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Students' mathematical conviction in Mathematics proof

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Abstract. Students are often not convinced with proofs that they have constructed in a valid manner. As a result, it may affect their mathematical understanding completely. A conviction is a broad definition of proof. Proof is not only a physical and mental activity individually, but it is also followed by an activity of convincing one self and others. This research will elaborate the components of the cause of the mathematical conviction based on the proof construction of a Plane Euclidean Geometry theorem and a convictional questionnaires about mathematical proof construction of the theorem. These components are described from five respondents found in this study. The mathematical components of conviction have been investigated in this study, it consists of: statements in the system of axioms (strength); inference flow and form of proof (logic); conformity of proof representation (formality); identification of proof statement (completeness); generalization and the extent of proof (consistency); and precision of symbols and writing (accuracy). These six components renew and fill components that are available previously. Considering to these components within the implementation of learning, it is implied to improve students' ability in understanding proof and can convince others.

1. Introduction

Proofing is an individual activity [1] in being convinced in the correctness of a proof construction [2]. When mathematical symbolic language has been constructed by the student in a proof, then it should have fulfilled the verification to be a tool that can convince and explain about a theorem [3], [4]. Similarly, the research on proof is also divided into two categories, namely: description of the proof construction when solving the problem about proof and description of the students' conviction on a statement in the proof [5].

Proof is a mathematical study that has become a trend and is widely being discussed by researchers in recent years [6], [7]; it is important for their provision in the globalization era [8]. However, most of those researches are still related to proof construction. Not many of those researches are related to proof conviction. In fact, there are some problems related to the proof conviction category that have not been able to be solved, for instance, even though the students are able to understand the proof and to perform a proofing activity, they are still unconvinced [9]. Dealing with this problem [10] and it is related to [11] students' ability in validating proof that they have constructed is uncertain, therefore, the problem has arisen [12].

Proof as the final product of proofing [6] is an artifact of the mathematical practice community [13]– [15]. The mathematical practitioner community is an intellectual effort in allowing minorities to counter the major opinions based on common rules [16], these refer to a convictional study, thus it is called as mathematical conviction.



The mathematical component of a proof is seen because it contains several elements, namely: logical, axiomatic, and formal [17] meanwhile, in proof evaluation, there are components of structure and concept in the proof [18]. Besides that in the context of formal logic, the analysis cannot be separated from the logical form of reasoning and a study into the features of concepts, propositions, and inferences [19]. It is exactly these components that become the basis of mathematical conviction in this research. Those components will be described in this study. The caused of the occurrence of mathematical conviction in mathematical proof will be investigated.

2. Method

To solve the problems in this study, the respondents of this study are students who are taking Plane Euclidean Geometry courses. The researchers give a theorem related to the course and respondents construct the proof of the given theorem. The researchers check the students' evidence construction and classify into valid and invalid. The valid respondents of evidence construction will resubmit verbally about the results of the evidence construction and the researchers interview them related to the conviction in the evidence construction.

In identifying and describing the questions in this study, the researchers used a mathematical conviction questionnaire. This questionnaire contains a number of questions that respondents must answer with yes or no answers. They also write their reason. The respondents fill the questionnaire after the interview process has been done. Then, the data are coded, and hereinafter are described in a narrative form. The results of the description would be interpreted by researchers to interpret and reveal the important values associated with this research question.

3. Result and Discussion

3.1 Proof construction

Students construct the proof from the question about Plane Euclidean Geometry. The problem is compiled by the researchers through considering the different forms and representations in the proof construction. The forms of proof in Geometry can be classified into two forms, they are direct and indirect proof. A proof is said to be a direct proof when it applies syllogistic law, both of them is direct implication or contrapositive. On the other hand, a proof is said to be an indirect proof if it applies modus Tollens [18]. The representation of proof will be in a two-column proof, paragraph proof, flow proof, and proof tree [20].

