j; 0, i \neq j, i, j = \overline{1, m}\}$$

has an inverse matrix $A^{-1} = \{\theta_{ij}\}$, then the system (8) and consequently the equation (6) have the only solution:

$$\tilde{z}(t) = \sum_{j=0}^m a_j(t) P_j + f(t),$$

or

$$\tilde{z}(t) = \int_0^{nT} R(t,s) \tilde{z}(s) ds + f(t), \quad (9)$$

where

$$R(t,s) = \sum_{i=0}^m \sum_{j=0}^m a_j(t) \theta_{ij} b_i(s). \quad (10)$$

We know from [2] that, given natural assumptions for the kernel $K(t,s)$ we can define the kernel $\tilde{K}(t,s)$ for any given $\varepsilon > 0$ as follows:

$$\int_0^{nT} \int_0^{nT} [K(t,s) - \tilde{K}(t,s)]^2 dt ds \leq \varepsilon^2. \quad (11)$$

Let matrix A built according to the functions $a_j(t), b_j(t), j = \overline{0, m}$, be invertible and $A^{-1} = \{\theta_{ij}\}$. If $\varepsilon < 1/r$,

where $r = 1 + \left\{ \int_0^{nT} \int_0^{nT} [R(t,s)]^2 dt ds \right\}^{1/2}$, and the function $R(t,s)$ is defined according to the equality relation

(10), then the equation (5) with the kernel $K(t,s)$, which meets the inequation (12), has a unique solution.

Thus, the boundary value problem (2), (3) is uniquely solvable, and its solution can be expressed as

$$\tilde{P}(t) = \int_0^t \tilde{z}(s) ds - \int_0^t B(s) P(0) ds,$$

with a precision of

$$\int_0^{nT} [z(t) - \tilde{z}(t)]^2 dt \leq \frac{\varepsilon^2 r^4}{(1 - \varepsilon r)^2} \cdot \int_0^{nT} [f(t)]^2 dt ,$$

and, in addition,

$$P(0) = \int_0^{nT} W(0, s) z(s) ds .$$

4. Solving the boundary value problem in the fuzzy case. Conclusions

We now consider the fuzzy variation of the problem. Just as in paragraph 2 above, we assume that the fuzzy parameters are such that the counterpart of equation (2) takes the form of (2'). Again, we take some natural number n and consider the equations (2') at $t \in [0, nT]$. We now formulate the problem of an ω -fold increase in the price of goods by the time nT in its fuzzy form, considering $P(t)$ and ω as triangular numbers: $\tilde{P}(nT) = \tilde{\omega} \tilde{P}(0)$. This condition splits into three: $P_i(nT) = \omega_i P_i(0)$, $i=1,2,3$. Thus, given (2') for each $i=1,2,3$, we come to the problems solved in paragraph 3 above. Thus, the formulated problem is uniquely solvable, and its solution can be expressed through (16).

In conclusion, we note that the study of the fuzzy dynamic Walrasian model for a single-product market with constant supply price adjustment coefficients has shown that, given the conditions for the model parameters as per the equations (2'), we can assert the existence of an equilibrium position and its exponential stability. As for the problem of an ω -fold increase in the price of goods by the time nT , we have defined a boundary value problem that is in a sense equivalent to the one formulated. Hence, we have given, with a guaranteed degree of accuracy, an approximate solution of said boundary value.

5. References

- [1] Azbelev N V, Maksimov V P, Rakhmatullina L V 1991 Introduction to theory of functional differential equations *Nauka* (Moscow)
- [2] Maksimov V P, Rumyantsev A N 1993 Boundary value problems and problems of pulse control in economic dynamics *Constructive study IzVUZ Mathematics* **5** 56-71
- [3] Suslov V I 2016 Economic dimensions: probability and reliability, mathematical modeling, Big Data, e-statistics *Voprosy Statistiki* **1** 4
- [4] Nikaido H 1968 Convex structures and economic theory *Academic Press* (NewYork)
- [5] Goodwin R M 1947 Dynamical coupling with especial reference to markets having production lags *Econometrica* **15** 181-204
- [6] Allen R 1963 Mathematical economics *Foreign Literature Publisher* (Moscow)
- [7] Kalman R, Falb P, Arbib M 1971 Essays on mathematical theory of systems *Mir* (Moscow)
- [8] Zadeh L A 1965 Fuzzy Sets *Inf. And Control* **8** 338-353
- [9] Wong C K 1975 Fuzzy Topology Fuzzy Sets and their Applications to Cognitive and Decision Processes *Academic Press* 171-190
- [10] Zadeh L A 1975 The concept of a linguistic variable and its application to approximate reasoning, Part I: Inf. Sci. **8** 199-249; Part II: Inf. Sci. **8** 301-357; Part III: Inf. Sci. **9** 43-80
- [11] Gusev Yu M, Efanov V N, Krymsky V G, Rutkovsky V Yu 1991 Analysis and synthesis of linear interval dynamic systems (the state of the problem).I, II. *AS Bulletin Engineering cybernetics* **1** 3-23 **2** 3-30
- [12] Ashchepkov L T, Davydov D V 2002 Stabilization of observable linear system of control with constant interval coefficients *IzVUZ Mathematics* **2**(477) 11-17
- [13] Ashchepkov L T, Kolpakova G E, Stegostenko Yu B 2002 Stabilization of non-stationary linear discrete system of control with interval coefficients by phase state observations *Automatics and teleautomatics* **5** 3-11
- [14] Davydov D V 2003 Local stabilization of observable linear system of control with uncertain parameters *computational technologies* **1**(8) 44-51

- [15] Ashchepkov L T, Kosogorova I B 2002 Minimization of quadratic function with interval coefficients *Journal of computational mathematics and mathematical physics* **5**(42) 653-664
- [16] Levin V I 1998 Comparison of interval numbers and optimization of uncertain systems *Information technologies* **7** 22-32
- [17] Ashchepkov L T, Gutorova S V, Karpachev A A, Li S 2003 Interval matrix games *Far East mathematical journal* **2** 276-288
- [18] Shashikhin V N 2001 Solution of an interval matrix game in mixed strategies *Izv Akad Nauk Theory and control systems* **5** 97-104
- [19] Buckley J J 1987 The fuzzy mathematics of finance *Fuzzy Sets and Systems* **21** 257-273
- [20] Kofman A, Hill Aulakh X 1992 Introduction to theory of fuzzy sets in enterprise management *Higher School* (Minsk)
- [21] Kuchta D 2000 Fuzzy capital budgeting *Fuzzy Sets and Systems* **111** 367-385
- [22] Lefley F, Sarkis J 2005 Applying the FAP model to the evaluation of strategic information technology projects *International Journal of Enterprise Information Systems* **1** 69-90
- [23] Huang X 2007 Optimal project selection with random fuzzy parameters *Int. J. Production Economics* **106** 513-522
- [24] Nedosekin A O 2002 Fuzzy multiple risk analysis of stock investments *Sezam* (Moscow)
- [25] Simonov P M 2003 Research of stability of solutions of some dynamical micro- and macroeconomic models *Bulletin of Perm University Mathematics Informatics Mechanics* **1** 88-93