IOP Conference Series: Materials Science and Engineering

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To cite this article: M Glovnea *et al* 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **591** 012048

IOP Conf. Series: Materials Science and Engineering **591** (2019) 012048 doi:10.1088/1757-899X/591/1/012048

Theoretical aspects regarding the straight rod with elliptical cross section subjected to torsion

M Glovnea, C Manolache-Rusu and C Suciu
Stefan cel Mare University of Suceava-Romania, Department of Mechanics and Technologies, 13th University Street, 720229, Suceava, Romania

E-mail: mglovnea@usm.ro

Abstract. Many engineering structures such as beams, shafts or thin-walled tubular rods are subject to twisting stress. Torsion of cylindrical and prismatic rods of various cross-section is of great practical utility and technical importance in engineering, structural design and mechanical work. Machine construction, steel bridges, ship buildings and brake systems are just some of the main areas of application. Calculation of the shear stress distribution due to a torsional moment is a complicated problem. Although several solutions have been formulated over time, only a few have analytical forms. In many situations the torsion problem was solved numerically using 2-dimensional Finite Element method. In the present paper, equations for the stress function and the torsional moment (Prantl's formula) are established for the considered geometry.

1. Introduction
In order to solve a twisting problem in a general case, a semi-inverse method is used, partially proposing the displacements in the rod and determining the rest of the solution from the condition of satisfying the linear elasticity fundamental equations: Cauchy equations, the integral relations between stresses and sectional stresses and contour conditions [1-6].

Several methods of solving based on this principle have been developed over time, namely [1-4]: Saint Vénant's twist function, Prandtl's stress-function method, complex potency function method, or methods based on various analogues (with membrane, fluid-dynamics, hydrodynamics, electrodynamics and optics). Of these, the simplest and most intuitive method is Prandtl's tension function. As a result, this is the method used in this paper.

2. Proposal for a solution in displacements
For the present study, a rod segment is considered with multiple linked cross section submitted to a torque $\vec{M}_t$, as illustrated in figure 1. The cross section'souter perimeter is bound by the contour (C_0), and inside there are several contours (C_i) where $i = 1\ldots n$. According to the current solving procedure, [3], in mathematics, the outer contour is followed in trigonometric sense, while the inner ones in a clockwise direction.

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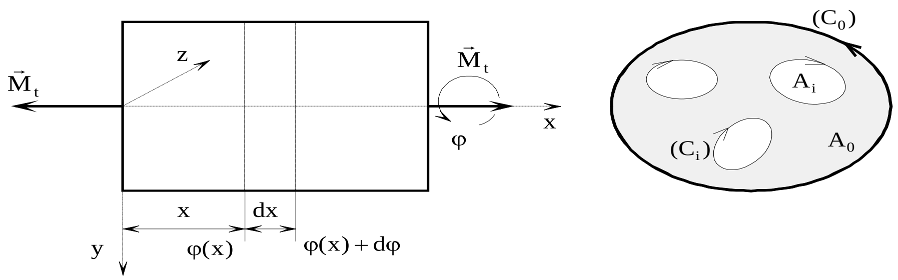

Figure 1. Twisting of a rod with multiple linked cross section.

The area of the surface enclosed by the outer contour (C_0) is marked with A_0, and the areas enclosed by the inner contours (C_i) with A_i. As a result, the cross-sectional area is, equations (1):

$$A = A_0 - \sum_{i=1}^{n} A_i \tag{1}$$

The applied torque causes elastic twists of the cross-sections with values increasing from 0 in the origin to φ at the free end, [3]. If $\varphi(x)$ is considered to be the twisting of the current cross-section of *x* abscissa,in the infinitely adjacent section, $x + dx$, the twisting becomes $\varphi(x) + \frac{\partial \varphi(x)}{\partial x}\, dx$. The relative twist of the two adjacent cross-sections that define a rod segment of *dx* length is, equations (2):

$$\delta\varphi(x) = \varphi(x) + \frac{\partial \varphi}{\partial x}\, dx - \varphi(x) = \frac{\partial \varphi}{\partial x}\, dx \tag{2}$$

The ratio:

$$\frac{\delta\varphi(x)}{dx} = \frac{\partial \varphi}{\partial x} = \frac{d\varphi}{dx} = \omega(x) \tag{3}$$

is the specific twist in section *x*. In equation (3), the partial derivative $\partial\varphi / \partial x$ was replaced by the total derivative because φ only depends on *x*.

