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Analytical Solution of the Advection-Dispersion Equation Using Asymmetrical Pollution Distribution

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Abstract. Solution of the pollution spreading in rivers is an engineering task, which could be solved by very different ways. For the simulation of the hydrodynamic dispersion in river (pollution spreading) can be 1D, as well 2D or 3D approach used. 2D (eventually 3D) approach requires much more data – physical proportions and model boundaries, velocities and dispersion parameters for two (or three) dimensions. Such data are not always available and it can be very difficult, time and money consuming task to collect necessary data. Because of this, simple analytical solution, derived for simplified initial and boundary conditions, are very often used in the practice. These analytical solutions, describing the 1D substance transport in streams, have of course many limitations and factors, which determine their accuracy. One of the very important factors is the assumption, that the pollution cloud (in case of an instantaneous pollution injection) is spreading downstream and upstream symmetrically. But in reality, almost in all streams are present the transient storage areas (dead zones), deforming the concentration distribution of the transported substance (pollution). For better adaptation to such real conditions, a simple 1D approximation method is presented in this paper. The proposed approximate method is based on the asymmetric probability distribution (Gumbel's distribution) and was verified on field experiments in Slovakia. Tracer experiments confirmed the presence of dead zones in various extents, depending mainly on the vegetation occurrence and extent in stream. Statistical evaluation confirms that the proposed method approximates the measured concentrations significantly better than methods based upon the symmetrical Gaussian distribution.

1. Introduction

Environmental problems of water streams pollution have received attention especially in connection with pollution spreading prediction. According to this the mathematical or numerical modelling becomes as a very important tool. One specific group of these models is hydrodynamic models, which are based on solutions of so-called advection-dispersion equation. The dispersion of the substances together with the advection in the flow are underlying mechanism of dissolved particles movement in an aqueous medium. These phenomena help reduce the maximum concentration values of the solute in the flow. Solute spreads gradual to the sides due to the pulsation rate and different concentrations of the substance. The key parameter of the dispersion rate are the dispersion coefficients in the corresponding direction. Determination of these dispersion parameters values, plays therefore an important role in tasks solving the transport of pollutants in streams and also in modelling water quality.



Mass transport in natural streams from hydrodynamic point of view is strongly related to river characteristics, such as mean flow velocity, velocity distribution, secondary currents and turbulence features. Those parameters are mainly determined by the stream morphology and the discharge conditions [1]. In natural channels, changing stream width, curvature, bed form, bed material and vegetation are the reason for flow condition changing and by this way for flow describing characteristics diversity. The small cavities existing in stream beds, side arms and embayments, can produce flows which interact on different scales with the flow in the main stream direction. These irregularities are designated as dead zones impact.

The goal of this study is to design a simple mathematical solution for prompt and sufficiently accurate one-dimensional modelling of the dispersion process in streams with the occurrence of dead zones. Two distributions were considered to approximate the dispersion process. One is classically applied the Gaussian distribution and the second alternative is the Gumbel distribution. In both cases there is presupposed injection from an instantaneous pollution source.

The simplest mathematical formulation and description of pollution spreading process in a surface stream from hydrodynamic point of view is the one-dimensional advection-dispersion equation (ADE). Its form is:

$$\frac{\partial C}{\partial t} + v_x \frac{\partial C}{\partial x} = D_x \left(\frac{\partial^2 C}{\partial x^2} \right) + M_s \quad (1)$$

where t is the time [s], C is the concentration of pollution [kg.m^{-3}], D_x is the dispersion coefficient in the longitudinal direction [$\text{m}^2 \text{s}^{-1}$], v_x is the flow velocity of water [m.s^{-1}], M_s is a function representing the sources of pollution [$\text{kg.m}^{-3} \text{s}^{-1}$], x is the spatial coordinate - distance [m].

This equation includes two basic transport mechanisms: advection caused by water flow and dispersion caused by gradient of mass concentration and velocity gradient.