There were 5 students who had constructed a valid proof about the problem. The first respondent presented the proof in the form of indirect proof and in a paragraph representation. The other four respondents presented the proof in form of direct proof. Meanwhile, the proof representation of the four respondents were paragraph proof, flow proof, paragraph proof and two-column proof, respectively.

3.2 Interview

The identification of conviction can be made by asking directly about students' conviction [12]. The respondents are interviewed to ask about their conviction. It is found that all the respondents in this study are uncertain with their proof construction.

The first respondent is uncertain of the results of his proof construction. Here is the result of the conversational data collection with the first respondent in the interview process.

AI : eee... I am not that certain.

P : Like what?

AI : in... in... for... CD is a segment. For example, I draw D point here and then the C point can be drawn arbitrarily right, sir...

P : ok...

AI : this is AB .. CD .. non-line segment... so what I am trying to say is...so I am uncertain with how it can intersect sir...

Based on the interview, the first respondent is uncertain of the statement “whether \overline{CD} is a segment, then \overline{AB} intersects with \overline{CD} (if it is collinear)” in the proof construction. The first respondent, can not imagine the statement which must be able to develop a graphical argument to verify why the statement is true [21].

The second respondent also shows his uncertainty about his proof construction. Here are the results of the second respondent's conversation data collection in the interview.

RZ: I am uncertain.

P: can you tell me why do you feel uncertain?

RZ: ya... that's... I am confused with the first number (of the questions) sir.

P: why do you feel confused?

RZ: here (by pointing at the picture)

P: why is that?

RZ: the three... the three angles... the three vertexes are collinear, forming a 180 degree angle, so it means that it has a shape of a triangle

P: a shape of a triangle

RZ: the condition of P , Q and R are collinear, thus they make me confused, sir

Based on the interview, the second respondent is uncertain of the statement if $P - Q - R$ are collinear and the point of P , Q and R are on the same line, it will connect them with other statements. The proofs are constructed. It appears that the student actually does not understand about the procedures of the direct form [18].

The third respondent also declares that he is uncertain with his proof construction. Here are the results of the third respondent's conversation data collection in the interview.

RI: actually, I am uncertain with this (by pointing at the picture)

P: what makes you uncertain?

RW: at first, I go here.....the way is here....stop here...how come it turn into this...

P: why does stop there?

RW: I am stuck, sir, I can't do anything from here. Then, I search in the picture again, then it appears that R angle is a straight angle, so it can be like this...it can be put into this, then, here.... But I am still uncertain.

Based on the interview, the third respondent is uncertain of the statement “ $\angle 4 \cong \angle x$ (postulate of alternate interior angle) $\rightarrow \angle R$ is straight angle (given)”. The problem is because of the logical relationship of the statement. In the proof creation, an active effort is required to identify the logical relationship in a proof [22].

The fourth respondent also declares that he is uncertain with his proof construction. Here are the results of the fourth respondent's conversation data collection in the interview,

SI: I am not that certain.

P: ooo why do you feel uncertain?

SI: because this is $m\angle 4$ and $m\angle 6$ this is congruent

P: he'em... where does it come from that $\angle 4 \cong \angle 6$

SI: from this... (by pointing his finger on the picture), if it is summed, the result should be 180, because this is congruent with this (while pointing on the picture).

Based on the interview, the fourth respondent is uncertain on the statement “ $\angle 4 \cong \angle 6$ of the definition of alternate interior angle”. It indicates that the respondents have a problem. The respondents are not able to define alternating interior angle [23].

The fifth respondent is uncertain with the results of his proof construction. Here is the result of the conversation with the fifth respondent in the interview process.

ZB: I am not that certain.

P: how come you are not that certain?

ZB: i mean eee... should the proofing... is it only $\angle B$ or $\angle A$, $\angle B$, $\angle C$

P: the $\angle B$

ZB: if it is drawn, it has already been not collinear, but what I mean here is...is the proofing is only $\angle B$ or this, this, and this. I prove all of them, Sir...

