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Optimal control of procurement policy optimization with limited storage capacity

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Abstract. This research aims to minimize the cost of inventory procurement, i.e raw material procurement. Companies must be able to determine when to order supplies, when to use inventory in the warehouse, and when to postpone buyers demand based on prices, demand, and limited storage capacity. Therefore, the optimal control theory is used to find the optimal solution of inventory procurement problem by combining JIT (just-in-time), warehousing, and backlogging procurement policy. The necessary conditions of Pontryagin's Maximum Principle (PMP) and Karush-Kuhn-Tucker (KKT) are met in order to obtain the optimal raw material procurement cost. The Hamiltonian is locally optimal along a singular arc. A numerical example is given to compare the generated optimal policy and simple procurement strategy. The cost of inventory procurement that is combining JIT, warehousing, and backlogging policies is more optimal than the cost of inventory procurement that just applies JIT procurement policy. The inventory procurement problem with JIT, warehousing, and backlogging policies can be divided into two cases, i.e. zero final stock and non-zero final stock. The cost of inventory procurement with zero final stock is more optimal than non-zero final stock.

1. Introduction

Recently, industrial world competition includes regional and global area. Each company tries to find the ways to be able to compete and have competitiveness in order to survive and grow. Companies are not only required to implement proven strategies in order to increase competitiveness but also make improvement and evaluation of management to increase overall performance and quality, one of which is the inventory management.

Inventory is a stock of materials that be used to facilitate production or to satisfy customer demand. Inventory can be either raw material, semi-finished goods (work in process), or finished goods. The availability of the main raw material is quite an important factor to ensure a smooth production process of the company. A shortage of raw material will stop the production process. Therefore, companies are required to provide the raw material on time in order to avoid backlogging, inability to meet customer demand because they do not have the supply of raw material. Too much stock of raw material (over stock) will increase the holding cost and emerge the waste that should not have occurred during storage in the warehouse as raw material stock is unused or awaiting to be produced. Some raw materials have an expiration date. If the prediction is wrong, it will cause many raw materials expire. Therefore, the raw material can only be removed. Besides, the limited capacity of warehouse make raw material procurement cannot be done in the desired amount.

Pandian and Lakshmi say that the primary function of inventory is to provide customer service considering factors such as availability of consumer goods at the factual time, in the correct place and at the actual cost [1].

Based on the problems, it needs proper planning and management in inventory management, especially procurement issues. Raw material procurement is done at a certain time period, especially



with fluctuating raw material prices and limited storage capacity, the company must take into account the interest rate prevailing at that time, and the economic value in the future (Net Present Value). The company must be able to determine when to order inventory, when to take inventory in the warehouse, and when to backlog buyers demand based on prices, demand, and limited storage capacity. Teeravaraprug and Potcharanthitikull show that the decision making may be changed with and without considering inventory cost. Considering variations of demand and purchase lead time, it is found that high demand variation tends to use buy option whereas high purchase lead time variation tends to use make option [2].

There are some literatures that discusses problem-solving of inventory management using optimal control theory. Minner and Kleber present an optimal control approach to optimize the production, remanufacturing, and disposal strategy with respect to dynamic demand and return [3]. Adida and Perakis present a continuous time optimal control model for studying a dynamic pricing and inventory control problem for a make-to-stock manufacturing system [4]. Arnold, Minner, and Eidam present a deterministic optimal control approach optimizing the procurement and inventory policy of a company that is processing a raw material when the purchasing price, holding cost, and the demand rate fluctuate over time [5]. Maity and Maiti develop advertising and production policies for a deteriorating multi-item inventory control problem [6]. The system is under the control of fuzzy inflation and discounting. Fang and Lin deal with a lean supply chain system where the production facilities operate under a just-in-time (JIT) environment, and the facilities consist of a raw material supplier, a manufacturer with multi-work-stage, and multiple buyers where inventories of raw material, work-in-process (WIP), and finished products are involved respectively [7].

Arnold and Minner analyze a commodity procurement problem under uncertain future procurement prices and product demands [8]. Arnold, Minner, and Morrocu present a deterministic continuous time approach minimizing the net present value of production and inventory holding cost with dynamic parameters [9]. Alshamrani considers a stochastic optimal control of an inventory model with a deterministic rate of deteriorating items [10]. Yang, Huang, and Xu consider a continuous review inventory system with finite production/ordering capacity and setup cost and show that the optimal control policy for this system has a very simple structure [11]. They also develop efficient algorithms to compute the optimal control parameters. Ivanov, Dolgui, Sokolov, and Ivanova develop an optimal control model of supply chain reconfiguration that has been previously investigated with the help of mathematical programming approach [12].

