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Numerical Methods for a Shallow Water Rosenau-Burgers Equation

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Abstract. We turn to study the numerical solution of a shallow water of Rosenau-Burgers equation. We propose a scheme to numerically solve this equation. A detailed analysis is carried out for the scheme, and analysis shows that the scheme is unconditionally stable; At last, we use the proposed methods to investigate the asymptotical behavior of the solutions to the shallow water wave equation.

1. Introduction (Heading 1)

Rosenau proposed the Rosenau equation in [1, 2], the existence and uniqueness of this equation was proved by Park [3]. Chung[4] propose a finite difference approximate solutions for the Rosenau equation, S.A. Manickam, A.K. Pani, S.K. Chung, [5] use a second order spilitting combined with orthogonal cubic spline collocation method for the Rosenau equation. Y.D. Kim, and H.Y. Lee [6], consider the finite element Galerkin method for the Rosenau equation. On the other hand, for the further consideration of the dissipation in space for the dynamic system, such as the phenomenon of bore propagation and the water waves. The viscous term $\alpha\partial_x^2 u$ needs to be included. with $\alpha > 0$. This equation is usually called the Rosenau-Burgers equation. Some other works have been focused on using numerical technique to solve Rosenau-Burgers type equations [7-8].

In this paper, we propose finite difference/spectral schemes for the Rosenau-Burgers equation. The proposed schemes combine a linearized finite difference method in time and Fourier spectral method in space. A detailed analysis is carried out for the scheme, and analysis shows that the scheme is unconditionally stable. We use the proposed methods to investigate the asymptotical behavior of the solutions to the shallow water wave equation.

2. Preliminaries

We consider the following Roseau- Burgers equation:

$$\partial_t u + \partial_t \partial_x^4 u - \alpha \partial_x^2 u + \partial_x u + u\partial_x u = 0,\ x \in \Lambda, \tag{2.1}$$

Satisfied the following initial conditions:

$$u(x,0) = u_0(x),\ x \in \overline{\Lambda}, \tag{2.2}$$

And the boundary conditions

$$u(x,t) = u(x+L,t), t \in (0,T], x \in \overline{\Lambda}, \tag{2.3}$$

With $\alpha > 0$.

Lemma 1. The solution of equation (2.1) - (2.3) satisfies the following energy inequality:

$$E(u) \le E(u_0),$$

Where

$$E(u) = \| u \|_0^2 + \| \partial_x^2 u \|_0^2 .$$

Proof: Equation (2.1) on both sides and the inner product can be:

$$\frac{1}{2}\frac{d}{dt}(\| u \|_0^2 + \| \partial_x^2 u \|_0^2) + \alpha \| \partial_x u \|_0^2 = 0.$$

Thus

$$\frac{d}{dt}E(u) \le 0.$$

This proved the theorem.

3. TIme discretization scheme

In this section, we introduce two linearized time semi discrete schemes for Roseau- Burgers equation and investigate their stability. For a given positive integer $K > 0, t = n\Delta t, . n = 0,1,\cdots, K, \Delta t = T / K$

3.1. First order semi implicit scheme consider the first order semi implicit scheme based on the Euler method：

$$\begin{aligned} &\frac{u^{n+1} - u^n}{\Delta t} + \frac{1}{\Delta t}(\partial_x^4 u^{n+1} - \partial_x^4 u^n) - \alpha \partial_x^2 u^{n+1} + \partial_x u^{n+1} \\ &+ \frac{1}{3}(2u^n \partial_x u^{n+1} + u^{n+1}\partial_x u^n) = 0, n \ge 0. \end{aligned} \tag{3.1}$$

The semidiscretized problem (2.9) is unconditionally stable in the sense that for all， $\Delta t > 0$ ， it hold that：

$$E(u^{n+1}) \le E(u^n), n \ge 0. \tag{3.2}$$

Proof: taking inner in () with $2\Delta t u^{n+1}$， noticed that.

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$$(2u^n\partial_x u^{n+1}+u^{n+1}\partial_x u^n,u^{n+1})=$$
$$(u^n\partial_x u^{n+1}+\partial_x(u^{n+1}u^n),u^{n+1})=0.$$

then：

$$\|u^{n+1}\|_0^2-\|u^n\|_0^2+\|u^{n+1}-u^n\|_0^2+\|\partial_x^2 u^{n+1}\|_0^2-\|\partial_x^2 u^n\|_0^2$$
$$+\|\partial_x^2 u^{n+1}-\partial_x^2 u^n\|_0^2+\alpha\|\partial_x u^{n+1}\|_0^2=0.$$

Thus we have：

$$\|u^{n+1}\|_0^2+\|\partial_x^2 u^{n+1}\|_0^2\le\|u^n\|_0^2+\|\partial_x^2 u^n\|_0^2, n\ge 0.$$

This proved the theorem.

