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Wave Equation in the Formulation of the Cauchy Problem with Respect to Spatial Variables for the Prediction of Catastrophic Events

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Abstract. To determine a priori the unknown structure of the Earth and the forecast of catastrophic events such as earthquakes, it is suggested to apply the formulation of the Cauchy problem when both the Cauchy stress vector and the displacement vector as a function of the boundary and time coordinates t are given on the same surface. The advantages of this formulation are demonstrated by the example of the solution of the dynamic problem for a semi-infinite rod where there are no initial conditions along the entire length of the rod, and only Cauchy conditions at the end of the rod imitating the Earth's surface are given. To construct a finite-difference scheme with the second order of approximation accuracy, both the Cauchy conditions and the wave equation are used at the end of the rod. The calculated dependences of displacements, displacement velocities along the rod, comparison of numerical and analytical solutions are given.

1. Introduction

For the purpose of predicting earthquakes, scientists have investigated the relationship of the forthcoming earthquake with the movement of the earth's crust [1], [2], changes in the groundwater table in the wells [3], soil temperature change [4], a change in ion concentration in the ionosphere [4] and so on. Scientists still do not know all the details of the physical processes associated with the earthquake, and methods that they can accurately predict. At the same time, because of the urgency of the problem in different countries, the issues of earthquake prediction continue to be studied. In Russia, satellite technologies are used to monitor the total electronic content of the ionosphere, and the temperature in the lower layers of the atmosphere [5]. Methods of active monitoring using vibrational sources [6], mathematical modeling of phenomena and digital processing of observations [7-11] are being developed.

2. Problem statement and its solution

In this paper, the problem is posed-for example, the solution of the wave equation in Cauchy's formulation, but not in the classical version, when the Cauchy conditions are set at the initial instant of time, and in another version, when the Cauchy conditions are set for a given value of the physical coordinate, events like earthquakes, mountain strikes in mine conditions and so on. This formulation is



different from the generally accepted ones, it requires the construction of new algorithms, schemes and numerical calculation programs. To prove this assertion, we consider an example of a one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}, \quad (1)$$

where $u = u(x, t)$ is an unknown function (displacement), t is time, x is a spatial variable, and a is the velocity (phase velocity). Equation (1) refers to the case of a homogeneous medium. However, in this paper we will assume that in this medium there are sources located at some points with coordinate x and acting in time according to the law $\varphi(x, t)$. These sources send signals in a homogeneous medium that move in a homogeneous medium with velocity a . The task is to determine the position of the sources and the laws of their radiation.

To illustrate the situation, imagine that there is a semi-infinite rod, which is the simplest model of the Earth. The rod is depicted in Figure 1. Here, $x=0$ corresponds to the Earth's surface, the growth of the x coordinate means an immersion into the Earth.

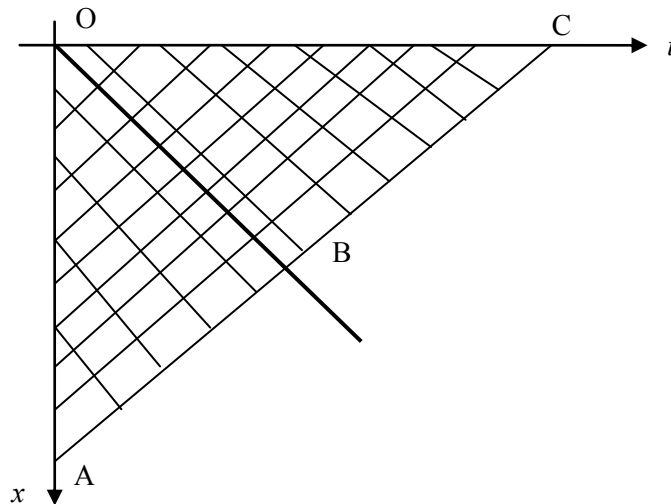


Figure 1. Semi-infinite rod ($x > 0$) as the simplest model of the Earth.

We consider two statements of the solution of equation (1). The first one is classical, it presupposes the specification of the initial Cauchy conditions on the whole bar:

$$u|_{t=0} = \alpha(x), \quad \frac{\partial u}{\partial t}|_{t=0} = a \cdot \beta' x, \quad (2)$$

where $\alpha(x)$, $\beta' x$ are the given functions of the coordinate x , $\beta' x$ is the derivative of the function βx . The general solution of (1) has the form (the d'Alembert solution)

$$u = f(x - at) + g(x + at) \quad (3)$$

The form of the functions f and g occurring in (3) is reconstructed using (2). As a result, solution (3) is represented as

$$u = \frac{1}{2} [\alpha(x - at) - \beta(x - at)] + \frac{1}{2} [\alpha(x + at) + \beta(x + at)]. \quad (4)$$