Analytical solution of ADE in form of eq. (1) for instantaneous point source and uniform steady flow condition can be written in the form [2]:

$$C(x, t) = \frac{M}{2A\sqrt{\pi D_x t}} \exp\left(-\frac{(x - \bar{v}_x t)^2}{4 D_x t}\right) \quad (2)$$

where $C(x, t)$ is a mass concentration [kg.m^{-3}] in a place and time; D_x is the longitudinal dispersion coefficient [$\text{m}^2 \text{s}^{-1}$]; A is a discharge area in a stream cross-section [m^2], M is the mass of pollutant or tracer [kg], \bar{v}_x is a mean flow velocity [m.s^{-1}], x is a distance [m], t is time [s].

Analytical solution by eq. (2) should have a form of Gaussian normal distribution with parameters of normal distribution (e.g. standard deviation σ , etc.) [3]. The form of the eq. 2 signifies, that the pollutant will be spread symmetrically inside of stream channel space.

This form of solution is currently considered as the standard one, but it is necessary to say that its validity is limited by precondition of flow without barriers and with symmetrical movement of pollutant in the stream channel. In real streams this assumption is not valid and the pollutant concentration distribution curve does not have the exact shape of a Gaussian normal distribution.

Using one-dimensional ADE in conditions of real streams confronts with certain limitations. Among them as the most significant limitation is the occurrence of dead zones. As there was indicated above, dead zones are parts of streams with occurrence of secondary fluxes or zones with the appearance of small or zero or even negative velocities. These can be found along the beds and banks of a stream, behind any obstacles in a channel or in the case of an aquatic vegetation occurrence during a growing season, etc. Transported pollution is trapped or separated from the main flux of the stream in these “dead zones” parts of a stream. Thereafter transported pollution is gradually released and integrated back into the main part of a stream.

The effect of these zones has been described in several studies (e.g. [4], [5], [1]). Their occurrence commonly causes deformation of pollution cloud shape and by this way deformation of the form of

concentration distribution curve because the pollution is released later and little by little from the "tanks" formed by irregularities of the stream channel or from parts with dense aquatic vegetation. The result effect is unsymmetrical shape of pollution concentration time course curve and it is noticeable, that the Fick's law cannot be applied even after a long period in natural streams, since the concentration distribution due to large irregularities of the stream bed will never be a Gaussian [6], [2].

2. Proposed method

Analytical solution of equation 1 for simplified conditions (instantaneous point source, prismatic streambed, steady and uniform flow) is shown as eq. 2. According various authors ([3], [7], [2]) this analytical solution can be expressed more generally:

$$C = \frac{M}{A\sqrt{D_x t}} f\left(\frac{x}{\sqrt{D_x t}}\right) \quad (3)$$

where A is a discharge area in a stream cross-section [m^2], M is a pollutant mass [kg], f is the unknown function. The unknown function f can be determined in two different ways [3]:

1. Derive a curve fit to real data from tracer experiment
2. Solve the equation 1 analytically.

Equation. 2 is in fact an analytical solution of the advection – dispersion equation (equation 1), assuming spatial symmetrical normal Gaussian distribution of the pollution in stream. As it was mentioned above, in the case of real conditions flow with occurrence of dead zones in a stream, it is appropriate to approximate the function f from equation 3 not in the form of the Gaussian normal distribution, but in different statistical distribution form with asymmetric shape.

As an appropriate approximation of the function f from the equation 3 the formula of the Gumbel's distribution is proposed. The general equation of the Gumbel's distribution is

$$p = \frac{1}{\xi} e^{-(z+e^{-z})} \quad z = \frac{x-\mu}{\xi} \quad (4)$$

where p is the distribution probability (density), the parameter μ is the location parameter and the parameter ξ is the scale parameter.

