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Saint-Venant Problems for Two-dimensional Elastic Solids

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Abstract. Using the symplectic system method, the classical Saint-venant problems in elasticity are studied. Based on the Laplace transformation and integral transformation and the symplectic character, all the Saint-Venant solutions, including the tension, bending and local solutions, are obtained. The final solution of the problem can be directly analysed in the time domain solution space. As applications of the proposed method, some stress concentration boundary condition problems are discussed.

1. Introduction

In the 1850's, Saint-Venant presented the famous theory of Semi-inverse method [1], in which he proposed that the resultant equilibrium can only satisfie boundary conditions approximately, and thus, the exact solution should include two groups, the classical Saint-Venant solutions and the local solutions. In the research on elasticity and and elastic structures, most of the correlation method adopted single variable of the force method or displacement one. From a mathematical point of view, they are included in the Lagrange system, and inevitably lead to difficulties in solving higher order partial differential equations [2].

Taking original variables (displacements) and their dual variables as the basic variables, Zhong [3] takes displacement and its dual stress variables as basic variables, and introduced Hamiltonian system into the field of mechanics, and establishes a direct method without any assumptions. For viscoelastic problems, the Hamiltonian system method can be implemented in Laplace space because the stress-strain relationship and the geometric equation in Laplace domain are formally identical with the corresponding elastic equation in time domain. By using this method, a complete solution describing boundary conditions can be found by separating variables. In this paper, the Hamiltonian system method for planar viscoelastic media and structures is discussed. According to the properties of dual equations, the solutions of the governing equations are divided into two categories. One is the zero eigenvalue solution which can satisfy the boundary conditions in the whole sense, and the other is the local effect solution which reflects the stress or deformation concentration.

2. The state space

By appling the variational method, dual equations of two dimensional problems are

$$\dot{\psi} = \mathbf{H}\psi \tag{1}$$

where

$$\psi = [u \quad v \quad \sigma \quad \tau]^T \tag{2}$$

and $\mathbf{H}$ is a differential operator:

$$\mathbf{H} = \begin{bmatrix} 0 & -\beta_3\partial_y & \beta_2 & 0 \\ -\partial_y & 0 & 0 & \beta_1 \\ 0 & 0 & 0 & -\partial_y \\ 0 & -\beta_4\partial_y^2 & -\beta_3\partial_y & 0 \end{bmatrix} \tag{3}$$

in which the coefficients $\beta_1 = 1/G$, $\beta_2 = 1/(\lambda+2G)$, $\beta_3 = \lambda\beta_2$, $\beta_4 = 4G(\lambda+G)\beta_2$. The dual equations are

$$\mathbf{H}\psi_j(y) = \mu_j\psi_j(y) \tag{4}$$

The solutions satisfy the homogeneous equations (3), can be solved as

$$\begin{Bmatrix} u \\ v \\ \sigma \\ \tau \end{Bmatrix} = C_1 \begin{Bmatrix} A_{11}\cos(\mu y) + A_{12} y \sin(\mu y) \\ A_{21}\sin(\mu y) + A_{22} y \cos(\mu y) \\ A_{31}\cos(\mu y) + A_{32} y \sin(\mu y) \\ A_{41}\sin(\mu y) + A_{42} y \cos(\mu y) \end{Bmatrix} e^{\mu x} \tag{5}$$

where

$$\begin{aligned} A_{11} &= A_{41} = (\beta_3 - 1)\sin\mu + (\beta_3 + 1)\mu\cos\mu \\ A_{12} &= A_{22} = (\beta_3 + 1)\mu\sin\mu \\ A_{21} &= -2\sin\mu - (\beta_3 + 1)\mu\cos\mu \\ A_{31} &= \mu(\mu\cos\mu - \sin\mu)\beta_4 \\ A_{32} &= -A_{42} = \beta_4\mu^2\sin\mu \end{aligned} \tag{6}$$

Let's consider the following boundary conditions

$$\begin{aligned} \mathbf{p}_{x=-l}(y) &= \mathbf{p}_{-l} \\ \mathbf{q}_{x=l}(y) &= \mathbf{q}_l \end{aligned} \tag{7}$$

The final solution can be expanded by the general solutions as

$$\psi = \sum[a_n e^{\mu_n x}\psi_{\alpha_n}(y) + b_n e^{-\mu_n x}\psi_{\beta_n}(y)] \tag{8}$$