The rod is considered homogeneous over its length and the specific twist ω is constant. Therefore, equation (3) takes the following form:

$$d\varphi = \omega\, dx \tag{4}$$

Equation (4) is a differential equation with separate variables, having the solution:

$$\varphi(x) = \omega \cdot x + c_1 \tag{5}$$

where c_1 is an integration constant determined by the condition that in absence of twist, i.e. $\omega = 0$, the free end does not rotate, respectively $\varphi = 0$. From this condition it follows that $c_1 = 0$ and thus the twisting of the x-section, $\varphi(x)$, given by equation (5) becomes:

$$\varphi(x) = \omega \cdot x \tag{6}$$

Equation (6) shows that the twist is proportional to the abscissa, so that twisting of the free end is:

$$\varphi = \varphi(\ell) = \omega \cdot \ell \tag{7}$$

By analogy to the rotation motion, twisting can be viewed as a vector directed along the axis of the rod:

$$\vec{\varphi}(x) = \varphi(x)\,\vec{i} = \omega\, x\, \vec{i} \tag{8}$$

By rotation of the *x*-section with an angle equal to $\varphi(x)$, an initial point $M(x,y,z)$ belonging to said section will be displaced by $\vec{\delta}$ in the x-plane, given by the vector product:

$$\vec{\delta} = \vec{\varphi}(x) \times \vec{r} \tag{9}$$

where $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ is the position radius of this point. The displacement $\vec{\delta}$ is written:

$$\vec{\delta} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega\, x & 0 & 0 \\ x & y & z \end{vmatrix} = -\omega\, x\, z\, \vec{j} + \omega\, x\, y\, \vec{k} \tag{10}$$

On the other hand, the displacement vector at the point M has the expression, u, v and w being axial displacements. Therefore, the component $\vec{\delta}$ can be written as:

$$\vec{\delta} = v\,\vec{j} + w\,\vec{k} \tag{11}$$

By identifying equations (10) and (11), the following expressions of the displacements along the *y* and *z* axes are obtained:

$$v = -\omega\, x\, z\,;\, w = \omega\, x\, y \tag{12}$$

According to Saint Vénant's hypothesis, the *u* displacement does not depend on *x*. Therefore, the transverse cross sections have to deform the same. Instead, this displacement may depend on *y* and *z*. When the rod is not loaded ($\omega = 0$), there are no deformations, and the displacement *u* can be written:

$$u = \omega \cdot \psi(y,z) \tag{13}$$

where $\psi(y,z)$ is called Saint Vénant's torsion function. It can be seen that the displacement*u* depends only on the coordinates *y* and *z*, being independent of *x*. Therefore, the displacement u_0 of the points on the rod axis is constant. As noted above, the center of the left end of the rod, of coordinates $x = 0$, $y = 0$, $z = 0$ remains fixed after the rod is deformed. As a result, the displacement u_0 is null, $u_0 = 0$, i.e. the rod axis points do not move along the axis direction after applying the twisting moment.

Finally, the displacement vector components are:

$$u = \omega \cdot \psi(y,z); \quad v = -\omega\, x\, z\,; \quad w = \omega\, x\, y \tag{14}$$

The unknown function $\psi(y,z)$ is to be determined from the condition of satisfying the fundamental equations of elasticity, [1-4].