Referring to Figure 1, solution (4) is a solution in the triangle AOB. In order to obtain a solution in the triangle OBC, it is necessary on the boundary of the OC, that is, for $x=0$, to specify the boundary

conditions-either to specify a function $u|_{x=0} = \gamma(t)$, or to specify the derivative along the normal $\frac{\partial u}{\partial x} = \tau$. Moving from the boundary OA to characteristics $x - at = \text{const}$, from the boundary OC by characteristics $x + at = \text{const}$, we obtain a solution in the triangle OBC. Here questions arise: how to define the functions $\alpha(x)$, $\beta' x$, entering into (2), (4). How is it possible to look inside the Earth to know the distribution of displacements, displacement velocities inside the Earth? It's impossible to do it exactly. As a rule, in such situations, one proceeds as follows: the functions $\alpha(x)$, $\beta' x$, are assumed to be zero. Obviously, with such a choice of functions $\alpha(x)$, $\beta' x$, with large errors, a solution is found in the triangle OBC itself, both in direct and in inverse formulations. Another way to solve the problem. Let there be equation (1), one-dimensional model of the Earth in Figure 1. Solution (1) we write in the form

$$u = f\left(t - \frac{x}{a}\right) + g\left(t + \frac{x}{a}\right), \quad (5)$$

where the functions f and g are found from the boundary conditions

$$u|_{x=0} = \alpha(t), \quad \frac{\partial u}{\partial x}|_{x=0} = \frac{\beta' t}{a}. \quad (6)$$

Then the form of the functions f and g in (5) is determined by the form of the functions α , β' in (6) (β' is the derivative of the function βt). In this case

$$u = \frac{1}{2} \left[\alpha\left(t - \frac{x}{a}\right) - \beta\left(t - \frac{x}{a}\right) \right] + \frac{1}{2} \left[\alpha\left(t + \frac{x}{a}\right) + \beta\left(t + \frac{x}{a}\right) \right]. \quad (7)$$

This solution can be interpreted as follows. There is a semi-infinite rod, depicted in Figure 2. It is required to find the value of the function u at a point x, t . To do this, from the value of t on the time axis, let's go left and right in Figure 2 by the amount δ , where is the time of arrival of the signal from the boundary of the semi-infinite rod with coordinate $x=0$ to the point of the rod with coordinate x . At the propagation velocity of the signal a , the time of its arrival at the point x is equal to $\delta = \frac{x}{a}$.

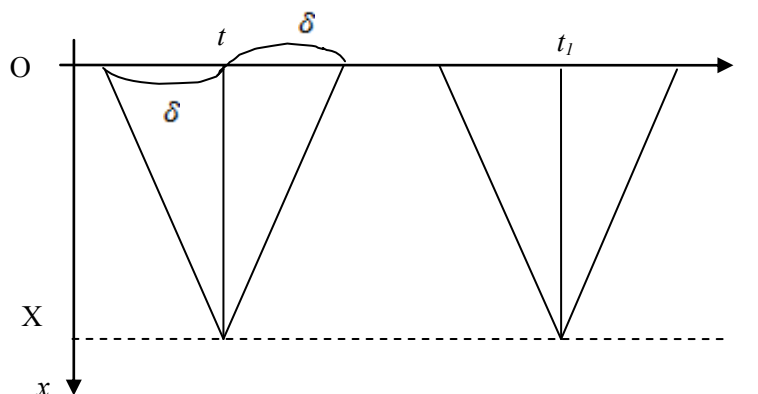


Figure 2. Scheme of the study of the state of the rod at a point with coordinate X .

Next, we find the value of the boundary functions α and β at the instant of time $t - \delta$ and $t + \delta$, we will compose from them the value of the function u at the point x, t . It is obvious that similar actions can be performed for the same depth, but for other times t_1 . Thus, the time is tracked for the state of displacements at a point with coordinate x . Changing the values of x , we control the state of the entire rod with a change in time t .

It may seem that the proposed scheme refers to a "homogeneous" environment, where there can not be sources (and sinks). In fact, within the framework of the formulation of the Cauchy problem in the form (6) it is possible to establish the positions and the intensity of the action of the sources in a homogeneous medium. Let us imagine that at some time t_* a source began to operate at a depth $x = x_*$. From the source, oscillations begin to emerge, which will pass through a homogeneous medium in which the propagation velocity of the perturbations is constant, equal to a . This information through time $\delta = \frac{x_*}{a}$ will come to the surface $x = 0$, reflected in the changes in the boundary functions α and β . These functions are registered by instruments on the boundary $x = 0$. The accuracy of further calculations of the position of the sources and their intensity of action depends only on the accuracy of measurements of the values of the boundary functions $\alpha(t)$, $\beta(t)$. It is clear that the accuracy of the prediction according to the scheme given above will be much higher than in the case of the application of (2).

