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The Hamiltonian System Method for a Cylinder under the Action of Gravity

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Abstract. The classical problem of a viscoelastic circular cylinder under its own weight is analysed in the Hamiltonian system. On the basis of the theory of state space for axisymmetric problems and the application of dual variables of stresses, the technology of variable separation can be used. Hence the fundamental solutions, zero eigenfunctions and non-zero eigenfunctions, are derived. Because of the integrity of the solution space, various boundary conditions can be described by certain combinations general eigenfunctions. By using the adjoint symplectic relations, the eigenfunction expansion approach is applied to satisfying the boundary conditions.

1. Introduction

A cylinder under the action of gravity is a classic problem for the description of the body loads [1-3]. Due to the property of time dependence of the materials, analytical solutions are difficult to be obtained. Among the researches of viscoealsticity, most of the existed techniques are based on the Laplace transformation. However, this procedure presents some difficulties when viscous parameters vary along time, or when complicated time dependent boundary conditions are imposed. A lot of inverse Laplace integral transforms can not be solved analytically. Thus, the numerical method of the inverse transform is developed rapidly.

It is well known that classical methods to discuss the governing equations of mechanical problems belong to Lagrange system. To derive the exact solutions, Zhong introduced Hamiltonian theory in elastic mechanics [4]. The Hamiltonian system method is developed on the basis of the mathematical theory on symplectic geometry, in which the variable separation method can be applied. Xu et al discussed the traditional Saint-Venant solutions, and noticed local effects ot the eigenfunctions[5]. In recent years, this method achieved more and more attention, and has been widely applied into various branches of mechanics.

A new Hamiltonian method is used in this paper to analyse a finite viscoelastic circular cylinder under the action of gravity. Using this approach, the complete eigenfunction space of the time domain, essential to satisfy the boundary conditions, is constructed, and various boundary conditions are described by expanding the eigenfunctions. In the numerical calculations, the results show that stress concentrations occur when the end is fixed, and the normal stress in the axial direction exhibits much more serious concentration than other stresses. Because of the time dependent property of viscoelastic solids, the deformation of the cylinder exhibits the creep characteristic.

2. Governing equations

A viscoelastic cylinder is considered in the cylindrical coordinates (r,θ,z). The radius and length are R and l, respectively. The constitutive relations for linear viscoelastic material can be expressed in an integral form

$$\sigma_{\mathrm{m}}(\rho,t)=3\int_0^t K(t-\tau)\frac{d\varepsilon_{\mathrm{m}}(\rho,\tau)}{d\tau}d\tau,\ s_{ij}(\rho,t)=2\int_0^t G(t-\tau)\frac{d\varepsilon_{ij}(\rho,\tau)}{d\tau}d\tau \tag{1}$$

in which ρ is a position vector, $K(t)$ and $G(t)$ are relaxation modulus. Using the Laplace transformation, the constitutive Eq. (1) are rewritten as

$$\bar{\sigma}_{\mathrm{m}}(\rho,s)=3\bar{K}(s)\bar{\varepsilon}_{\mathrm{m}}(\rho,s),\ \bar{s}_{ij}(\rho,s)=2\bar{G}(s)\bar{\varepsilon}_{ij}(\rho,s) \tag{2}$$

The strain energy density is

$$\bar{L}=r\gamma+\frac{r(3K-2\bar{G})}{6}\left(\frac{\partial\bar{u}}{\partial r}+\frac{\bar{u}}{r}+\dot{\bar{w}}\right)^2+\bar{G}r\left[\left(\frac{\partial\bar{u}}{\partial r}\right)^2+\frac{\bar{u}^2}{r^2}+\dot{\bar{w}}^2+\frac{1}{2}\left(\frac{\partial\bar{w}}{\partial r}+\dot{\bar{u}}\right)^2\right] \tag{3}$$

and its dual vector

$$\bar{\mathbf{p}}=\{r\bar{\tau}_{rz},\ r\bar{\sigma}_z\}^{\mathrm{T}} \tag{4}$$

Using the principle of minimum total potential energy, we get the governing equations of the Hamiltonian system