In this research, the optimal control theory is used to find the optimal solution of inventory procurement problem by combining just in time procurement, warehousing, and backlogging. Assuming there is no initial stock. The final stock is divided into two conditions, there is no final stock and there is final stock or the remaining stock in the warehouse. The necessary conditions of Pontryagin's Maximum Principle (PMP) and Karush-Kuhn-Tucker (KKT) are met in order to obtain raw material procurement costs by obtaining the optimal Net Present Value (NPV). The obtained Hamiltonian is locally optimal along the singular arc if meet the second order Legendre Clebs general condition. In the final part, a numerical example is given to compare the resulting optimal policy with the simple procurement strategy.

2. Literature review

2.1. Net present value

Net Present Value (NPV) is the present value of the amount of money that would be acceptable in the future and converted to present by using the interest rate (discount rate) [13]. NPV is used to analyze the Discounted Cash Flow (DCF) and the standard method for estimating the financial condition of long-term projects. Discounting is done to obtain NPV by a discount $\frac{1}{(1+r)^t}$ for discrete time system or e^{-rt} for continuous time system.

$$NPV = \frac{R_t}{(1+r)^t} \quad (1)$$

Or

$$NPV = e^{-rt} R_t \quad (2)$$

for continuous time system, where

t = investment time

R_t = net cash flow at time t

r = interest rate.

2.2. Optimal control problem

Fig. 1 describes how to obtain optimal control $u^*(t)$ (the sign * states the optimal condition) which will encourage and regulate the system P from the initial state to a final state with constraints [14]. Control with same state and time can determine the optimum value based on the objective function.

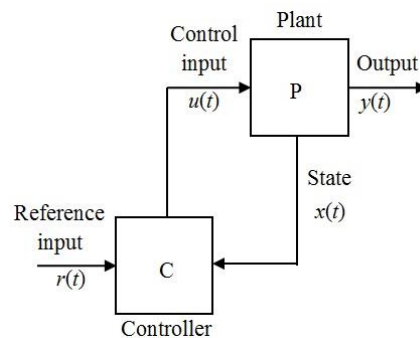


Figure 1. Control Scheme.

Formulations in optimal control problem [14] are as follows:

- Describing the mathematical process means getting the mathematical method of controlling process (generally in the form of a state variable).
- Specification of the objective function.
- Determining the boundary conditions and physical constraints on the state and or control.

Generally, the optimal control problem in the form of mathematical expression can be formulated as objective function and the constraints. With the aim of seeking control $u(t)$ which optimizes (maximize or minimize) the objective function

$$J = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (3)$$

with constraints

$$\dot{\mathbf{x}} = f(\mathbf{x}(t), \mathbf{u}(t), t) \quad (4)$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 \quad (5)$$

$$\psi(\mathbf{x}(t_f), t_f) = 0 \quad (6)$$

$$\mathcal{C}(\mathbf{x}(t), \mathbf{u}(t)) \leq 0 \quad (7)$$

$$\mathcal{S}(\mathbf{x}(t)) \leq 0 \quad (8)$$

2.3. Current value formulation

In management science and economic problems, the objective function is usually formulated in terms of the time value of money or equipment. The ownership of money or equipment in the future is usually discounted.

Suppose we assume $r \geq 0$ that is the constant continuous discount rate. The discounted objective function is a special case of the objective function with the assumption that the dependence of the function of the time just happen because of discount factor [15].

$$J = e^{-rt} \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} e^{-rt} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (9)$$

With discounted objective function, the necessary condition to achieve optimal conditions is

a. Stationary condition

$$\frac{\partial H^k}{\partial \mathbf{u}} = 0 \quad (10)$$

b. State and co-state equation

$$\dot{\mathbf{x}} = \frac{\partial H^k}{\partial \boldsymbol{\lambda}} \quad (11)$$

$$\dot{\boldsymbol{\lambda}}^T = r\boldsymbol{\lambda}^T - \frac{\partial L}{\partial \mathbf{x}} \quad (12)$$

with $\mathbf{x}(t_0) = \mathbf{x}$ and $\boldsymbol{\lambda}^T(t_f) = 0$.

2.4. Raw material procurement model

Just-in-time procurement policy, warehousing and backlogging strategy are implemented in this model. Just-in-time procurement policy is purchasing raw material to meet the production demand and using in the production process directly without storage process in the warehouse. Warehousing is storage of raw material in the warehouse or using raw material in the warehouse. Raw material procurement based on limited storage capacity will cause the amount of raw material inventory $x(t)$ can not exceed the warehouse capacity w . Backlogging is a policy to delay or do not meet the needs of production demand but accumulated demand to be met next time.