The partial derivatives of the displacements, $\partial u / \partial x$, $\partial v/\partial y$ and $\partial w / \partial z$, are null. Therefore, the divergence of the displacement vector $\vec{U}$ and implicitly the specific volume strain ε_v, have null values:

$$div\vec{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \varepsilon_v = 0 \tag{15}$$

which shows that torsion occurs without volume variation. From equation (15), immediately follows that u, v and w displacements given by equations (14) meet the particularities of the Lamé's equation in the case of null mass forces:

$$\Delta div\vec{U} = \Delta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{16}$$

$$\Delta\Delta\vec{U} = 0 \tag{17}$$

In equation (16), the order of the "Laplace" and "Divergence" differential operators is reversed, resulting in:

$$div\ \Delta\vec{U} = 0 \tag{18}$$

or:

$$\Delta\vec{U} = \Delta u\ \vec{i} + \Delta v\,\vec{j} + \Delta w\,\vec{k} = 0 \tag{19}$$

The displacements along the axes v and w, expressed by equation (14) lead to the conclusion that $\Delta v = 0$ and $\Delta w = 0$. Equation (19) is only met if, $\Delta u = 0$ respectively, if:

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{20}$$

This equation shows that Saint Vénant's torsion function is a harmonic function over the cross section of the rod.

3. Components of the stress tensor

Knowing the displacements u, v, w, the specific strains in the rod are calculated by the differential relations between specific displacements and strains:

$$\begin{gathered}\varepsilon_x = \frac{\partial\, u}{\partial\, x} = 0\,; \quad \varepsilon_y = \frac{\partial\, v}{\partial\, y} = 0\,, \quad \varepsilon_z = \frac{\partial\, w}{\partial\, z} = 0\,; \quad \gamma_{xy} = \frac{\partial\, v}{\partial\, x} + \frac{\partial\, u}{\partial\, y} = \omega\left(\frac{\partial\ \psi}{\partial\, y} - z\right); \\ \gamma_{yz} = \frac{\partial\, w}{\partial\, y} + \frac{\partial\, v}{\partial\, z} = \omega\, x - \omega\, x = 0\,; \quad \gamma_{zx} = \frac{\partial\, u}{\partial\, z} + \frac{\partial\, w}{\partial\, x} = \omega\left(\frac{\partial\ \psi}{\partial\, z} + y\right).\end{gathered} \tag{21}$$

Stress tensor components are obtained by substitution of specific strain in Hooke's generalized law:

$$\sigma_x = \frac{E}{1+\nu}\left(\varepsilon_x + \frac{\nu}{1-2\nu}\ \varepsilon_v\right) = 0\,; \quad \sigma_y = \frac{E}{1+\nu}\left(\varepsilon_y + \frac{\nu}{1-2\nu}\ \varepsilon_v\right) = 0\,; \tag{22}$$

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$$\sigma_z = \frac{E}{1+\nu}\left(\varepsilon_z + \frac{\nu}{1-2\nu}\ \varepsilon_v\right) = 0; \qquad \tau_{xy} = G\gamma_{xy} = G\omega\left(\frac{\partial\ \psi}{\partial\ y} - z\right);$$

$$\tau_{yz} = G\ \gamma_{yz} = 0;\ \tau_{xz} = G\ \gamma_{xz} = G\omega\left(\frac{\partial\ \psi}{\partial\ z} + y\right).$$

Equations (22) show that the only non-null stresses are the orthogonal and tangential tensile stresses τ_{xy} and τ_{xz}. On the cross-section, these are the components of the stress vector, which in this case is a tangential stress vector $\vec{\tau}$:

$$\vec{\tau} = \tau_{xy}\ \vec{i} + \tau_{xz}\ \vec{k}\ . \tag{23}$$

4. Prandtl's stress function

In elasticity, the notion of stress function is frequently used, [1-6]. Generally, this is a coordinate function, whose partial derivatives are proportional to the components of the stress tensor. For twisting, the stress function is denoted by $\Phi(y,z)$, and its partial derivatives over the cross section provide the orthogonal tangent stresses through the following relations, [3]:

$$\tau_{xy} = G\omega\frac{\partial\Phi}{\partial z}; \qquad \tau_{xz} = -G\omega\frac{\partial\Phi}{\partial y}. \tag{24}$$