We give a numerical scheme for solving the problem (6). We have the wave equation (1), the difference analog of which

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = \frac{1}{a^2} \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}, \quad (8)$$

where τ, h - steps along time and coordinate. The initial conditions (6) are given for (8). The first condition in (6) is written as

$$u_0^n = \alpha(\tau n), \quad (9)$$

where n is the number of steps in time such that $t = n \cdot \tau$. As for the second equation (6), it should be noted that (8) is the approximation (1) to within $O(\tau^2)$, $O(h^2)$, or $O(\tau^2 + h^2)$. To maintain the same accuracy in the approximation of the second boundary condition (6), we write it in the form

$$\frac{u_1^n - u_{-1}^n}{2h} = \frac{1}{a^2} \beta'(\tau n). \quad (10)$$

In order to find the values u_{-1}^n, u_1^n , we require that (8) be satisfied at $i = 0$. Then we have

$$\frac{u_1^n - 2u_0^n + u_{-1}^n}{h^2} = \frac{1}{a^2} \frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\tau^2}. \quad (11)$$

From (10) and (11) we find u_{-1}^n, u_1^n . In particular u_1^n , we obtain the expression

$$u_1^n = u_0^n + h \frac{\beta'(\tau n)}{a} + \frac{h^2}{2} \frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{a^2 \tau^2}, \quad (12)$$

which essentially coincides with the expansion of the function $u(x, t)$ in a Taylor series at a point with coordinate $x = 0$ with order $O(h^2)$ with the use of the second derivative with respect to x taken

from (1). Thus, the value u_0^n is determined by (9), the value u_1^n is based on (12). The values u_{i+1}^n for $i \geq 1$ are restored based on (8):

$$u_{i+1}^n = 2u_i^n - u_{i-1}^n + \frac{h^2}{a^2 \tau^2} u_i^{n+1} - 2u_i^n + u_i^{n-1}. \quad (13)$$

Note one feature of the solution (13): to determine the value u_{i+1}^n , it is necessary to know the values of u_i^{n+1} and u_i^{n-1} , i.e. values u at times $n+1 \tau$ and $n-1 \tau$. This means narrowing the time interval for studying what is happening at the depth of x . To increase it, an increase in the interval on the surface $x=0$ is required. In fact, for the definition u_1^1 we use (12). From (13) follows that the u_2^1 value has to be defined by u_1^0 value, but this value u on the first layer in initial timepoint is unknown. Therefore the u_2^1 value can't be defined.

Therefore, from (13) we can find u_2^2 using u_1^1 , u_1^2 , u_1^3 . The constriction occurs both from the left side, and from the right. In this case, as follows from Figure 2, we consider

$$\frac{h}{a} = \tau. \quad (14)$$

To test the calculation scheme, we will compile a test case. We put in (6)

$$\alpha t = A \sin t, \quad \beta t = \text{const}, \quad \beta' t = 0. \quad (15)$$

According to (7), the solution of (1) is the function

$$u = A \sin t \cdot \cos \frac{x}{a}. \quad (16)$$

We now present the results of a numerical calculation and their comparison with an analytical solution for depths $2h, 3h, 4h, \dots$

We have

$$u_0^n = A \sin n\tau,$$

$$u_1^n = \frac{1}{2} A \sin n+1 \tau + \sin n-1 \tau = A \sin n\tau \cos \tau,$$

$$\begin{aligned} u_2^n &= u_1^{n+1} + u_1^{n-1} - u_0^n = A \sin n+1 \tau \cos \tau + \sin n-1 \tau \cos \tau - A \sin n\tau = \\ &= A \sin n\tau \cdot 2 \cos^2 \tau - 1 = A \sin n\tau \cdot \cos 2\tau, \end{aligned}$$

$$u_3^n = u_2^{n+1} + u_2^{n-1} - u_1^n = A \sin n\tau \cos \tau \cdot 2 \cos 2\tau - 1 = A \sin n\tau \cos 3\tau,$$

because $\cos \tau \cos 2\tau = -\sin 2\tau \sin \tau + \cos \tau$.

Similarly, we get that $u_k^n = A \sin n\tau \cos k\tau$, those we have an ideal coincidence of the numerical solution under the condition (14) with the analytical solution (16), where $t = n\tau$, $\frac{x}{a} = \frac{kh}{a} = k\tau$.

Let us now consider the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}, \quad (17)$$

As the region of variation of the displacement $u = u(x, y, z, t)$, we take the half-space defined by the inequality $z \leq 0$. Let the following Cauchy conditions be given on the boundary $z = 0$:

$$u = u(x, y, 0, t) = \alpha(x, y, t), \quad \frac{\partial u}{\partial z}(x, y, 0, t) = \beta(x, y, t). \quad (18)$$

Similarly, we find $u = u(x, y, h, t)$, the value $u = u(x, y, 0, t)$, determined by the boundary condition (18). To restore the value u on layers $2h, 3h$, etc. equation (19) is used.

3. Conclusions

A new formulation of the Cauchy problem is proposed, when the desired function and its derivative along the normal are given on the boundary of the body.

A finite-difference scheme for solving a one-dimensional wave equation in a new formulation of the Cauchy problem is constructed.

It is shown how it is possible to solve a three-dimensional wave equation for a half-space with given Cauchy conditions on its boundary.

4. References

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