If warehousing policy is applied, it will cause storage cost $h_w(t)$. Meanwhile, if the backlogging policy is applied, it will cause penalty cost $h_b(t)$, so the holding cost is

$$h(t) = \begin{cases} h_w(t); & 0 \leq x(t) \leq w \\ -h_b(t); & x(t) < 0 \end{cases} \quad (13)$$

Raw material procurement model consists of the raw material procurement cost and inventory holding equations. Raw material procurement cost consists of the purchase cost (item cost), ordering cost, holding cost, and out of stock cost (stock out cost). Because it is assumed that there is no ordering cost, so raw material procurement cost can be written as follows:

$$\text{Raw material procurement cost} = p(t)u(t) + h(t)x(t) \quad (14)$$

where

$p(t)$ = raw material price at time t

$u(t)$ = raw material procurement rate at time t

$h(t)$ = holding cost at time t

$x(t)$ = raw material stock rate at time t

NPV minimum of raw material procurement cost is as follows [16]

$$\min_{u(t)} NPV = \int_0^T e^{-rt} [p(t)u(t) + h(t)x(t)] dt \quad (15)$$

with dynamic system

$$\dot{\mathbf{x}} = \mathbf{u}(t) - \mathbf{d}(t), \mathbf{u}(t) \geq 0 \quad (16)$$

and constraints

$$k_1(x(t), t) = x(t) - b \geq 0 \quad (17)$$

$$k_2(x(t), t) = w - x(t) \geq 0 \quad (18)$$

$$x(0) = x_0 \quad (19)$$

$$x(T) = 0 \quad (20)$$

where

$d(t)$ = demand at time t
 w = warehouse capacity
 b = backlogging lower bound
 r = interest rate
 t = time

2.5. Pontryagin's maximum principles with bounded control

The maximum principle is a condition that can obtain optimal control solution in accordance with the purpose, i. e. to minimize the objective function where the control $u(t)$ is limited ($u(t) \in \mathcal{U}$). This principle is stated informally that the Hamiltonian equation is maximized throughout the \mathcal{U} which is a set of possible controls [17]. Here is the Hamiltonian equation

$$H = \mathcal{L}(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t) \quad (21)$$

Since the bounded control $u(t)$ is limited ($a < u(t) < b$), then the Hamiltonian-Lagrangian equation is formed as follows

$$L = \mathcal{L}(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t) + w_1(b - u(t)) + w_2(u(t) - a) \quad (22)$$

Then the necessary conditions to achieve optimal condition are

- a. Stationary condition

$$\frac{\partial L}{\partial u} = 0 \quad (23)$$

- b. State dan co-state equation

$$\dot{x} = \frac{\partial L}{\partial \lambda} \quad (24)$$

$$\dot{\lambda}^T = -\frac{\partial L}{\partial x} \quad (25)$$

with $x(t_0) = x_0$ and $\lambda^T(t_f) = 0$

2.6. Bang-bang and

Difficulties in implementing the Pontryagin's Minimum Principle can be solved using bang-bang and singular control. This occurs when the Hamiltonian equation depends on the control $u(t)$ linearly. If the control $u(t)$ appears linearly in the Hamiltonian, optimal control $u(t)$ cannot be determined from the condition $H_u = 0$. Since $u(t)$ is bounded, it can determine the maximum Hamiltonian as below [18]

$$\begin{aligned}
 u(t) &= u_{max}, \text{ if } H_u < 0 \\
 u(t) &= u_{sing}, \text{ if } H_u = 0 \\
 u(t) &= u_{min}, \text{ if } H_u > 0
 \end{aligned} \quad (26)$$

H_u is called a switching function that can be positive, negative, or zero. So this solution is called bang-bang control. The change of control $u(t)$ from u_{max} to u_{min} occurs when H_u changed from negative to positive value. In this case, H_u is zero at finite time interval. It is called singular control. In that interval, the control $u(t)$ can be found from the recurring derivative of H_u depending on the time until control $u(t)$ appears explicitly. So that the control on this interval is called condition requirement of continuous singularity.

This control will obtain the optimal singular arc if it meets [15]

- a. Hamiltonian equation $H \equiv 0$
 b. Kelley condition expressed by the following equation

$$(-1)^k \frac{\partial}{\partial u} \left[\left(\frac{d}{dt} \right)^{2k} H_u \right] \geq 0, \quad k = 0, 1, \dots \quad (27)$$

This condition is called Legendre Clebs general condition that guarantees Hamiltonian equation to be optimal along the singular arc.