The parameters from the equation 4 can be defined as it follows:

$$\xi = \sqrt{D_{x,G} t} \quad z = \frac{\bar{v}_x t - x}{\sqrt{D_{x,G} t}} \quad (5)$$

where the $D_{x,G}$ is the dispersion coefficient in the longitudinal direction [$\text{m}^2 \text{s}^{-1}$], used in the proposed model. For the reason of dimensional consistency, z is a dimensionless parameter and ξ has the dimension of a length [m].

By substitution from equation 4 and 5 into equation 3, the proposed one-dimensional analytical solution has the form:

$$c(x, t) = \frac{M}{A\sqrt{D_{x,G} t}} \exp \left[\frac{x - \bar{v}_x t}{\sqrt{D_{x,G} t}} - \exp \left(\frac{x - \bar{v}_x t}{\sqrt{D_{x,G} t}} \right) \right] \quad (6)$$

3. Field measurements

The tracer experiments were performed at the stream in south Slovakia –the Malina stream. This stream is situated at lowland areas and therefore it has a low slope. For lowland type streams is also typical low flow velocities and bottom sediment occurrence [8], [9].

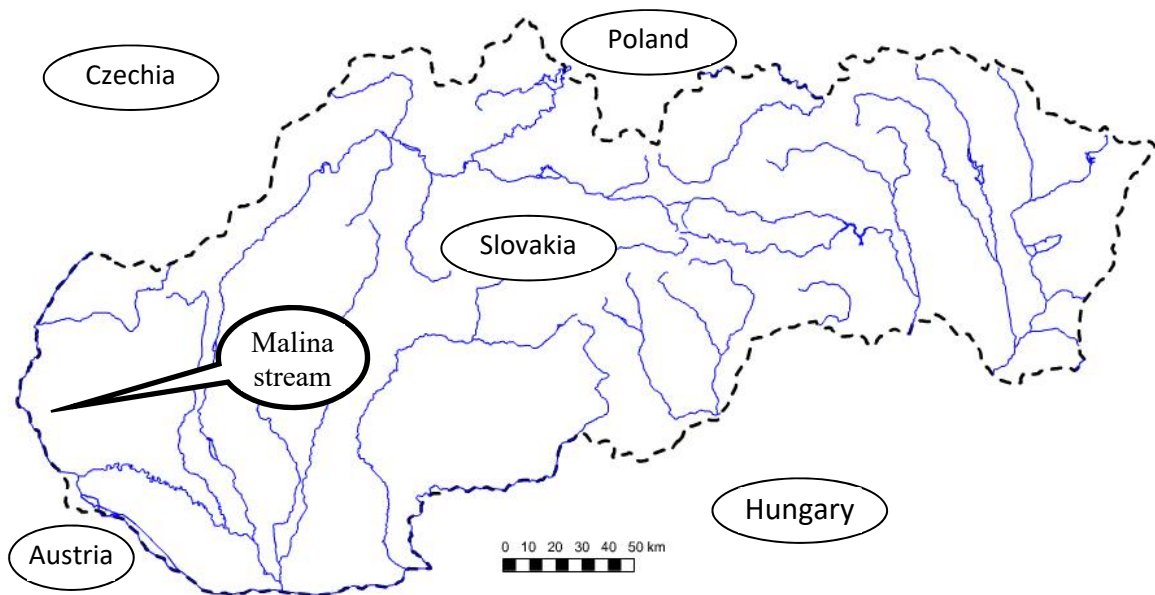


Figure 1. Map of the field tracer experiments

The location of the tracer experiments at the Malina stream was located in the cadastral areas of Lab and Zohor municipalities (N+48.334771°, E+16.967445°). The experiments were carried out on the stream reach with a length of 1415 m. It was a straight reach of the stream, without significant directional changes. Originally constructed cross section shape was significantly influenced by vegetation. Discharge during the experiments was $0,408 \text{ m}^3\text{s}^{-1}$. The slope, specified by levelling measurements, was 0.45 ‰. The channel shape in the examined reach can be considered as a prismatic one, the width at water level was about 5 m, the average depth was 0.88 m.

