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# The Attractive Distribution of the Renormalized Permeability in 3D Case

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**Abstract.** In this article, the renormalization process for the permeability and the attractive distribution of the renormalized permeability are investigated. It is known that the attractive distribution is log-normal for 2D case but it isn't for 3D case. It is discovered that the renormalized permeability from infinite system is independent of the attractive distribution variance. Making use of this important property, numerical simulation is then performed to find the specific form of the attractive distribution. Based on the numerical results, it is shown that the attractive distribution is 1/3-normal in the 3D case.

## 1. Introduction

The application of geostatistics is vastly used in reservoir engineering, hydrology, geology, environmental science, etc. and the data set considered is normally distributed which is the primary assumption of many geostatistics. Deviations from this assumption may degrade statistical estimations [1]. For this reason, any nonnormally distributed data set should be transformed to normal for further processing.

For a long time, reservoir permeability has been perceived as a random-valued property of the formation. Law [2] was among the first to analyze three horizons from a sandstone reservoir permeability had a log-normal probability distribution function. After Law's significant study, lots of researchers, showed interest to analyze the permeability distribution in different reservoir. While, [3-4] analyzed a large number of sandstone and limestone permeabilities from Canadian reservoirs and they determined that the probability density function permeability data set is skewed to the right. After that, [5] studied permeability data from 689 wells in 22 fields and she exhibited that 285 wells had approximately the probability density function was normally, 297 log-normal, and 102 wells had the probability density function was exponential distribution functions. Based on the studies of Law's and Bennion indirect proof, some of the researchers in hydrology field like [6-7] determined the permeability was log-normally distributed. According to the Central Limit Theorem, permeability data were normal distributed which is the attractive distribution of iterative summation [8-9]. It can be seen as the particular case of upscaling permeabilities for the parallel-layer case (an extreme anisotropic media) and converging to a p-Normal distribution with  $p=1$ . However, [10] selected the value of  $p$  by using the trial and error method which is so time consuming but finally it gives the straightest line on a normal probability paper. A type curved approach to estimate the values of  $p$  of p-normal transformation which approximately normal transformation is secured for any single population data set [11]. The representative elemental volume (REV) grid upscaling technique which showed the flow simulation result of 3D multiphase real reservoir consistency behaviour before and after upscaling



where he [12] assumed the variance of properties in REV grid was the smallest within each block and largest between the blocks.

In this article, we propose that the attractive or fixed distribution of the renormalized permeability is properly be the 1/3-normal distribution.

## 2. Statistical Approach

### 2.1 Renormalization in 2D case

Let's recall the 2D case first. It is well known that for the infinite or enough large system consisting of elements with independent and identical log-normal distributed permeability, the system average permeability satisfies

$$\overline{K} = \lim_{n \rightarrow \infty} \sqrt[n]{k_1 k_2 \dots k_n} = \exp \left[ \lim_{n \rightarrow \infty} \frac{1}{n} (\ln k_1 + \ln k_2 + \dots + \ln k_n) \right] = \exp (\ln K_g) = K_g \quad (1)$$

where  $k_i$  is a sample of the random variable  $K = K_g e^{\sigma X}$  and  $X$  is standard normally distributed random variable. It is noticeable that  $\overline{K}$  is independent of the variance  $\sigma$  here.

Define  $\omega$ -average permeability as

$$\overline{K}_\omega = \lim_{n \rightarrow \infty} \left( \frac{k_1^\omega + k_2^\omega + \dots + k_n^\omega}{n} \right)^{1/\omega} \quad (2)$$

The relation (1) be equivalent to

$$\overline{K} = \lim_{\omega \rightarrow 0} \overline{K}_\omega = \overline{K}_{\omega=0} \quad (3)$$

Consider a renormalization process in 2D case. Suppose now the elements' permeabilities are of independent and identical, but arbitrary random distribution. The system average permeability  $\overline{K}$  must corresponds to a certain value of  $\omega$ , namely  $\omega^{(1)}$ , satisfying  $\overline{K} = \overline{K}_{\omega^{(1)}}$ ,

$$\overline{K}_{\omega^{(1)}} = \lim_{n \rightarrow \infty} \left( \frac{k_1^{\omega^{(1)}} + k_2^{\omega^{(1)}} + \dots + k_n^{\omega^{(1)}}}{n} \right)^{1/\omega^{(1)}} \quad (4)$$