2.7. Nonlinear programming approach

Nonlinear Programming is used for discretization of optimal control problem and interpret the result as an infinite-dimensional optimization problem [18]. Suppose that there is optimization problem such as

$$\min f(x) \quad (28)$$

with $g_i(x) \leq 0, i = 1, \dots, m$. Where the objective function and constraint function are assumed to be differentiable continuous. The problem is to find a solution x^* that minimizes the objective function to satisfy the constraints. Here's a Lagrange function $\mathcal{L}(x, \lambda)$

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) \quad (29)$$

where, $\lambda_i = 1, \dots, m$, are Lagrange multiplier.

In order x optimal locally, must meet the first order necessary conditions Karush-Kuhn-Tucker (KKT). KKT conditions are the generalization of Lagrange multiplier method for inequality constraints. Here's the KKT condition to be met [19].

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial x_j}, \quad j = 1, \dots, m \quad (30)$$

$$\lambda_i g_i(x) = 0, i = 1, 2, \dots, m \quad (31)$$

$$g_i(x) \leq 0, i = 1, 2, \dots, m \quad (32)$$

$$\lambda_i \geq 0, i = 1, 2, \dots, m \quad (33)$$

3. Methodology

3.1. Literature Study

Literature study is to analyze critically a segment of a published body of knowledge through summary, classification, and comparison of prior research studies, reviews of literature, and theoretical articles about optimal control, raw material procurement model, Pontryagin's Maximum Principle, Karush-Kuhn-Tucker condition, etc.

3.2. Completion of the raw material procurement model

In this stage, the raw material procurement model is solved using the optimal control theory, Pontryagin's Maximum Principle that is synthesized with bang-bang control and singular control. The steps to solve the raw material procurement model are as follows.

- Formulate the Hamiltonian and Hamiltonian-Lagrangian equations.
- Meet the necessary conditions of Pontryagin's Maximum Principle (PMP) and Karush-Kuhn-Tucker (KKT) to obtain an optimal objective function.
- Determine the optimal control.
- Determine each period of the optimal policy.
- Determine the extreme points, i.e. the switching point from one policy to another policy
- Determine minimum NPV of raw material procurement cost (optimal objective function).

3.3. Numerical example

In this stage, the solution is simulated using parameter data. The simulation is divided into two cases, i.e. zero final stock and nonzero final stock.

3.4. Conclusion and suggestion

The conclusion and suggestion is presented based on the completion of the raw material procurement model and numerical example.

4. Results and discussions

4.1. Application of optimal control theory for raw material procurement model

To find the solution of the raw material procurement model using optimal control theory, the first thing to do is determining Hamiltonian and Hamiltonian-Lagrangian functions.

$$\begin{aligned} H &= -[p(t)u(t) + h(t)x(t)] + \lambda(t)[u(t) - d(t)] \\ &= -p(t)u(t) - h(t)x(t) + \lambda(t)[u(t) - d(t)] \end{aligned} \quad (34)$$

Since control $u(t)$ is bounded, then Hamiltonian-Lagrange function $L(u(t), x(t), \lambda(t), t)$ is obtained from the current value Hamiltonian $H(u(t), x(t), \lambda(t), t)$ plus the Lagrange multipliers $a_1(t)$ and $a_2(t)$ multiplied by lower bound $u(t)$ and $x(t)$ respectively. While $a_3(t)$ multiplied by capacity warehouse constraint where $a_1(t) \geq 0$, $a_2(t) \geq 0$, and $a_3(t) \geq 0$.

$$\begin{aligned} L &= H + a_1(t)u(t) + a_2(t)[x(t) - b] + a_3(t)[w - x(t)] \\ &= -p(t)u(t) - h(t)x(t) + \lambda(t)[u(t) - d(t)] + a_1(t)u(t) \\ &\quad + a_2(t)[x(t) - b] + a_3(t)[w - x(t)] \end{aligned} \quad (35)$$

The necessary conditions that are established by PMP, are

a. Stationary condition

$$\frac{\partial L}{\partial u(t)} = -p(t) + \lambda(t) + a_1(t) = 0 \quad (36)$$

b. State Equation

$$\dot{x}(t) = \frac{\partial L}{\partial \lambda(t)} = u(t) - d(t) \quad (37)$$

c. Co-state Equation

$$\begin{aligned} \dot{\lambda}(t) &= r\lambda(t) - \frac{\partial L}{\partial x(t)} \\ &= r\lambda(t) - [-h(t) + a_2(t) - a_3(t)] \\ &= r\lambda(t) + h(t) - a_2(t) + a_3(t) \end{aligned} \quad (38)$$

The necessary conditions are established by KKT that must be satisfied in order to obtain the optimum condition as follows

$$\begin{aligned} a_1(t)u(t) &= 0 & u(t) &\geq 0, a_1(t) \geq 0 \\ a_2(t)[x(t) - b] &= 0 & x(t) &\geq b, a_2(t) \geq 0 \\ a_3(t)[w - x(t)] &= 0 & x(t) &\leq w, a_3(t) \geq 0 \end{aligned} \quad (39)$$

