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To cite this article: Hongli Qi *et al* 2019 *IOP Conf. Ser.: Earth Environ. Sci.* **233** 032029

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# An Efficient Finite Element For Restrained Torsion Of Thin-Walled Beams Including The Effect Of Warping And Shear Deformation

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**Abstract.** An efficient torsion element is developed to describe the restrained torsion of thin-walled beams based on the first-order torsion theory. The element has two nodes, each with two degree of freedoms, namely, total rotation and warping function of cross section. The element stiffness matrix is directly obtained on the basis of first-order torsion theory without the assumption of interpolation function and the derivation of classical finite element method, which is more convenient and applicable than the initial parameter method. This element can consider not only the warping deformation but also the shear deformation of cross section due to restrained torsion. Finally, numerical examples are presented to demonstrate the validity and reliability of current torsion element, and the comparison between the results obtained from current method and those from other approaches is also illustrated.

## 1. Introduction

When TW beams are subjected to restrained torsion, the warping deformation and its restrained effects must be considered, especially for open short thin-walled (OTW) beams [1, 2]. In order to deal with the restrained torsion of TW beams including warping deformation and torsion shear deformation, several theories and numerical methods are proposed.

The classical thin-walled beam theory was developed by Vlasov [3]. In this theory, it is assumed that the contour of cross section does not deform in its own plane and the shear deformation in middle surface can be negligible. As the assumption given above, Vlasov theory is only applicable for a slender beam. However, when the beam is OTW short-deep beam and closed thin-walled (CTW) beams, the shear deformation should be taken into account [4, 5]. The widely used torsion theory is developed by Bencoter, and in this theory, the twist rate function of rotation of cross section was substituted by another function [6]. Pavazza [7] developed an analytical approach to the torsion of OTW beams with effect of shear deformation according to the assumption that the shear stress was a constant along the length of an OTW beam. Mokos and Sapountzakis[8] proposed a non-uniform torsion theory and its extended form of doubly symmetrical arbitrary cross section including secondary torsion moment deformation effect. Wang et al. [2, 5] proposed the first-order torsion theory for TW beams and the corresponding initial parameter solution method; however, the initial parameter is tedious in practical application and is not applicable to numerical analysis.

Apart from the theoretical researches, several TW beam torsion element were proposed to analyze the torsion behavior of TW beam. Mohareb and Nowzartash [9] developed a finite element



formulation which included both the St. Venant and warping torsional effects of OTW beams. Murin and Kutíš [10] proposed a finite element of constant stiffness for torsion with warping of TW cross sections, which is based on the similarity to the derivation of transfer matrices for 2nd order beam theory for constant cross-section under the shear force deformation effect. Erkmén and Mohareb [11] also derived a force-based finite element, in which bimoment fields are assumed to be linear and hyperbolic ones. Shin et al. [19] developed a new C0 continuous tapered higher-order beam element. The above TW beam element stiffness matrices are obtained based on the classical finite element derivation with assumed postulated interpolation function.

The objective of this paper is to propose an efficient and practical finite torsion element for the analysis of restrained torsion of TW beams based on the first-order theory. According to relationship between the displacement parameters and force parameters at both two beam ends, the beam stiffness matrix can be obtained without delicate derivation, which can consider the effect of warping and shear deformation of cross section of TW beams. The validity and applicability of the current torsion element is demonstrated through numerical examples.

## 2. Restrained torsion of thin-walled beams

The total rotation of cross section can be divided into free warping rotation  $\theta_F$  and restrained shear rotation  $\theta_R$  [2,5] as shown in figure 1.

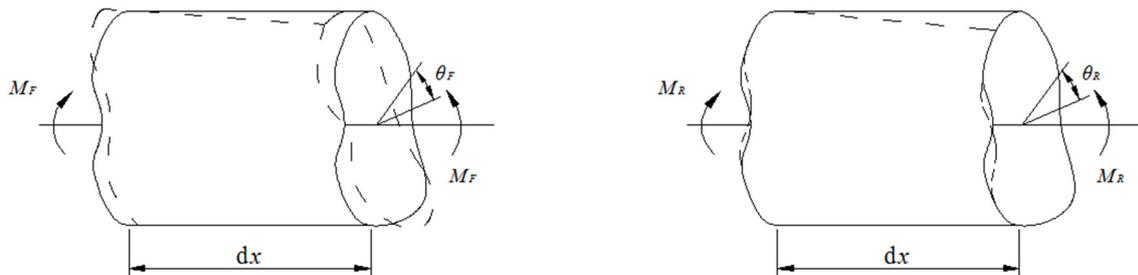


Figure 1. Rotation of TW beams subjected to restrained torsion.