The function $\Phi(y,z)$ is called the Prandtl's stress function. Its use is more convenient than Saint Vénant's torsion function. The two functions, $\Phi(y,z)$ and $\psi(y,z)$, are not independent. The link between them is obtained by identifying equations (22) and (24) of tangent stresses τ_{xy} and τ_{xz} :

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\psi}{\partial y} - z; \tag{25}$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial\psi}{\partial z} - y. \tag{26}$$

By derivation of equation (25) in relation to *z*, and equation (26) in relation to y and adding the results we obtain:

$$\Delta\Phi = -2. \tag{27}$$

Prandtl's stress function is no longer harmonic, as Saint Vénant's torsion function, but is bi-harmonic and satisfies the differential equation (27)with Poisson-type partial derivatives.

5. Determination of unspecified elements of the solution

In the above presented work, a partial solution to the problem has been proposed, in which displacements and stresses are expressed either by Saint Vénant's harmonic function or by Prandtl's bi-harmonic one. None of the two functions involved, Φ or ψ, is not specified as a concrete form, [3]. In order to specify the form of these functions, it is necessary to satisfy the fundamental equations of elasticity, namely Cauchy’s equations of equilibrium, the contour conditions of the problem and the integral equations between the stresses and the sectional stresses on the cross section of the rod, [1-4].

5.1. Meeting Cauchy's equations

The substitution of equations (24) in Cauchy equations for the translational equilibrium of the volume element leads to:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0+G\omega\frac{\partial^2\Phi}{\partial y\partial z}-G\omega\frac{\partial^2\Phi}{\partial y\partial z}=0\,;$$
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0+0+0=0\,;\quad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0+0+0=0\,. \tag{28}$$

It can be noticed that the proposed solution satisfies Cauchy's equations regardless of the actual form of the function.

5.2. Meeting the contour conditions

The contour of the rodis limited by the lateral and front surfaces. According to Saint Vénant's hypothesis, away from the front surfaces, stress and strain states do not depend on the actual mode of loading, but only on the resultant load value, [1-4]. Therefore, loading is considered to be distributed on the front surfaces as tractions that automatically satisfy the corresponding contour conditions, e.g., tractions identical to the twisting stresses of the current cross-section, [3].

Since external forces are not applied to the lateral surface, and the normal are perpendicular to the x-axis, that is their directional dimension relative to the *x*-axis is null, $\ell=0$, at the points of the contour of any cross-section the relations are valid:

$$m\tau_{yx}+n\tau_{zx}=0\,;\;\; m\sigma_y+n\tau_{zy}=0\,;\quad m\tau_{yz}+n\sigma_z=0\,, \tag{29}$$

where *m* and *n* are non-normal non-linear directing parameters satisfying the condition $m^2+n^2=1$.

The last two equations are identically satisfied because the stresses involved are null and the first equation becomes:

$$m\frac{\partial\Phi}{\partial z}-n\frac{\partial\Phi}{\partial y}=0\,. \tag{30}$$

According to figure 2, on the contour of the cross-section the normal's directing parameters of the normal are expressed according to the arc element by the relations:

$$m=cos\,\alpha=\frac{dz}{ds}\,;\quad n=sin\,\alpha=-\frac{dy}{ds}\,, \tag{31}$$

where the minus sign in the expression of *n* takes into account that dz and ds increase in trigonometric sense, while dy decreases along the same direction.

By substituting equation (31) into equation (30), it results that over the contour of the cross-section, the following condition is satisfied:

$$\frac{\partial\Phi}{\partial y}dy+\frac{\partial\Phi}{\partial z}dz=0\,. \tag{32}$$

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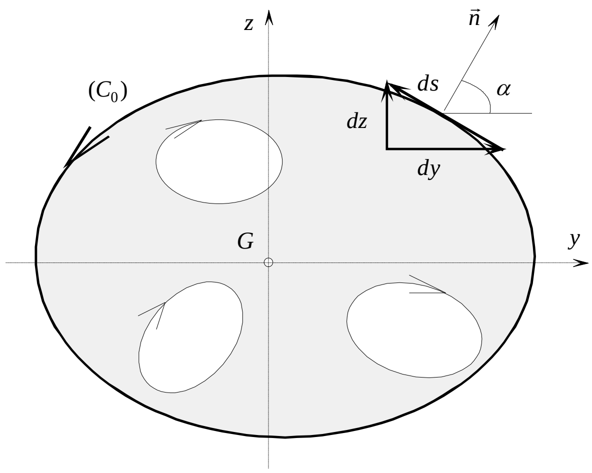

Figure 2. Contour conditions.