Control $u(t)$ appears linearly in the Hamiltonian so that the optimal $u(t)$ cannot be determined from the condition $H_u = -p(t) + \lambda(t) = 0$ (switching function). Because $u(t)$ is bounded, the maximum Hamiltonian can be determined as below:

$$u^*(t) = \begin{cases} 0 & ; \text{ if } \lambda(t) < p(t) \\ \text{undefined} & ; \text{ if } \lambda(t) = p(t) \\ \infty & ; \text{ if } \lambda(t) > p(t) \end{cases} \quad (40)$$

From the optimal control (40), $u^*(t) = d(t)$ can be determined when $\lambda(t) = p(t)$ in order to meet demand. Therefore, it can be rewritten as

$$u^*(t) = \begin{cases} 0 & ; \text{ jika } \lambda(t) < p(t) \\ d(t) & ; \text{ jika } \lambda(t) = p(t) \\ \infty & ; \text{ jika } \lambda(t) > p(t) \end{cases} \quad (41)$$

4.2. The optimal solution of raw material procurement model

4.2.1. The properties of raw material procurement model solution

1st Property: If the price of raw material increased exceeds the interest rate (discount rate) times the current price of raw material plus storage cost ($p(t).rp(t) + h_w(t)$), then warehousing policy is better than JIT procurement, and the two of them cannot occur simultaneously.

Conclusions of the 1st property are

1. Warehousing and JIT policies cannot occur at the same time.
2. The condition

$$\dot{p}(t) = rp(t) + h_w(t), t \in [0, T] \quad (42)$$

identifies the candidate point to enter the interval of warehousing policy.

2nd Property:

If the price of raw material decreased exceeds backlogging profit $\dot{p}(t) = rp(t) + h_b(t)$, then the backlogging is better than JIT procurement. At the moment there is a demand but the supply of raw material do not exist, then it should be accumulated to be met in the future. It means that the stock is negative ($x(t) < 0$) and raw material procurement equals to zero ($u(t) = 0$). In other words, backlogging and JIT do not occur simultaneously.

Conclusions of the 2nd property are

1. Backlogging and JIT policies cannot occur at the same time.
2. The condition

$$\dot{p}(t) = rp(t) - h_b(t), t \in [0, T] \quad (43)$$

identifies the candidate point to finish backlogging or leave the negative stock interval.

3rd Property:

Warehousing policy ($x(t) > 0$) and JIT procurement ($x(t) > 0$) can occur simultaneously if $x(t) = w$

Conclusions of the 3rd property are

1. Warehousing and JIT policy can occur at the same time if $x(t) = w$.
2. Warehouse capacity constraint is used since order time is started or since stock interval is positive.

Those properties identify transition time between policies intervals.

4.2.2. Optimal policy period

Using the analysis result of optimal control theory, the optimal trajectory can be obtained in each condition that $u^*(t), x^*(t), \lambda(t), a_1(t), a_2(t)$. The optimal trajectory of JIT procurement period is presented in Table 1. The optimal trajectory of warehousing period is presented in Table 2. The optimal trajectory of the backlogging period is presented in Table 3.

Table 1. The optimal trajectory of JIT procurement period

Variable	JIT Procurement
$u^*(t)$	$d(t)$
$x^*(t)$	0
$\lambda(t)$	$p(t)$
$a_1(t)$	0
$a_2(t)$	$rp(t) + h_w(t) - p(t)$
$a_3(t)$	0

Table 2: Optimal trajectory of warehousing period

Variable	Warehousing
$u^*(t)$	0
$x^*(t)$	$-\int d(t) dt$
$\lambda(t)$	$e^{rt} \int_{t^{JD}}^t h_w(t) e^{rt} dt + e^{r(t-t^{JD})} p(t^{JD})$
$a_1(t)$	$p(t) - \lambda(t)$

$a_2(t)$	0
$a_3(t)$	0 if $w - x(t) \neq 0$ $\lambda(t) - r\lambda(t) - h(t)$ if $w = x(t)$

Table 3: Optimal trajectory of backlogging period

Variable	Backlogging
$u^*(t)$	0
$x^*(t)$	$-\int d(t) dt$
$\lambda(t)$	$e^{rt} \int_t^{t^{BJ}} h_w(t) e^{rt} dt + e^{r(t-t^{BJ})} p(t^{BJ})$
$a_1(t)$	$p(t) - \lambda(t)$
$a_2(t)$	0
$a_3(t)$	0

Those optimal policy periods can be found by obtaining the critical points of the equation that are obtained from 1st, 2nd, and 3rd properties, i. e.

$$p(t) = rp(t) + h(t) \quad (44)$$

$$\lambda(t) = r\lambda(t) + h(t) \quad (45)$$

$$\lambda(t) = p(t) \quad (46)$$

Extreme points that mark the change of JIT procurement, warehousing, and backlogging policies are

- t^{JD} is the time when JIT turned into warehousing policy.
- t^{DJ} is the time when warehousing turned into JIT policy.
- t^{JB} is the time when JIT turned into backlogging policy.
- t^{BJ} is the time when backlogging turned into JIT policy.
- t^{DB} is the time when warehousing turned into backlogging policy.
- t^{lot} is the time of raw material replenishment.