4. Numerical results

4.1. Verification of convergence order

In this section, the numerical simulations are reported to confirm the theoretical Results. We start with some implementation details. By applying the Fourier transformation to (3.3), we obtain a linear system for the Fourier modes $\{\hat{u}_k^{n+1}\}_{k=-N/2}^{N/2-1}$.

Euler/ Fourier scheme：

$$\frac{1}{\Delta t}(\hat{u}_k^{n+1}-\hat{u}_k^n)(1+(2\pi k/L)^4)+(i2\pi k/L+\alpha(2\pi k/L)^2)\hat{u}_k^{n+1}$$
$$+\frac{1}{3}\{2u_N^n\partial_x u_N^{n+1}+u_N^{n+1}\partial_x u_N^n\}_k=0.$$

We present numerical approximations obtained by the proposed finite difference/spectral method to support our theoretical arguments. The main purpose is to check the convergence order of the numerical solution with respect to time step. The convergence rate in time is measured through computing the quantity:

$$Rate=\log_2\left(\frac{\|u_N^{n,2\Delta t}-u_N^{2n,\Delta t}\|_0}{\|u_N^{2n,\Delta t}-u_N^{4n,\Delta t/2}\|_0}\right)$$

Where $u_N^{n,2\Delta t}$ means the solution obtained with the time step size Δt and polynomial degree N.

All the results presented in this convergence rate test correspond to the numerical solution captured at $u(x,0)=\sin(2\pi x)$, $T=L=1$ and $N=50$. In Table 1 we list the computed rates (4.2) for the scheme (3.3) for several time step sizes. From this table, it is observed that for $\alpha=0$ and $\alpha=1$, the convergence order is approximately 1.

Table 1. Convergence rate in time

$\alpha \backslash \Delta t$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.01$	$\Delta t = 0.005$
0	0.9995	0.9998	1.0005	1.0009
1	0.9839	0.9919	0.9966	1.0019

4.2. Asymptotic behavior of solutoin

This subsection is devoted to numerically investigate the asymptotic behavior of the solutions. The numerical results are realized by using:

$$u(x,0) = \sin(2\pi x),\ L = 20, \Delta t = 0.01, N = 50.$$

Figure.1-4 show convergence of the solution $u(x,t)$ to its initial value. From Figure 5-8 we can see that L have a significant impact on solution. If we fix $L = 10, T = 10,\ \Delta t = 0.01,\ N = 50$, we can see from Figure 9-12, with the increase of α, the solution asymptotic convergence to zero more faster.

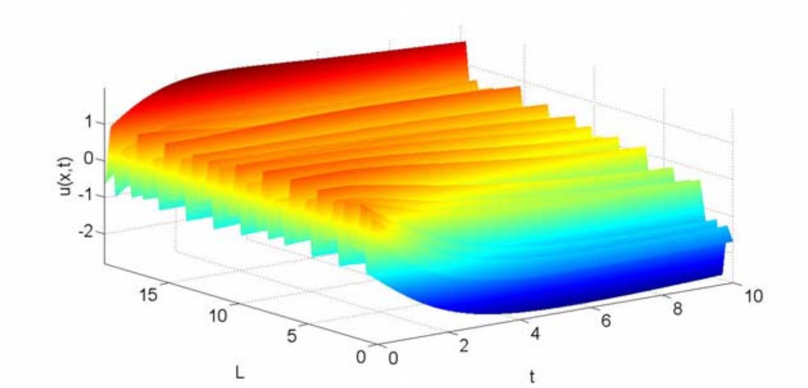

Figure 1. Solution for $T = 10$.

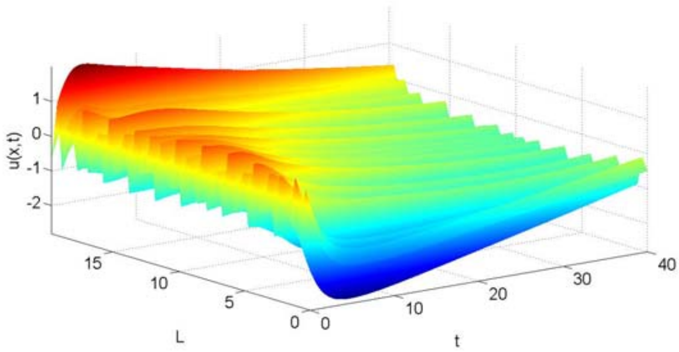

Figure 2. Solution for $T = 40$.