The total free warping torque  $M_F$  and total restrained torque  $M_R$  are corresponding to free warping rotation  $\theta_F$  and restrained shear rotation  $\theta_R$  respectively. The bimoment  $B$  is another internal force on TW section. These internal forces can be expressed as:

$$M_F = GI_F \theta_F' = GI_k \theta_F' + GI_B \theta_F' \quad (1)$$

$$M_R = -EI_R \theta_R''' = \frac{1}{f_R} GI_{\rho_s} \theta_R' = -EI_{\omega} \theta_F''' + GI_B \theta_F' = \frac{1}{f_{\omega}} GI_{\rho_s} \theta_R' + GI_B \theta_R' \quad (2)$$

$$B = -EI_{\omega} \theta_F'' \quad (3)$$

where,  $I_F = I_k + I_B$  and  $I_R = I_{\omega} / \alpha$  denote the stiffness of free warping part and restrained shear part of CTW beam respectively, and  $I_k$  is St. Venant constant;  $I_B$  is the Bredt's second formula;  $I_{\omega}$  is sectorial inertia moment of cross section of CTW beam;  $\alpha = I_{\rho_s} (I_{\rho_s} + f_{\omega} I_B)^{-1}$  reflects the effect of closed section type on the restrained torsion of CTW beams;  $f_{\omega}$  is the torsion shear coefficient of OTW sections corresponding to CTW sections and can be used to obtain the torsion shear coefficient for CTW sections;  $I_{\rho_s}$  is tangential polar moment of inertia.

The differential equation of restrained torsion of thin-walled beams can be obtained by:

$$GI_F \theta_F'' - EI_R \theta_F'''' = -m \quad (4)$$

When external distributed torque  $m=0$ , the homogeneous solution of equation (4) and total rotation are obtained:

$$\theta_F = C_1 + C_2 x + C_3 \sin \kappa x + C_4 \cosh \kappa x \quad (5)$$

$$\theta = C_5 + C_2 x + C_3 \mu_1 \sin \kappa x + C_4 \mu_1 \cosh \kappa x \quad (6)$$

where,  $\mu_1$  is the torsion shear influence parameter.

### 3. Restrained torsion element of TW beams

In order to solve the tediousness and inapplicability, an efficient torsion element is developed based on the first order torsion theory formulae. The current efficient torsion element has two nodes as shown in figure 2, in which all the rotations and stress resultants at both nodes are given and the directions of both rotations and resultants are positive. In current element, each node has two degree of freedoms, namely, total rotation  $\theta$  and free warping twist rate  $\theta'_F$  of TW section. The corresponding stress resultants are total torque  $M$  and generalized bimoment  $\bar{B}$ . The formula of bimoment  $\bar{B}$  will be given below.

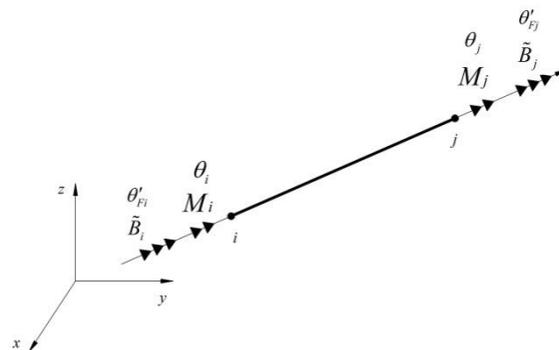


Figure 2. Torsion element of TW beams

The nodal displacement vector  $\mathbf{u}$  can be expressed as:

$$\mathbf{u}^T = \{\theta_i \quad \theta'_{Fi} \quad \theta_j \quad \theta'_{Fj}\} \tag{7}$$

The corresponding nodal force vector  $\mathbf{F}$  is given by:

$$\mathbf{F}^T = \{M_i \quad \bar{B}_i \quad M_j \quad \bar{B}_j\} \tag{8}$$

where,  $\bar{B}_i$  and  $\bar{B}_j$  are generalized bimoment at node  $i$  and  $j$  respectively. In first-order torsion theory, bimoment  $B$  is related to the second derivative of free warping rotation  $\theta_F$  rather than the total rotation  $\theta$ . In order to reflect the stress resultant in torsion element on the total rotation level, the generalized bimoment  $\bar{B}$  is defined as:

$$\bar{B} = -\mu_1 B / \alpha \tag{9}$$

According to equations (5), (6) and (7), the relationship between nodal displacement vector and constant vector is:

$$\mathbf{u} = \mathbf{T}_u \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \mu_1 / \kappa \\ 0 & 1 & 1 & 0 \\ 1 & l & \mu_1 \sinh(\kappa L) / \kappa & \mu_1 \cosh(\kappa L) / \kappa \\ 0 & 1 & \cosh(\kappa L) & \sinh(\kappa L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{10}$$