$$\dot{\bar{\psi}}=\mathbf{H}\bar{\psi}+\bar{\mathbf{f}} \tag{5}$$

where $\bar{\psi}=\{\bar{u},\ \bar{w},\ r\bar{\tau}_{rz},\ r\bar{\sigma}_z\}^{\mathrm{T}}$, $\bar{\mathbf{f}}=\{0,\ 0,\ 0,\ r\gamma\}^{\mathrm{T}}$, and the Hamiltonian operator matrix H is

$$\frac{1}{\bar{E}r(1-\bar{v})}\begin{bmatrix} 0 & \bar{E}(\bar{v}-1)\alpha_1 & 2(1-\bar{v}^2) & 0 \\ -\bar{v}\bar{E}\alpha_2 & 0 & 0 & (1+\bar{v})(1-2\bar{v}) \\ \bar{E}^2\alpha_3 & 0 & 0 & -\bar{v}\bar{E}\alpha_4 \\ 0 & 0 & \bar{E}(\bar{v}-1)\alpha_1 & 0 \end{bmatrix} \tag{6}$$

where $\alpha_1=r\partial/\partial r$, $\alpha_2=r\partial/\partial r+1$, $\alpha_3=(-r^2\partial^2/\partial r^2-r\partial/\partial r+1)/(1+\bar{v})$, $\alpha_4=r\partial/\partial r-1$, $\bar{E}(s)=9K\bar{G}(s)/[3K+\bar{G}(s)]$, and $\bar{v}(s)=[3K-2\bar{G}(s)]/[6K+2\bar{G}(s)]$.

3. Fundamental solutions

The eigenequation is

$$\mathbf{H}\bar{\varphi}(r)=\mu\bar{\varphi}(r) \tag{7}$$

where μ is an eigenevalue. The solution is

$$\begin{aligned} \bar{u}' &= h_1\mu J_1\beta_1+h_1[4(1-\bar{v})J_1-\mu r J_0]\beta_2 \\ \bar{w}' &= h_1\mu J_0\beta_1+h_1\mu r J_1\beta_2 \\ r\bar{\tau}_{rz}' &= h_2\mu r J_1\beta_2 \\ r\bar{\sigma}_z' &= h_3\mu^2 r J_0\beta_1+h_3\mu r[\mu r J_1+2\bar{v}J_0-4\bar{v}^2J_0]\beta_2 \end{aligned} \tag{8}$$

in which $h_1=1/(4\bar{E}-4\bar{E}\bar{v})$, $h_2=1/(2+2\bar{v})$, $h_3=1/[4(1-\bar{v}^2)(1-2\bar{v})]$.

4. Numerical computations

In this section, the geometrical data and the parameter of the Burgers viscoelastic model are selected as: and $l/R=2$, $2G_1=4G_2=K$, $\eta_1=2\eta_2=\eta$. In the numerical example, the first ten eigenfunctions are employed to achieve enough accuracy of the results.

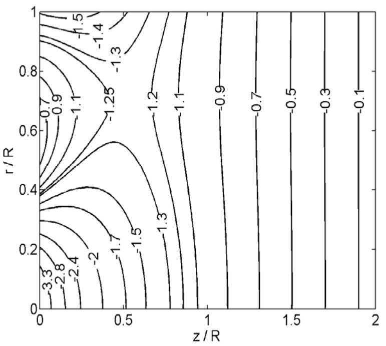

Figure 1. Contour lines of the stress component $\sigma_z/(\gamma R)$

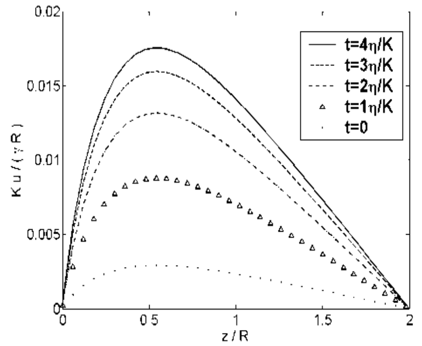

Figure 2. The displacement at $r=R$

It is clearly shown in figure 1 that the stress concentration appears near the fixed end, and decrease rapidly along the axial direction. Figure 2 exhibits the creep character of the circular cylinder due to the viscous action.

5. Conclusion

A circular cylinder under its own weight is analysed by using proposed symplectic method, and the final solution is be expressed by the combination of eigenfunctions. Numerical results indicate that local effects appear when the end is fixed, and the normal stress in the axial direction exhibits much more serious concentration than other stresses.

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