The left member of equation (32) represents the total differential $d\Phi$ of $\Phi(y,z)$ function, so that this equation becomes:

$$d\Phi = 0 . \tag{33}$$

Equation (33) shows that, on any contour of the rod, the differential of $\Phi(y,z)$ is null, i.e. this function has a constant value. According to equations (24), the stress state components depend only on the derivatives of $\Phi(y,z)$. Consequently, the constant can be arbitrarily chosen for any one contour of the rod, preferably the outer one. It is therefore convenient to consider that this function $\Phi(y,z)$ has a null value over this contour. Over the inner contours (C_i), the stress function $\Phi(y,z)$ takes constant values, denoted by k_i. Consequently, the contour conditions of the problem are fulfilled if the next relationships are true:

$$\Phi_{(C_0)} = 0 ; \Phi_{(C_i)} = k_i . \tag{34}$$

5.3. Meeting of the integral relationships between stresses and sectional efforts
Since the only non-null sectional effort is the torque M_t, the integral relationships between stresses and sectional efforts take the form:

$$\begin{gathered} \int_A \sigma_x \, dA = N = 0 ; \quad \int_A \tau_{xy} \, dA = T_y = 0 ; \quad \int_A \tau_{xz} \, dA = T_z = 0 ; \\ \int_A y \; \sigma_x \, dA = M_z = 0 ; \quad \int_A z \; \sigma_x \, dA = M_y = 0 ; \quad \int_A (y\tau_{xz} - z\tau_{xy}) \, dA = M_t . \end{gathered} \tag{35}$$

The relations of the sectional efforts, N, M_y and M_z are satisfied in the same way, as $\sigma_x = 0$. It remains to check the relationships for cutting forces T_y and T_z that of the twisting moment M_t.

5.3.1. Cutting forces
By substituting in the relations of forces T_y and T_z stresses τ_{xy} and τ_{xz} by their expressions from equations(24), it follows that:

$$\iint_A \frac{\partial\Phi}{\partial z}\,dydz = 0\,;\quad \iint_A \frac{\partial\Phi}{\partial y}\,dydz = 0\,. \tag{36}$$

Taking into account equation (1), equations(36) become:

$$\iint_{A_0} \frac{\partial\Phi}{\partial z}\,dydz - \sum_{i=1}^{n} k_i \iint_{Ai} \frac{\partial\Phi}{\partial z}\,dydz = 0\,;\quad \iint_{A_0} \frac{\partial\Phi}{\partial y}\,dydz - \sum_{i=1}^{n} k_i \iint_{Ai} \frac{\partial\Phi}{\partial y}\,dydz = 0\,. \tag{37}$$

The double integrals from equations(37) are transformed into the curvilinear integrals over the contours of the cross-sections using Green's formula. In a general case of two functions, $f = f(y,z)$ and $h = h(y,z)$, Green's formula has the following form:

$$\iint_A \left(\frac{\partial f}{\partial y} + \frac{\partial h}{\partial z}\right) dydz = \oint_{(C)} (fdz - hdy)\,. \tag{38}$$

where (C) is the contour of domain A.

Applying this formula to equations (37), it produces:

$$-\oint_{(C_0)} \Phi\,dy + \sum_{i=1}^{n} \oint_{(C_i)} \Phi\,dy = 0\,;\quad \oint_{(C_0)} \Phi\,dz - \sum_{i=1}^{n} \oint_{(C_i)} \Phi\,dz = 0\,. \tag{39}$$

On the outer contour (C_0), the function Φ is null, while over the inner contours, its values are denoted by k_i. As a result, equations (39) become:

$$\sum_{i=1}^{n} k_i \oint_{(C_i)} dy = 0\,;\quad \sum_{i=1}^{n} k_i \oint_{(C_i)} dz = 0\,. \tag{40}$$

According to known mathematical properties, the curvilinear integrals on the closed contours (C_i), which intervene in equations (40) are null, so that these equations are equally satisfied. It follows that the proposed solution satisfies the integral relationships of the cutting forces T_y and T_z.