Based on the interview, the fifth respondent indicates that he is still confused by the support of the statement " $\angle A \cong \angle C$ " to the conclusion of the proof construction of the questions in this study. It implies that the respondent is not able to know about the proper statement to keep constructing the proof [9].

3.3 Mathematical proof conviction

The analysis to describe the mathematical proof conviction is based on the components shown in Table 1. From the classification in Table 1, 5 components of mathematical conviction are obtained. These components are then used to construct a questionnaire about mathematical proof conviction. The questionnaire about mathematical proof conviction is filled by respondents after constructing the proof and conducting the interviews.

Table 1. Mathematical proof conviction components.

| | [24] | [25]–[27] |
|--------------|----------------------------------|---|
| Strength | Set of accepted statements | A set of true axioms or postulates as the foundation for reasoning |
| Logic | Modes of argumentation | A set of rules of proof with which to derive truths |
| Formality | Modes of argument representation | - |
| Completeness | - | A guarantee that the rules of proof are adequate to establish all the truths (completeness) |
| Consistency | - | A guarantee that the rules of proof are safe in warranting only truths (consistency) |

3.3.1 *Strength.* These mathematical components observe the statement in the system of axioms. The statements used in the proof construction are all contained in a mutually agreed axiom system and the precision between the statement and the reason given in the proof construction.

AY and RZ respondent are indicated on this *strength* component. Statement accuracy and reasons indicator can contribute in the proof construction.

| | | | |
|---|---|----|-------|
| 2 | Apakah dalam bukti tersebut sudah tepat antara pernyataan dengan alasan yang diberikan? | Ya | Tidak |
| | Jika tidak, bagaimana pernyataan atau alasan yang tepat kurang tepat karena AB dan CD merupakan segmen bukan garis (jika garis maka jelas berpotongan) | | |

Figure 1. AY's answer on *strength* component.

AY writes in Figure 1 above that "is less accurate, since \overline{AB} and \overline{CD} are segments, not lines (if the lines are clearly intersected)". The answer is the continuation of the interview that is previously discussed in the previous section of this study. AY intends to correct the statement "where \overline{CD} is a segment, then \overline{AB} intersects with \overline{CD} (if it is collinear)" in the proof construction, and in fact, the correction in his answer is false.

| | | | |
|---|--|----|--|
| 2 | Apakah dalam bukti tersebut sudah tepat antara pernyataan dengan alasan yang diberikan? | Ya | <input checked="" type="radio"/> Tidak |
| | Jika tidak, bagaimana pernyataan atau alasan yang tepat? Tidak, karena ketiga titik tersebut collinear jika ketiga titik harus berada pada garis yang sama sedangkan bukti yang diberikan ternyata tidak collinear. Dan cara membuktikan bukti tersebut ternyata tidak sesuai dengan pernyataan yang ada. | | |

Figure 2. RZ's answer on *strength* component.

It is similar with AY, RZ also experiences the same thing. His answer has a relation with the previous interview. RZ write “No, because the three points are called collinear whether all three points must be on the same line. Meanwhile, the given proof is not collinear. The way to verify the proof is not based on the existing statement”, as shown in Figure 2. Actually, the proof construction proposed by RZ is already valid.

- 3.3.2 *Logic*. This component focuses on the flow of inference and form of proof. The desired method of argumentation related to the inference flow of the statement in preparing the proof construction and the form of proof chosen to construct it that has been in accordance with the necessary rules.

In this component, it identifies RZ. His answer on questionnaire of mathematical proof conviction in number 3 and 4, they are “is the proof form in verifying process appropriate? and is the inference rule used in the proof is in accordance with the proof form?”, RZ circled the answer “yes” to both numbers. RZ's proof construction begun by negating the statement to be proved, “if $P - Q - R$ is collinear then the point of P , Q and R are on the same line”. RZ intends to use the direct form with the contrapositive in constructing the proof. It can be seen from RZ's answer to number 4 questions of mathematical proof conviction questionnaire but the proof construction does not precede from the supposition. Instead, it used the implications in the question.