Using adjoint relations, we can establish linear algebraic equations about the coefficients:

$$\int_{-b}^{b} \mathbf{p}_{-l}\cdot\mathbf{q}_{\beta_j}\,dy = \int_{-b}^{b}\left(\sum a_{\mathrm{n}} e^{-\mu_{\mathrm{n}} l}\mathbf{p}_{\alpha_n}\cdot\mathbf{q}_{\beta_j} + \sum b_{\mathrm{n}} e^{\mu_{\mathrm{n}} l}\mathbf{p}_{\beta_n}\cdot\mathbf{q}_{\beta_j}\right)dy \tag{9}$$

and

$$\int_{-b}^{b} \mathbf{q}_{l}\cdot\mathbf{p}_{\alpha_j}\,dy = \int_{-b}^{b}\left(\sum a_{\mathrm{n}} e^{\mu_{\mathrm{n}} l}\mathbf{q}_{\alpha_n}\cdot\mathbf{p}_{\alpha_j} + \sum b_{\mathrm{n}} e^{-\mu_{\mathrm{n}} l}\mathbf{q}_{\beta_n}\cdot\mathbf{p}_{\alpha_j}\right)dy \tag{10}$$

3. Numerical results

In this section, we suppose the strip is fixed at the *x*=*l* end, and in the end of *x*=-*l*, the material is subjected the stress load $\sigma_{-l} = 1$.

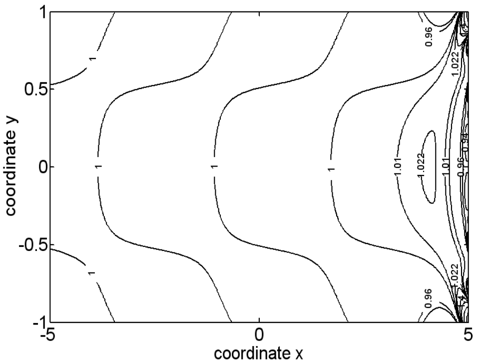

Figure 1. Distribution of normal stress

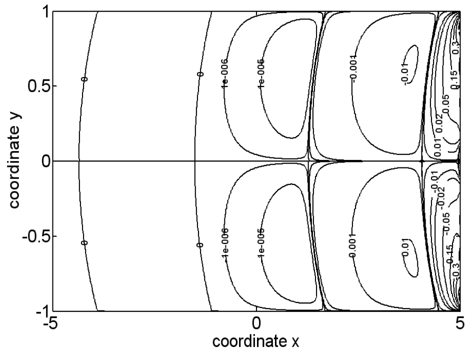

Figure 2. Distribution of shear stress

Figures 1 and 2 describe the distribution of normal stress and shear stress, respectively. It can be seen in the figure that the stress is almost constant far from the fixed end, which shows that the zero eigenvalue plays the most important role in these regions, while the non-zero eigenvalue is almost negligible. However, near the clamping end, the non-zero eigenvalue plays a dominant role due to the local effect. Figure 3 shows the strain distribution in the region. It can be seen that the strain and stress are similar, and the deformation concentration will also occur at the fixed end.

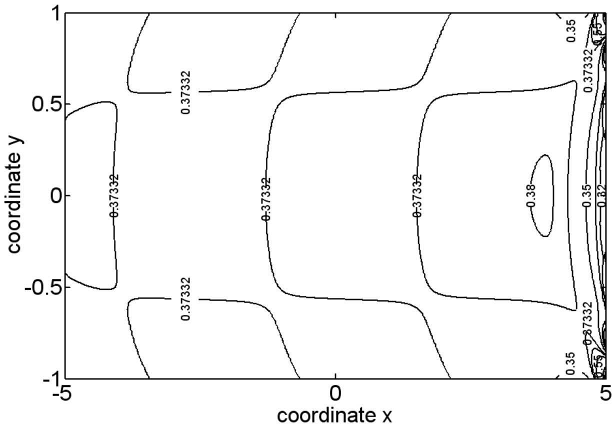

Figure 3. Distribution of normal stress.

4. Conclusion

Because of the energy loss in viscoelastictity, the problem belongs to an energy non-conservative system. However, in Laplace domain, they can be transformed into energy conservation form by using integral transformation, and Hamilton theory can be applied. The method presented in this paper is simple and direct, and can be applied to a wide field of applied mechanics.

Acknowledgments

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