This is the result of the first renormalization. Continue the renormalization. According to the Central Limit Theorem,  $\overline{K}_{\omega^{(1)}}$  is  $\omega^{(1)}$ -normal distributed (though with nearly zero variance) and denote its sample as  $\overline{k}_{\omega^{(1)}}$ . The system average permeability is still  $\overline{K}$  and it corresponds to another certain value of  $\omega$ , namely  $\omega^{(2)}$ , satisfying  $\overline{K} = \overline{K}_{\omega^{(2)}}$

$$\overline{K}_{\omega^{(2)}} = \lim_{n \rightarrow \infty} \left( \frac{(\overline{k}_{\omega^{(1)}})_1^{\omega^{(2)}} + (\overline{k}_{\omega^{(1)}})_2^{\omega^{(2)}} + \dots + (\overline{k}_{\omega^{(1)}})_n^{\omega^{(2)}}}{n} \right)^{1/\omega^{(2)}} \quad (5)$$

Continue this renormalization process, and we will get a set of values of  $\omega$ , namely  $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(n)}, \dots$  obviously, the limit of this set is  $\lim_{n \rightarrow \infty} \omega^{(n)} = 0$ . The reason is that the relation (3) holds.

In other words, the attractive distribution in 2D is Log-normal.

For the attractive distribution, there is an important property:

$$\frac{d\overline{K}}{d\sigma} = 0 \quad (6)$$

Thanks to this property, after performing the renormalization enough times, the distribution form of the renormalized permeability will be invariable, and only the variance will be reduced. From the relation (1), we can see that log-normal distribution satisfy the relation (6) in 2D case. So it is just the attractive distribution.

## 2.2 Renormalization in 3D case

We return to the 3D case and do the same renormalization as above. Similarly, a set of values  $\omega^{(1)}, \omega^{(2)}, \dots, \omega^{(n)}, \dots$  can be obtained. The problem is what the limit  $\lim_{n \rightarrow \infty} \omega^{(n)}$  equals in 3D case. Obviously, in 3D,  $\lim_{n \rightarrow \infty} \omega^{(n)} \neq 0$  because the relation (3) does not hold. Remember there is still some arguments about this problem. Someone supposes  $\bar{K} = K_g e^{\frac{\sigma^2}{6}}$  or  $\bar{K} = K_g \left(1 + \frac{\sigma^2}{6}\right)$  or others. The point is that for log-normally distributed permeability in 3D case  $\frac{d\bar{K}}{d\sigma} \neq 0$ .

Denote the fixed or attractive point  $\omega^* = \lim_{n \rightarrow \infty} \omega^{(n)}$ . After enough times of renormalization, we get  $\bar{K} = \bar{K}_{\omega^*}$  and  $\bar{K}_{\omega^*}$  does not change its distribution form ( $\omega^*$ -normal distributed) and only reduce its variance during renormalization. Thus we get for the fixed or attractive  $\omega^*$ -normal distributed distribution and its property  $\frac{d\bar{K}}{d\sigma} = 0$ .

The property (6) for the attractive  $\omega^*$ -normal distribution can be indicated directly. Suppose random variable  $X$  satisfies the standard normal distribution. The probability density function (pdf) is  $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ . The random variable  $K$  satisfies the  $\omega^*$ -normal distribution with its average  $\mu$  and variance  $\sigma$ . That is  $K = g(X) = (\omega^* \sigma X + \mu^{\omega^*})^{1/\omega^*}$ . Its inverse function is  $X = h(K) = \frac{1}{\sigma \omega^*} (K^{\omega^*} - \mu^{\omega^*})$ , so the pdf of  $K$  is

$$l(k) = f(h(k)) h'(k) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2(\sigma \omega^*)^2} (k^{\omega^*} - \mu^{\omega^*})^2} k^{\omega^*-1} \quad (7)$$

During the renormalization, the renormalized average value is calculated by  $\omega^*$ -average algorithm (2). Therefore,

$$\bar{K} = \left( \overline{K^{\omega^*}} \right)^{1/\omega^*} = \left[ \int_{-\infty}^{+\infty} k^{\omega^*} l(k) dk \right]^{1/\omega^*} = \mu \quad (8)$$

This leads to the property (6) directly.