- 3.3.3 *Formality*. Formality is a component of mathematical proof conviction that view the conformity of the proof representation. The proof construction is organized based on the use of the proof representation and it may be represented again in another form of representation.

RW stated that the proof representation used in his proof construction is appropriate. But in the proof construction, it is stated that “ $\angle 4 \cong \angle x$ (postulate of alternate interior angle) $\rightarrow \angle R$ is straight angle (given)”. It is indicated by arrows in the flow chart of the proof construction which must not be available.

- 3.3.4 *Completeness*. In this component, the mathematical proof is the one that identified the statement of the validator of the proof construction. The indicator in this component is the statements in the proof construction which is required to be added or reduced.

In the indicator related with wheter there may be additional statements in the proof construction, this has been done by 3 respondents, they are AY, RZ, and RW. The three respondents fill the indicator. It is only RZ and RW who can be categorized into this component because the additional statements will actually make the proof construction became long-winded.

| | | | |
|---|---|-------------------------------------|-----------------------------|
| 5 | Apakah dari bukti tersebut sudah tidak ada pernyataan yang perlu ditambahkan? | <input checked="" type="radio"/> Ya | <input type="radio"/> Tidak |
| | Jika ya, pernyataan apa yang perlu ditambahkan beserta alasannya Jika A, B, D segaris, dimana D adalah segmen titik D adalah perpanjangan di AB . Maka AB jelas berpotongan di D . | | |

Figure 3. AY's answer on *completeness* component.

It can seen from Figure 3 above that AY plans to add “(If A, B, D are collinear) where \overline{CD} is a segment, D point is an extension of \overline{AB} then \overline{AB} clearly intersected with \overline{CD} on the previous proof

construction. AY intends to provide a solution to the problem when it is conveyed in previous interview.

| | | | |
|---|---|--|-------|
| 5 | Apakah dari bukti tersebut sudah tidak ada pernyataan yang perlu ditambahkan? | Ya | Tidak |
| | Jika ya, pernyataan apa yang perlu ditambahkan beserta alasannya | <p>Ya, pernyataan yang ditambahkan seharusnya dari $m\angle 2 + m\angle 3 \neq 180$ itu membuktikan bahwa tidak segaris</p> | |

Figure 4. RZ's answer on *completeness* component.

| | | | |
|---|---|---|-------|
| 5 | Apakah dari bukti tersebut sudah tidak ada pernyataan yang perlu ditambahkan? | Ya | Tidak |
| | Jika ya, pernyataan apa yang perlu ditambahkan beserta alasannya | <p>$m\angle 2 + m\angle 3 \neq 180 \rightarrow m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 \neq 180$</p> | |

Figure 5. RW's answer on *completeness* component.

Figure 4 above shows that RZ adds the statement of “ $m\angle 2 + m\angle 3 \neq 180$ ” his previous statement proves that it is not collinear. Actually, there is a statement of “ $m\angle 2 + m\angle 3 \neq 180$ ”. Therefore, it can be concluded that the three points are not collinear. Figure 5 above shows that RW would add “ $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 \neq 180$ ” after “ $m\angle 2 + m\angle 3 \neq 180$ ”. It can be seen that RZ and RW planned to supplement their proof of construction with these statements. However, it makes the proof construction long-winded.

In the indicator related to the possibility of a statement on the proof construction to be reduced/omitted, all respondents say that there are no statements that needed to be reduced. When the proof construction has been analyzed again, RZ, SI and ZB can be included in this component. This happens because there are statements in each construction of the proof that make the proof construction long-winded.

Based on RZ's interview data collection, a statement makes RZ confuse “if $P - Q - R$ is collinear then the point of P , Q and R are on the same line”. It also makes the form of the proof confusing as described in the logic component. Actually, if the statement is omitted, it makes the proof construction excessiveless. SI is also similar to the statement of $\angle MNP$, $\angle NOP$ and $\overline{MP} \cong \overline{NP} \cong \overline{OP}$, they are not connected with other statements in the proof construction. In line with the RZ and SI, the statement of “ $\angle A \cong \angle C$ ” can make ZB confused that is shown in the interview above. If it is omitted, it will make the proof construction excessiveless.