Results of tracer experiments showed the concentration time courses deformations, which indicated significant presence of transient storage zones. These were formed by the stream bed irregularities, but mainly by the vegetation occurrence along the stream banks and on the bed of streams.



Figure 2. Water vegetation in Malina stream

In the field experiments the colouring agent – E133 (brilliant blue, food colour) was used and the tracer concentration was measured and determined using field spectrophotometry device.

In all field experiments an instantaneous tracer injection to the centre of the stream cross-section profile was used.

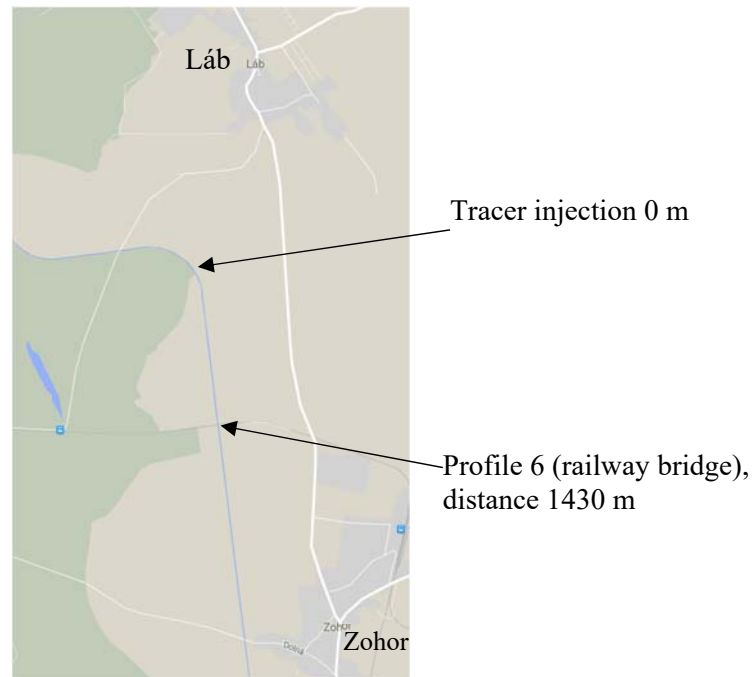


Figure 3. Map of the field tracer experiments at the Malina stream (N48.334771°, E16.967445°).

4. Results and discussions

Data from all tracer experiment showed deformations of the concentration distribution. These results indicated significant presence of dead zones, which was confirmed also during the locality reconnaissance.

For illustration, some results of measured data and concentration distribution approximated by equation 2 and equation 6 are shown and compared in figure 4. This figure visibly demonstrates, that the proposed method (eq. 6) approximates the measured very well, particularly in the increasing part of the curve.

The goodness of fit between measured values and the results from equation 2 and equation 6 was evaluated by comparing the sum of differences square between the measured and approximated values. The sum of differences square (referred as $\sum(\Delta y)^2$) for each measurement and their basic statistical evaluation are presented in table 1.

Table 1. Comparison of the longitudinal dispersion coefficients values from eq. 2 and eq. 6.

Site	X distance	D_x eq. 2	$D_{x,G}$ eq. 6	$\sum(\Delta y)^2$ eq. 2	$\sum(\Delta y)^2$ eq. 6
unit	[m]	[m ² .s ⁻¹]	[m ² .s ⁻¹]	[-]	[-]
Experiment Malina stream	1415	0.95	2.41	592.5	69.7