Since the original  $\omega^*$ -normal distribution does exclude the existence of negative value, which is unphysical, one can consider the truncated  $\omega^*$ -normal distribution. In the truncated distribution, the pdf of the random variable  $X$  is modified to

$$f(x) = \frac{c}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, |x| < \frac{\mu^{\omega^*}}{\sigma \omega^*} \quad (9)$$

with  $c$  being the normalization constant for ensuring  $\int_{-\mu^{\omega^*}/\sigma \omega^*}^{\mu^{\omega^*}/\sigma \omega^*} f(x) dx = 1$ . Then the truncated  $\omega^*$ -normal distribution for the variable  $K$  is:

$$l(k) = \frac{c k^{\omega^*-1}}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2(\sigma \omega^*)^2} (k^{\omega^*} - \mu^{\omega^*})^2 \right\}, 0 < k < 2^{1/\omega^*} \mu \quad (10)$$

Similarly, we still obtain the relation:

$$\bar{K} = \left( \overline{K^{\omega^*}} \right)^{1/\omega^*} = \left[ \int_0^{2^{1/\omega^*} \mu} k^{\omega^*} l(k) dk \right]^{1/\omega^*} = \mu \quad (11)$$

In the next section, we will try to determine the value of  $\omega^*$  by numerical test.

### 2.3 Numerical Test

We will estimate the value of  $\omega^*$  in 3D case by performing numerical simulation. The question is the true normalized value  $\bar{K}$  corresponds to which value of  $\omega^*$ . In 2D case, it is known that the true normalized value  $\bar{K}$  corresponds to  $\omega^* = 0$  and the relation (1) holds. In 3D case, it is not clear yet. Actually, the true normalized value  $\bar{K}$  can be estimated through numerical simulation. Consider a 3D cubic block with size  $L^3$ . Each sub-cube with size  $1^3$  has its own permeability. The upscaled or the average permeability can then be calculated by solving the quasi-Laplace equation:

$$\nabla \cdot (K \nabla P) = 0 \quad (12)$$

Under proper boundary conditions. When the size  $L$  tends to infinity, the average permeability is just the normalized one defined in previous section. Of course, when performing the numerical simulation, only limited value of  $L$  can be used and this will bring some error when determining  $\omega^*$ .

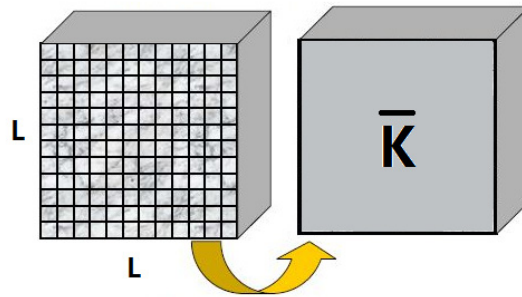


Figure 1. The sketch map of the upscaled or average permeability  $\bar{K}$  of the cubic block consisting of  $L^3$  sub-cubes.

The numerical simulation is performed as follows. First, we get a series sample  $\{x_i\}$  from the standard normal distribution. Choose proper value for the parameter  $\omega$ ,  $\sigma$  and  $\mu$  and the permeability of each sub-cube is designated as  $k_i = (\omega \sigma x_i + \mu^\omega)^{1/\omega}$ . Then the finite analytical method (FAM) proposed by Wang [13] is used to calculate the average permeability  $\bar{K}$  by solving the quasi-Laplace equation (12). The FAM method can solve the quasi-Laplace equation efficiently, especially when the media is strongly heterogeneous. Different values for  $\sigma$  are tried to test whether  $\left. \frac{d\bar{K}}{d\sigma} \right|_{\omega} = 0$ . Based on the relation  $\left. \frac{d\bar{K}}{d\sigma} \right|_{\omega=\omega^*} = 0$ , the value of  $\omega^*$  can be found. The numerical results are shown in Table 1 and 2.

The value of  $\omega^*$  seems to be 1/3. That is, the attractive distribution is probably 1/3-normal in 3D case.

Table 1. Estimation of  $\omega^*$  from numerical simulation ( $L=16$ )

	Sample1	Sample2	Sample3	Sample4	Sample5
$\omega^*$	0.3405	0.3610	0.3555	0.3431	0.3599

Table 2. Estimation of  $\omega^*$  from numerical simulation ( $L=32$ )

	Sample1	Sample2	Sample3	Sample4	Sample5
$\omega^*$	0.3445	0.3521	0.3483	0.3497	0.3484

### 3. Conclusion

In this article, we consider the renormalization process for the permeability and the attractive distribution of the renormalized permeability is investigated. The attractive distribution is the invariant one after enough times of renormalization. In 2D case, it is known that the attractive distribution for renormalized permeability is log-normal. It is easy to see that in 3D case, the attractive distribution is p-normal but not the log-normal. The purpose of this article is trying to find the value of 'p'.

One important property during the renormalization is that at the attractive distribution the renormalized permeability from infinite system is independent of the distribution variance  $\sigma$ . That is,  $d\bar{K}/d\sigma = 0$ . Numerical simulation is then performed and based on this property the value of 'p' is estimated. It seems that, in 3D case, the attractive distribution is 1/3-normal.

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