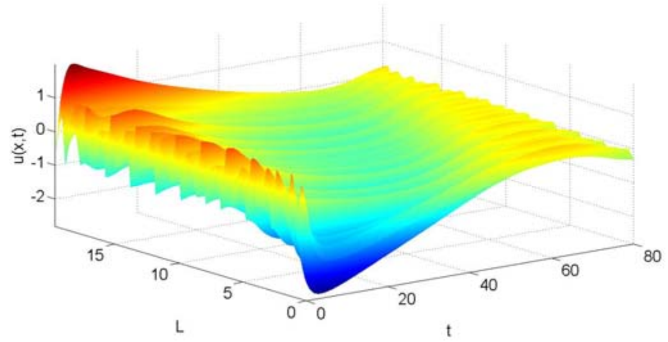

Figure 3. Solution for $T = 80$.

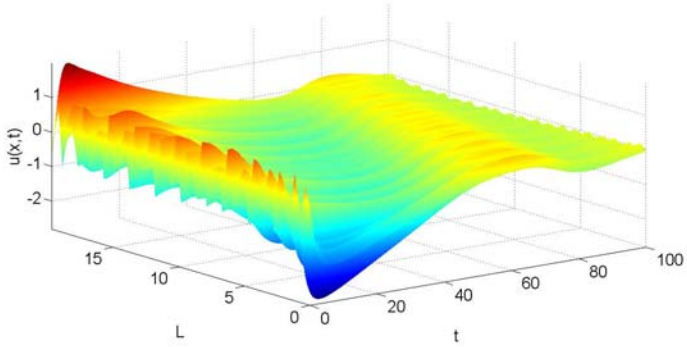

Figure 4. Solution for $T = 100$.

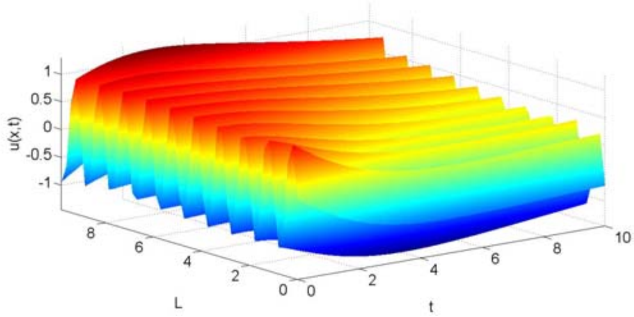

Figure 5. Solution for $L = 10$.

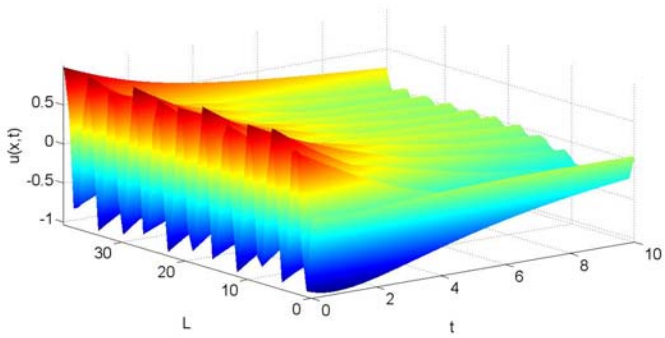

Figure 6. Solution for $L = 40$.

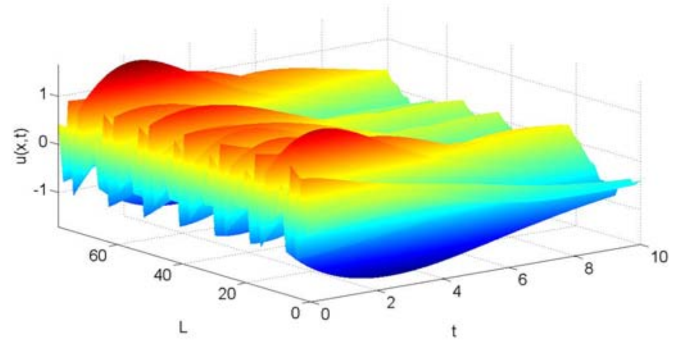

Figure 7. Solution for $L = 80$.