In view of equations (1), (2), (8) and (9), the relationship between nodal force vector and constant vector can be obtained as:

$$\mathbf{F} = \mathbf{T}_F \mathbf{C} = \begin{bmatrix} 0 & -GI_F & 0 & 0 \\ 0 & 0 & 0 & -EI_\omega \kappa \mu_1 / \alpha \\ 0 & GI_F & 0 & 0 \\ 0 & 0 & EI_\omega \kappa \sinh(\kappa L) \mu_1 / \alpha & EI_\omega \kappa \cosh(\kappa L) \mu_1 / \alpha \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \tag{11}$$

Substituting equation (11) into equation (10) yields:

$$\mathbf{F} = \mathbf{K} \mathbf{u} \tag{12}$$

where,  $\mathbf{K}$  is the stiffness matrix of current torsion element, which is given by:

$$\mathbf{K} = \mathbf{C} \begin{bmatrix} \kappa \sinh(\kappa L) & \mu_1 (\cosh(\kappa L) - 1) & -\kappa \sinh(\kappa L) & \mu_1 (\cosh(\kappa L) - 1) \\ L\mu_1 \cosh(\kappa L) - \frac{\mu_1^2 \sinh(\kappa L)}{\kappa} & -\mu_1 (\cosh(\kappa L) - 1) & \kappa \sinh(\kappa L) & \frac{\mu_1^2 \sinh(\kappa L)}{\kappa} - L\mu_1 \\ \text{symmetry} & & & \\ & & & \mu_1 L \cosh(\kappa L) - \frac{\mu_1^2 \sinh(\kappa L)}{\kappa} \end{bmatrix} \quad (13)$$

where  $L$  is the length of torsion element; the formula of  $C$  is written as

$$C = \frac{GI_F}{\kappa L \sinh(\kappa L) - 2\mu_1 (\cosh(\kappa L) - 1)} \quad (14)$$

#### 4. Numerical examples

On the basis of the above derivation, a torsional element analysis program is developed. Numerical examples of TW beams are given to verify the accuracy and applicability of the current efficient

##### 4.1. OTW example

An I-shaped beam with two ends fixed is provided to demonstrate the accuracy and applicability of the current efficient to OTW beams. The torsion shear coefficient of this I-shaped section is 1.2. As shown in figure 3, the external concentrated torque  $M = 3600\text{Nm}$  is applied on the shear center of I-shaped section at midspan. The elastic modulus  $E$  and Poisson's ratio  $\nu$  are 206GPa and 0.3 respectively. The analysis results of current element are compared with those of other methods, such as Vlasov theory, Benscoter theory, first-order torsion theory and ABAQUS. The finite element model is shown in figure 4; 19200 S4R shell elements are generated and kinematic coupling type is used in shell element analysis.

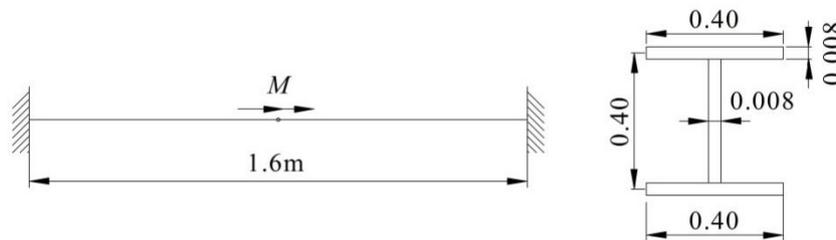


Figure 3. Torsion element of TW beams

The variation of rotations along beam length is plotted in figure 5, and it can be seen that the total rotations given by efficient element are closer to those from Abaqus than other methods. For OTW sections, the results obtained from Vlasov theory do not consider the effect of restrained shear deformation. Therefore the results of Vlasov theory can be used as a reference to study the effect of restrained shear rotation.