- 3.3.5 *Consistency*. This component identifies the generalized ability of the proof construction as well as its validity. Indicator in this component can be seen by asking other alternatives from pictures which have other shapes but it is the same condition with the previous problem. And, it checks the breadth of the theorem by replacing hypothesis problem.

| | |
|--|---------------------------|
| <p>Tetap valid karena apabila segitiga nya adalah sama sisi, siku kati dan kedua segitiga akan tetap berimpit. Dan apabila diganti menjadi segitiga sama sisi tidak akan berpengaruh. Karena syarat untuk alarnya adalah segitiga bukan berimpit. Berbeda kalau yang berimpit bukan kati dan segitiganya melainkan alarnya. Hasilnya adalah alarnya berimpit tetapi nanti kutunya akan segaris</p> | <p>Terdapat alasannya</p> |
|--|---------------------------|

Figure 6. RW's answer on *completeness* component.

Figure 6 shows that “it is still valid because if the triangles are equilateral triangles, the foot sides and the two triangles will remain coincide. If it is replaced into an equilateral triangle, it will not affect the

base, because the requirement for the base is parallel rather than coincident. It is different if the ones are coincident, it is not the foot sides and the triangles but it is the base. Then, the result is the base but later the foot sides are not parallel." Actually, the choice of RW is correct in answering the question number 10 of mathematical proof conviction questionnaire. But his statement indicated that RW is not very familiar with the definition of an equilateral triangle and its elements (there are no foot sides and base elements on equilateral triangles [28]). Then for the statement "... the base coincides but later the foot sides are not parallel", it is possible that RW can not imagine the condition of the statement.

- 3.3.6 *Accuracy*. This component intends to see the accuracy of symbols and writing in the proof construction. This indicator is the use of the symbols that are given less precisely in the statement in the proof construction or even wrong in the writing in the proof construction.

This component is a new finding in this study as it is not mentioned in the components in Table 1 above. This component is identified from the proof construction and the results of the conversation in the SI's interview. The proof construction stated that " $\angle 4 \cong \angle 6$ from the definition of alternate interior angle, then $m\angle 4 = m\angle 6$ " and this statement is repeated in the interview by stating that "because this is $m\angle 4$ and $m\angle 6$ is congruent". The statement is the one that make him uncertain with his proof construction.

Problems in mathematical notation may provide difficulties in constructing the proof [23]. It shows that the indicator on this component is incorrect on the writing in the proof construction. It is even explained deeper. Even though the student can properly write the notation, they will not hear the "mathematical music" [15]. This opinion is the one which is intended in the indicator component, that is the use of the given symbols are less precise in the statement in the proof construction.

4 Conclusion

Based on the results and discussion above, the cause of mathematical uncertainty of a proof construction can be categorized into several components. There are 6 components of mathematical conviction, they are *strength* where the students are uncertain with their proof construction. It is due to the statement of proof construction in the axiom system. *Logic* is caused by the inference flow and the form of proof. *Formality* is the cause of the uncertainty related to the conformity of the proof representation. *Completeness* is the identification of the composition of the proof statement. *Consistency* is seen from the generalization and the extent of proof construction, and; *accuracy* is in terms of the precision of symbols and writing. Uncertainty about mathematical proof construction is caused by one or more of these components. On the other hand, the proof conviction must fulfill these six mathematical components. These six components renew and fill components that are available previously. Considering to these components within the implementation of learning, it is implied to improve students' ability in understanding proof and can convince others. For further studies, proof of mathematics can be examined from psychological views or other reviews. Changes in students' conviction also need to be examined further, whether dynamic or permanent. The evidence of conviction which is carried out in this study is obtained from individual construction. Furthermore, it could also be investigated related to the conviction of the evidence constructed together.

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