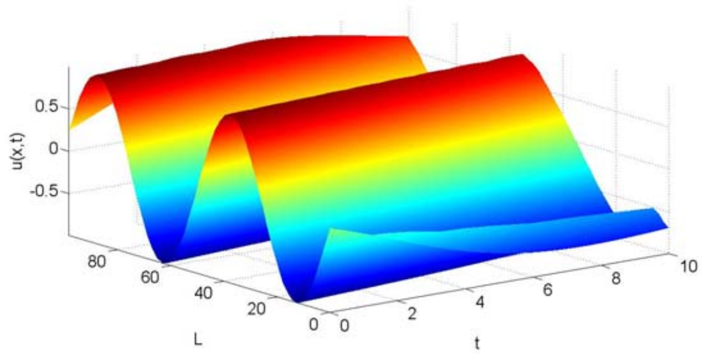

Figure 8. Solution for $L = 100$.

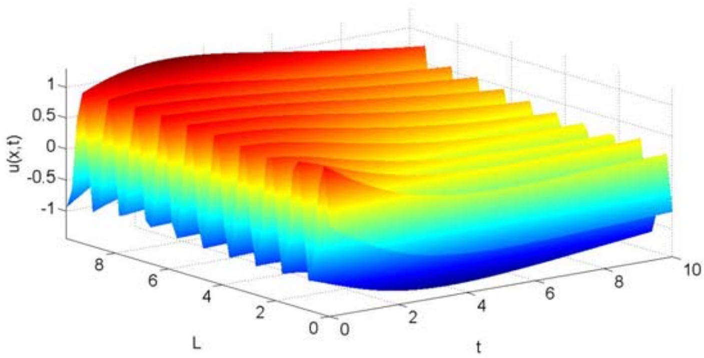

Figure 9. Solution for $\alpha = 1$.

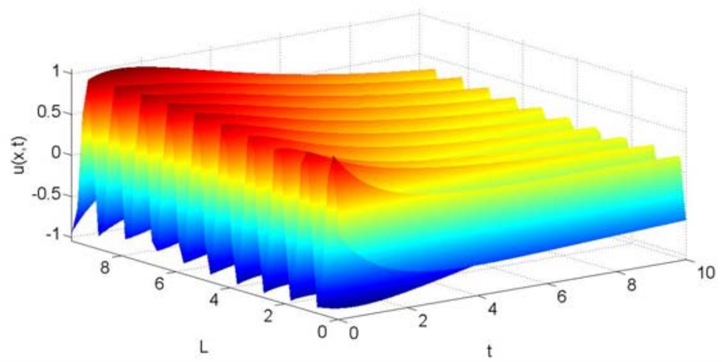

Figure 10. Solution for $\alpha = 4$.

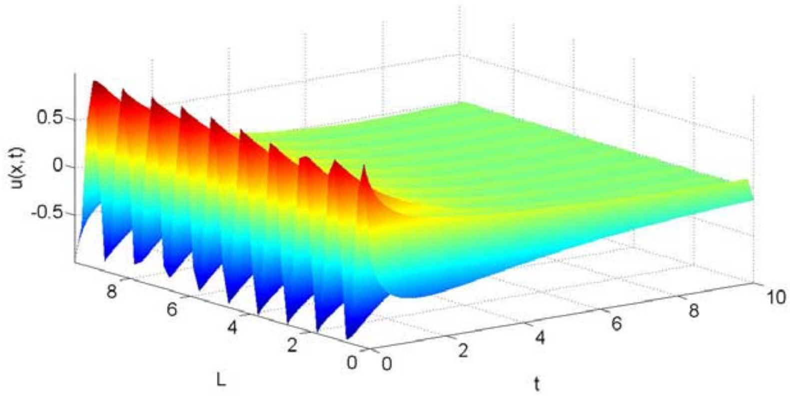

Figure 11. Solution for $\alpha = 20$.

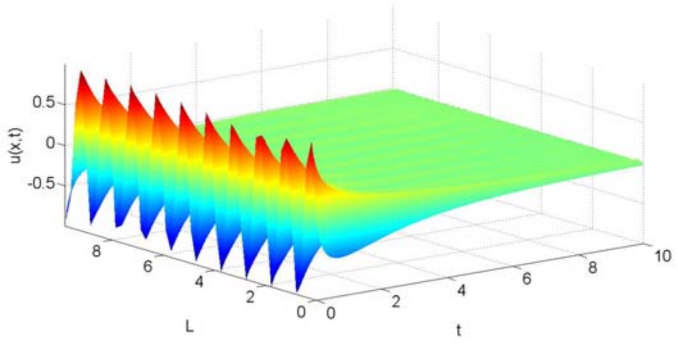

Figure 12. Solution For $\alpha = 40$.

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