5.3.2. Twisting moment (Prandtl's formula)

By substituting tangent stress equations (24) in the integral twist moment from equations (35), the following expression is obtained:

$$M_t = -G\omega \iint_A \left(y\frac{\partial\Phi}{\partial y} + z\frac{\partial\Phi}{\partial z} \right) dydz\,. \tag{41}$$

By writing the integrand from equation (41) as:

$$y\frac{\partial\Phi}{\partial y} + z\frac{\partial\Phi}{\partial z} = \frac{\partial}{\partial y}(y\Phi) + \frac{\partial}{\partial z}(z\Phi) - 2\Phi\,, \tag{42}$$

equation (41) becomes:

$$M_t = -G\omega \iint_A \left[\frac{\partial}{\partial y}\left(y\Phi\right) + \frac{\partial}{\partial z}\left(z\Phi\right) \right] dydz + 2G\omega \iint_A \Phi(y,z)dydz \ . \tag{43}$$

Taking into account equation (1) of the area A on which the first integral of equation(43) is defined, it follows that:

$$\begin{aligned} M_t &= -G\omega \iint_{A_0} \left[\frac{\partial}{\partial y}(y\Phi) + \frac{\partial}{\partial y}(y\Phi) \right] dydz + \\ &+ G\omega \sum_{i=1}^{n} \iint_{A_i} \left[\frac{\partial}{\partial y}(y\Phi) + \frac{\partial}{\partial y}(y\Phi) \right] dydz + 2G\omega \iint_A \Phi(y,z)\, dydz \ . \end{aligned} \tag{44}$$

As in the case of cutting forces, the double integrals of the first two terms of the right-hand side of equation (44) are transformed into curvilinear integrals on the contours of the sections, using Green’s formula:

$$\begin{aligned} M_t &= -G\omega \oint_{(C_0)} \left(-z\Phi dy + y\Phi dz\right) + G\omega \sum_{i=1}^{n} \oint_{(C_i)} \left(-z\Phi dy + y\Phi dz\right) \\ &+ 2G\omega \iint_A \Phi(y,z)dydz. \end{aligned} \tag{45}$$

Because the function Φ is null on the contour (C_0) and takes the values k_i on the inner contours (C_i), equation (45) becomes:

$$M_t = G\omega \sum_{i=1}^{n} k_i \oint_{C_i} \left(-zdy + ydz\right) + 2G\omega \iint_A \Phi(y,z)dydz. \tag{46}$$

The curvilinear integral over one of the inner contours (C_i) is twice the area A_i enclosed by said contour:

$$\oint_{(C_i)} \left(y\ dz - z\ dy \right) = 2A_i \ . \tag{47}$$

Therefore, equation (46) yields the following simple final form:

$$M_t = 2G\omega \sum_{i=1}^{n} k_i A_i + 2G\omega \iint_A \Phi(y,z)\, dydz. \tag{48}$$

Equation (48) is called Prandtl's formula for multiple linked domains, and must be satisfied for any straight rod subjected to twisting.

In the case of simply linked sections, Prandtl's formula becomes:

$$M_t = 2G\omega \iint_A \Phi(y,z)\, dydz. \tag{49}$$

6. Method of solving twisting problems

In a twisting problem, the length of the rod ℓ , the contours of the cross-section (C_0) and (C_i), the material and its elastic properties E, G, and ν , as well as the applied torque M_t, are usually given. It is required to determine the stress and strain states of the rod in order to impose rigidity and strength conditions. In essence, the stress function $\Phi(y,z)$ is determined so that it meets the following conditions:

- $\Phi_{(C_0)} = 0$; $\Phi_{(C_i)} = k_i$;

- $\Delta\Phi = -2$;
- $M_t = 2G\omega\sum_{i=1}^{n} k_i A_i + 2G\omega\iint_A \Phi(y,z)dydz$ or $M_t = 2G\omega\iint_A \Phi(y,z)dydz$.