4.2.3. The algorithm to obtain the optimal solution of raw material procurement model

An algorithm to obtain the optimal solution of raw material procurement model is presented below. Flow chart of the process to obtain an optimal solution can be seen in Figure 2.

1. t^{JD} is determined by solving equation $p(t) = rq(t) + h_w(t)$ with condition $p(t) > rq(t) + h_w(t)$
2. t^{DJ} is determined by solving equation $p(t^{DJ}) = \lambda_w(t^{DJ})$, $t^{DJ} > t^{JD}$
3. t^{BJ} is determined by solving equation $p(t) = rq(t) + h_b(t)$ with condition $p(t) < rq(t) - h_b(t)$
4. t^{JB} is determined by solving equation $p(t^{JB}) = \lambda_B(t^{JB})$, $t^{JB} > t^{BJ}$
5. Checking the existence of t^{DB} It said overlap if $t^{DJ} > t^{JB}$

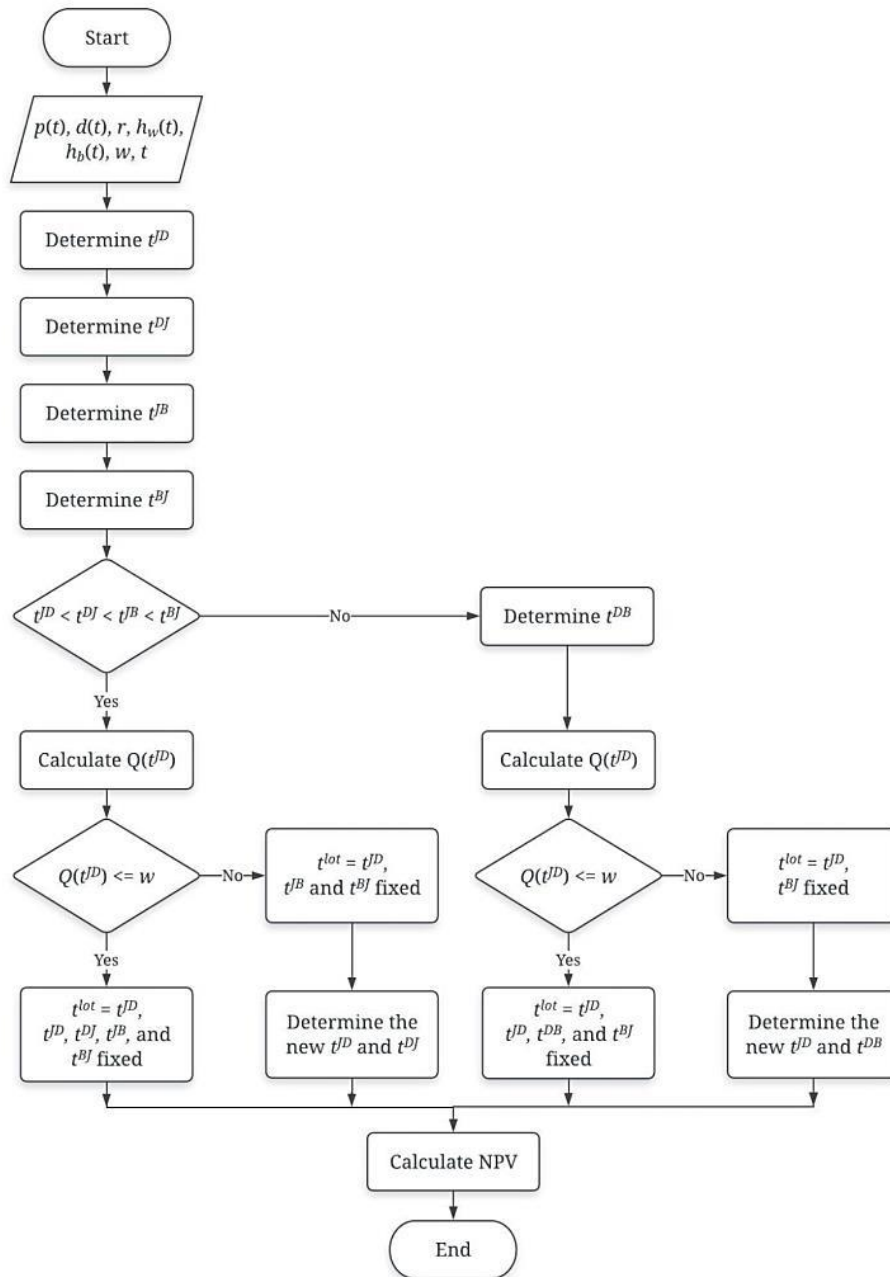


Figure 2. Flowchart of the algorithm to obtain the optimal solution.