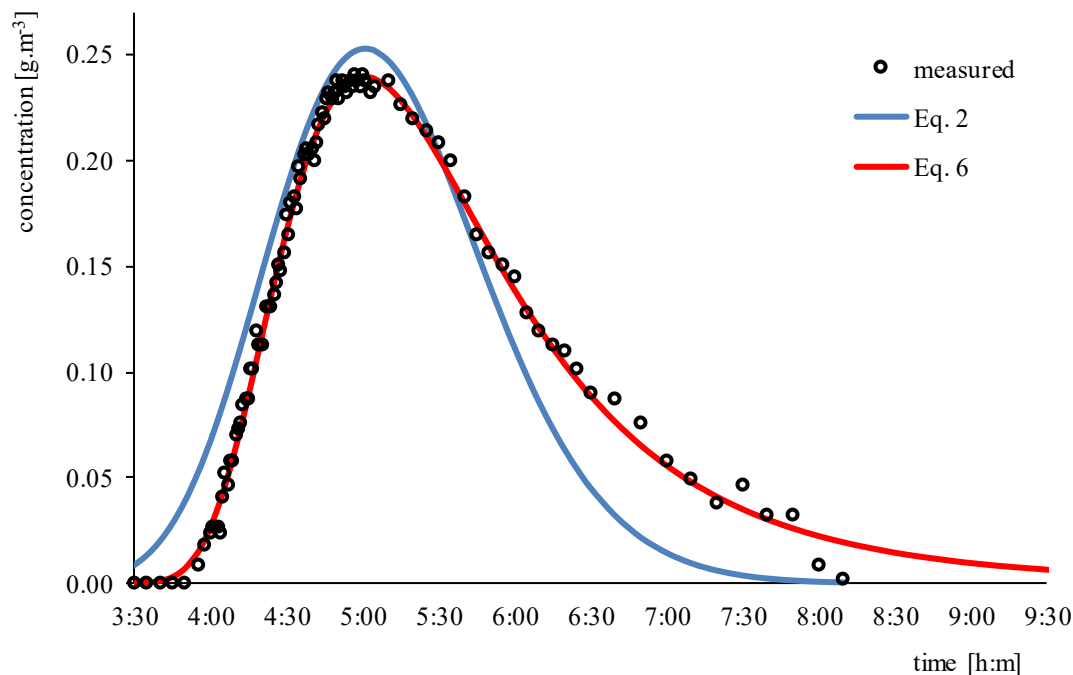


Figure 4. Comparison of approximation results, the Malina stream (approximation using eq. 2 and eq. 6)

Although lot of literature sources are based on the application of the equation 2, it is evident that proposed approximation by Gumbel distribution brings significant increase of the accuracy of the dispersion processes simulation in natural stream with dead zones based on application of analytical solution of ADE. This proposed approximation could be very useful also in case of some special applications linked with extensive consumption of computer time (e.g. iterative tasks in case of pollution source localization).

5. Conclusions

The goal of this study was to design a simple mathematical solution for prompt and sufficiently accurate one-dimensional modelling or prediction of the dispersion process in streams with the occurrence of dead zones. There was derived an analytical solution of the one-dimensional advection- dispersion equation using asymmetrical distribution in the form of eq.6. The proposed approximation was verified using the experimental data from tracer field experiments, which were performed at the stream in Slovakia –Malina stream. Data obtained from the field tracer experiment has shown considerable asymmetry of the tracer concentration distribution, which clearly indicates extensive presence of dead zones in the stream.

Results of the study confirmed, that proposed one-dimensional analytical solution approximates the tracer or pollutant concentration distribution very well and is fully applicable in the practice for simple tasks (simulations) of the pollution spreading in streams. Its advantage is simple application in case of programming and also the fact, that in case of practical using of proposed approximation for simulation of pollutant spreading in natural streams less computational time is needed. This could be very advantageous especially for iterative tasks solving, e.g. the inverse task as a pollution source localization. For full and reliable application, a simple verification and calibration is sufficient, but it is not necessary to determine the parameters of dead zones specifically. Moreover. the estimation of these parameters is quite difficult in practice. Finally, as it was documented, such analytical solution has

significantly increased the accuracy of the results obtained by it in comparison with real data from natural stream flow condition.

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