Of the several methods developed for solving, it is preferable to construct this function by using the cross section contour equation.

The following steps are taken in order to solve the problem:

1. Determination of the stress function $\Phi(y,z)$:

a. The following form of the stress function is proposed:

$$\Phi(y,z) = k\ C_0(y,z), \tag{50}$$

where:

$$(C_0) \equiv C_0(y,z) = 0 \tag{51}$$

is the equation of the outer contour of the rod, and *k* is a constant to be determined. This form automatically ensures that the stress function is canceled on the outer contour of the rod, $\Phi_{(C_0)} = 0$.

b. Determine the constant k from the $\Delta\Phi = -2$ condition.

c. Determine the values k_i that the stress function takes on the inner contours (C_i) of the rod. If there is only one inner contour, $i = 1$, concentric and similar to the outer contour, the value is obtained by substituting the curve (C_1) coordinates in the stress function. If there is only one outline, eccentric to the outside, or there are several inner contours, the values of the constants k_i are determined by the theorem of the movement of the tangential stresses applied to these contours.

2. Determination of the specific twist and rigidity check of the rod:

Calculating the specific twist using Prandtl's formula as:

$$\omega = \frac{M_t}{2G\left[\sum_{i=1}^{n} k_i A_i + \iint_A \Phi(y,z)dydz\right]}, \tag{52}$$

for multiple linked domains, or:

$$\omega = \frac{M_t}{2G\iint_A \Phi(y,z)dydz}, \tag{53}$$

for simply related domains. Equations(52) and (53) can be written in the following simple form:

$$\omega = \frac{M_t}{GI_t}, \tag{54}$$

where I_t is the conventional twisting moment of inertia of the rod. It has one of the following expressions:

$$I_t = 2\left[\sum_{i=1}^{n} k_i A_i + \iint_A \Phi(y,z)dydz\right], \tag{55}$$

for multiple related domains, or:

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$$I_t = 2\iint_A \Phi(y,z)\,dydz\,, \tag{56}$$

for simply related domains.

3. Determining the stress state and checking the rod's strength

The components of the stress vector are determined by replacing the specific twist given by equation (54) in equation (24):

$$\tau_{xy} = G\omega\frac{\partial\Phi}{\partial z} = \frac{M_t}{I_t}\frac{\partial\Phi}{\partial z};\quad \tau_{xz} = -G\omega\frac{\partial\Phi}{\partial y} = -\frac{M_t}{I_t}\frac{\partial\Phi}{\partial y}\,. \tag{57}$$

4. Determination of $u(y,z)$ displacement over the cross-section

The displacement $u(y,z)$ is proportional to Saint Vénant's twisting function, according to equation(13). The partial derivatives of $\psi(y,z)$ are determined by the derivatives of $\psi(y,z)$ by means of equations (25) and (26). These are then integrated, the contour conditions are imposed over the contour of the rod and on the twisting axis in order to identify the solution and find the $\psi(y,z)$ function.

7. Conclusions

The work presented herein describes the resolution of a torsion problem using a semi-inverse method. Rod displacements were partially proposed, while the remainder of the solution was determined by imposing it to meet the fundamental conditions of linear elasticity.

The particular interest of the studied problem is given by the considered straight rod'scross section shape, which is a multiple connected domain.

By particularization of the cross section and considering it filled and bound by a closed conical curve (ellipse/circle), the presented method yields known calculus relations for torsion, which validates the obtained equations.

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Acknowledgements

This work was partially supported from the project "Integrated Center for Research, Development and Innovation in Advanced Materials, Nanotechnologies, and Distributed Systems for Fabrication and Control", Contract No. 671/09.04.2015, Sectoral Operational Program for Increase of the Economic Competitiveness co-funded from the European Regional Development Fund.