a. JIT Period

$$NPV_1 = \int_0^{t^{JD}} e^{-rt} p(t) d(t) dt + \int_0^{t^{JB}} e^{-rt} p(t) d(t) dt + \int_{t^{BJ}}^T e^{-rt} p(t) d(t) dt \quad (47)$$

If there is an overlapping period then

$$\int_{t^{DJ}}^{t^{JB}} e^{-rt} p(t) d(t) dt = 0$$

$$NPV_1 = \int_0^{t^{JD}} e^{-rt} p(t) d(t) dt + \int_{t^{BJ}}^T e^{-rt} p(t) d(t) dt \quad (48)$$

b. Warehousing Period

$$NPV_2 = - \int_{t_1^{JD}}^{t^{DB}} e^{-rt} h_w(t) x(t) dt - \int_{t_2^{JD}}^T e^{-rt} h_w(t) x(t) dt \quad (49)$$

where

$$x(t) = D(t^{JD}) - \int d(t) dt$$

and

$$D(t^{JD}) = \int d(t) dt | t = t^{JD}$$

c. Backlogging period

$$d. NPV_3 = - \int_{t^{DB}}^{t^{BJ}} e^{-rt} h_b(t) x(t) dt \quad (50)$$

e.

Where

$$x(t) = D(t^{DB}) - \int d(t)$$

And

$$D(t^{DB}) = \int d(t) dt | t = t^{DB}$$

f. Raw material Replenishment

$$NPV_4 = e^{-rt^{lot}} p(t^{lot}) \int_{t^{lot}}^{t^{JD}} d(t) dt \quad (51)$$

g. Fulfilling the Demand due to backlogging

$$NPV_5 = p(0) \int_0^{t_1^{BJ}} d(t) dt + \int_{t^{DB}}^{t^{BJ}} e^{-rt} d(t) dt \quad (52)$$

So, NPV value of raw material cost is

$$NPV = \sum_{i=1}^5 NPV_i \quad (53)$$

4.3. Numerical example

This numerical example is using the following parameter data

$$p(t) = 4 + \sin(t - \pi) + 0,2t$$

$$time\ interval = 0 \leq t \leq 10$$

$$d(t) = 1 + 0,1t$$

$$r = 0,05$$

$$w = 3$$

$$b = -5$$

$$h_w(t) = 0,2$$

$$h_b = 0,5$$

NPV of raw material procurement cost with JIT policy is as follows

$$NPV_{jIT} = \int_0^T e^{-rt} p(t) dt$$

$$= \int_0^{10} e^{-0,05t} (4 + \sin(t - \pi) + 0,2t)(1 + 0,1t) dt$$

$$= 55,7026$$

While, NPV of procurement cost with combination of JIT, ware- housing, and backlogging policy can be divided into 2 cases, i.e.

- a. There is no final stock or no remaining stock in the warehouse ($x(T) = 0$).

Extreme points in this case are $t_1^{lot} = 1,7627$, $t_2^{lot} = 8,0991$, $t_1^{JD} = 2,6613$, $t_2^{JD} = 8,4447$, $t^{DB} = 4,9194$, $t_1^{BJ} = 1,0283$, and $t_2^{BJ} = 7,3848$. From those extreme points, NPV is obtained in every condition, i.e. when applying JIT, warehousing, backlogging policy, t^{lot} , and demand fulfillment due to backlogging. Then, it is summed up to get the total NPV as follows

$$NPV_{total} = NPV_1 + NPV_2 + NPV_3 + NPV_4 + NPV_5$$

$$= 12,1111 + 0,8404 + 1,9868 + 5,3456$$

$$+ 22,8909$$

$$= 43,1752$$

Thus NPV of raw material procurement cost with JIT, warehousing, and backlogging policy is 43,1752. The result is smaller than the NPV of raw material procurement cost with only JIT policy that is 55,7026.

Fig. 3 is the graph of the optimal solution of raw material procurement with JIT, warehousing, and backlogging policy. At time T it shows that the final stock $x(T)$ is zero or there is no remaining stock in the warehouse. The optimal policy is backlogging-JIT-warehousing-backlogging-JIT-ware-housing in accordance with the times that were determined before i.e. t^{lot} , t^{JD} , t^{DJ} , t^{DB} , t^{JB} , and t^{BJ} .