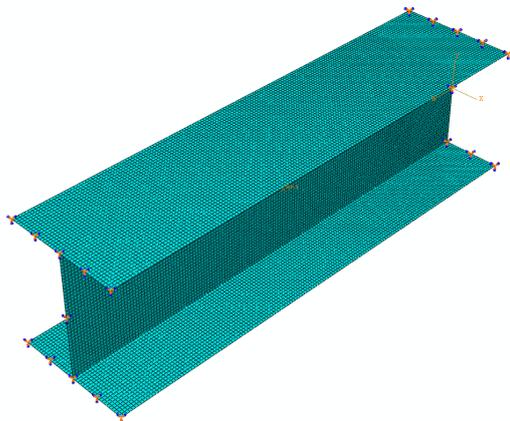


Figure 4. Torsion element of TW beams

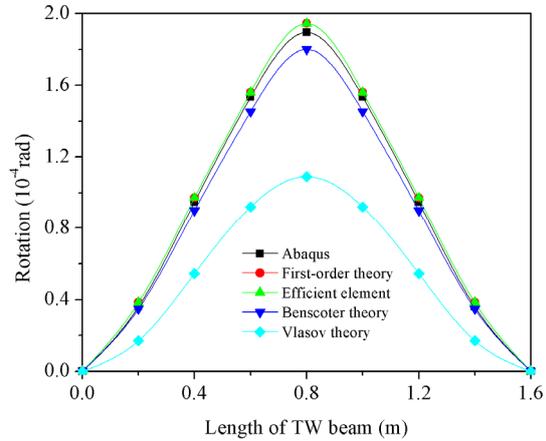


Figure 5. Torsion element of TW beams

4.2. CTW example

In order to investigate the accuracy and applicability of the current efficient element to the restrained torsion of CTW beam, a two-end-fixed beam with box cross section is subjected to external concentrated torque, as shown in Fig. 10. The concentrated torque  $M$  is 30000Nm, and the elastic modulus  $E$  and shear modulus  $G$  are  $2.1 \times 10^{11}$  N/m<sup>2</sup> and  $8.0 \times 10^{10}$  N/m<sup>2</sup> respectively. The length of thin-walled beam  $l$  is 7m. The dimensions of cross section are given in figure 6, and the torsion shear coefficient of this box section is 1.00295. 3640 S4R shell elements are generated and kinematic coupling is used in shell element analysis.

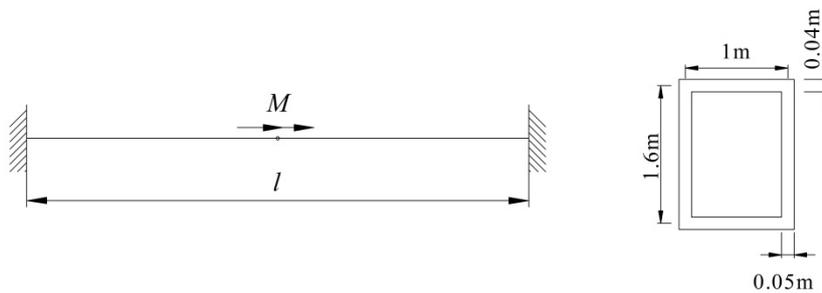


Figure 6. Torsion element of TW beams

The variations of rotation obtained from Abaqus, first-order theory, efficient element, Bencoter theory are illustrated in figure 7. And the deformed shape of box beam obtained from Abaqus is shown in figure 8. It can be seen from figure 7 that the rotations from these four methods are very close.

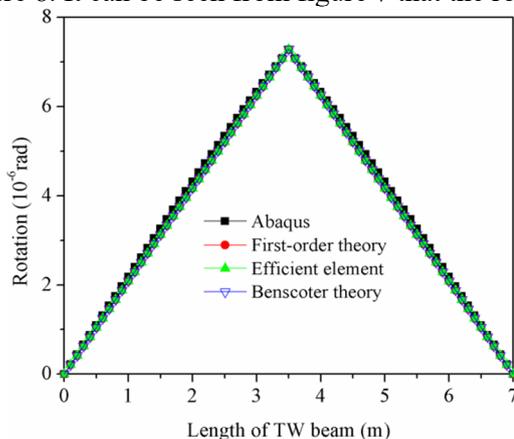


Figure 7. Variation of rotation along beam length

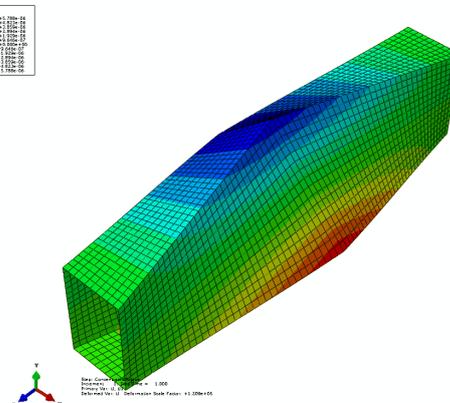


Figure 8. Deformed shape from Abaqus

## 5. Numerical examples

In this paper, an efficient torsion element of TW beams is developed. This element is not based on the derivation of classical finite element method. The element stiffness matrix is directly obtained through the relationship of end displacements and forces which have been obtained by first-order torsion theory. The efficient torsion element could consider the effect of warping and shear deformation and is more convenient and applicable than the initial parameter method. The expression form of current element stiffness is concise. Numerical examples have demonstrated the validity and applicability of the current efficient torsion element.

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