The blue graph represents control or the procurement level $u(t)$. The red graph represents the stock level in the warehouse $x(t)$. The green graph represents price $p(t)$. The purple graph represents $\lambda(t)$. At time $t^{lot} < t < t^{JD}$ applied JIT and warehousing simultaneously as in 3rd property. It can occur when purchase amount is equal to the warehouse capacity. When $\lambda(t) < p(t)$ there is no raw material procurement ($u(t) = 0$) i.e. when $t^{JD} < t < t^{DB}$. In other words raw material procurement occurs when $\lambda(t) > p(t)$. While when $\lambda(t) = p(t)$, there is a switching point, that is the change of value on the control $u(t)$. In this optimal solution, there are two switching points, i.e. at t_1^{BJ} , t_1^{JD} , t_2^{BJ} and t_2^{JD} .

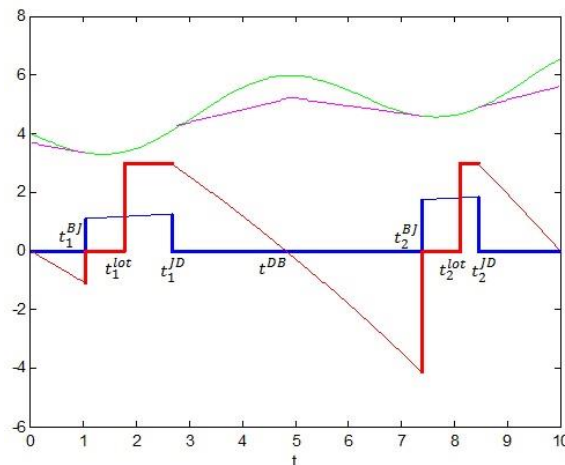


Figure 3. The optimal solution of raw material procurement model with zero final stock.

- b. There is final stock in the warehouse or there is remaining stock ($x(T) \neq 0$).

The extreme points, in this case, are same as the a) case except for point $t_2^{JD} = 8,9977$. NPV of each condition is obtained from this extreme point. Then it summed up to get the total NPV as follows

$$\begin{aligned}
 NPV_{total} &= NPV_1 + NPV_2 + NPV_3 + NPV_4 + NPV_5 \\
 &= 12,1111 + 0,6751 + 1,9868 + 8,5565 \\
 &\quad + 22,8909 \\
 &= 47,221
 \end{aligned}$$

Thus the NPV cost of raw material procurement with JIT, warehousing, and backlogging policies with non-zero final stock is 47,221 which result is less than NPV cost of raw material procurement with JIT policy, i.e 55,7026. The other way, its greater than NPV cost of raw material procurement with zero final stock.

Fig. 4 shows that the initial stock is zero but the final stock is not zero. Thus when $T = 10$, there is remaining stock in warehouse ($x_T = 1,0264$).

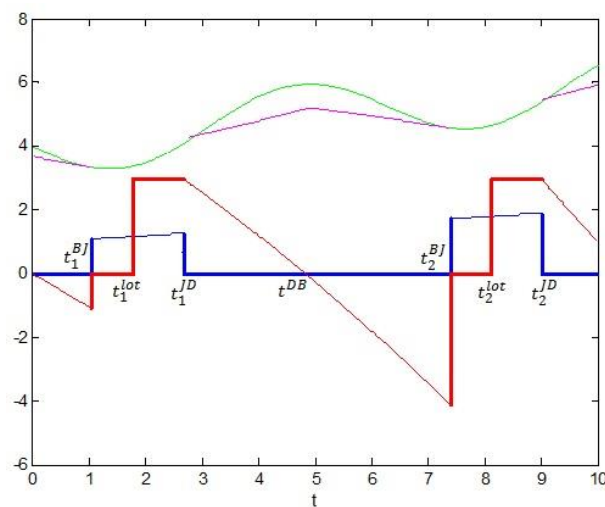


Figure 4. The optimal solution of the raw material procurement model with non-zero final stock.

5. Conclusion

The problem of raw material procurement can be solved using optimal control theory. The cost of raw material procurement that is combining JIT procurement, warehousing, and backlogging policies is more optimal than the cost of raw material procurement that just applies JIT procurement policy. The cost of raw material procurement with zero final stock is more optimal than non-zero final stock.

The raw material procurement model can be developed by adding the ordering cost and lead time. In addition, it can be specialized in certain types of raw materials, such as metal, flour, or wood. The raw material procurement model can also be developed so that it can be used for optimization of raw material procurement multi-item with limited warehouse capacity.

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