Modeling of stress and strain states induced by torsion of straight rods with elliptical cross sections

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Modeling of stress and strain states induced by torsion of straight rods with elliptical cross sections

M Glovnea, C Manolache-Rusu and C Suciu
"Stefan cel Mare" University of Suceava, Department of Mechanics and Technologies, 13th University Street, 720229, Suceava, Romania

E-mail: mglovnea@usm.ro

Abstract. Torsion of bars of various cross-section is of great practical utility in engineering, structural design and mechanical work. It is applied in machine construction, steel bridge or railway construction. Many publications can be found in literature that present the stress-strain analysis in the case of straight rods with constant profiles subjected to torsion. Most of these works were however directed to the case of rods with simple connected domains as cross sections. The present work uses a semi-inverse method which permits to determine general equations for stress-strain analysis in the case of rods having a multiple connected domain as cross section, subjected to torsion. For the present study, the general equations were customized for two particular cases that of an elliptical cross section and that of an elliptical ring.The determined analytical equations were implemented to a specific situation by aid of Mathcad software. This permitted to graphically represent the distribution of torsion stresses and cross-section deflection.

1. Introduction
Many publications can be found in literature, [1-3], that present the stress-strain analysis in the case of straight rods with constant profiles subjected to torsion. Most of these works were however directed to the case of rods with simple connected domains as cross sections.

Based on previous research by the authors, [4-6] the present study uses a semi-inverse method which permits to determine general equations for stress-strain analysis in the case of rods having a multiple connected domain as cross section, subjected to torsion. For the present study, the general equations were customized for two particular cases that of an elliptical cross section and that of an elliptical ring.

2. Torsion of rods with elliptic cross section

2.1. Stress function
A straight rod with filled elliptical cross section with the a and b half-axes is considered, as illustrated in figure 1.

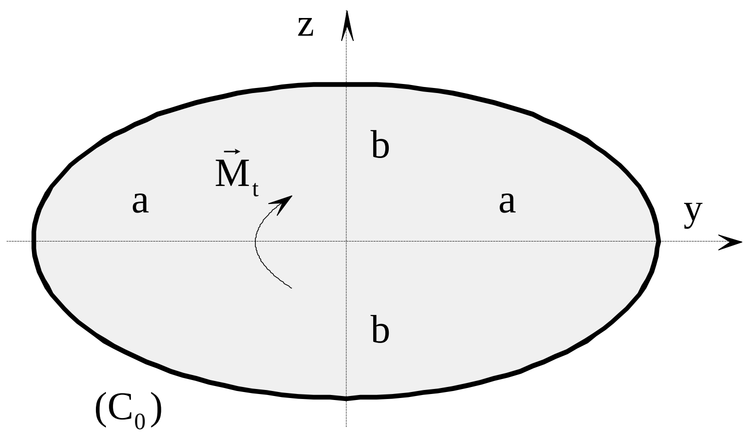


Figure 1. Rod with elliptical cross-section.

The equation of the corresponding limit ellipse, given by:

$$\frac{y^2}{a^2}+\frac{z^2}{b^2}-1=0 \tag{1}$$

allows to write the following outer contour equation (C_0):

$$(C_0)\equiv a^2b^2-a^2z^2-b^2y^2=0 \tag{2}$$

In accordance with the general expression of the stress function, [4]:

$$\Phi(y,z)=k\ C_0(y,z) \tag{3}$$

the stress function corresponding to an elliptical cross section is written as:

$$\Phi(y,z)=k\left(a^2b^2-a^2z^2-b^2y^2\right) \tag{4}$$

Where k is determined by the condition $\Delta\Phi=-2$. By computing the Laplace operator for the Φ function, it follows that:

$$\Delta\Phi=-2k\left(a^2+b^2\right)=-2 \tag{5}$$

leading to:

$$k=\frac{1}{a^2+b^2} \tag{6}$$

The final expression of the stress function is:

$$\Phi(y,z)=\frac{1}{a^2+b^2}\left(a^2b^2-a^2z^2-b^2y^2\right) \tag{7}$$

At the center of the ellipse, the stress function gets the value $\Phi_0=a^2b^2/(a^2+b^2)$. A constant value of the function Φ, be it $\chi\in(0,\Phi_0)$, converts equation (7) to:

$$\frac{y^2}{a^2}+\frac{z^2}{b^2}=1-\frac{\chi}{\Phi_0} \tag{8}$$

Equation (8) represents the stress lines equation. These are coaxial and concentric ellipses with the contour ellipse of the section (C_0).

2.2. Specific twist

As the considered cross-section is simply related, Prandtl's formula for a multiple connected domain:

$$M_t=2G\omega\iint_A \Phi(y,z)\,dydz, \tag{9}$$

becomes, [4]:

$$M_t=\frac{2G\omega}{a^2+b^2}\iint_A\left(a^2b^2-a^2z^2-b^2y^2\right)dydz, \tag{10}$$

where G represents the transverse elasticity modulus and ω is the specific twist.

To solve the above shown double integral, the following substitutions are made:

$$y=a\rho\cos\alpha\,;\quad z=b\rho\sin\alpha\,,\quad \rho\in[0,1]\,,\quad \alpha\in[0,2\pi] \tag{11}$$

The $dydz$ area element is in this case:

$$dydz=\begin{bmatrix}\frac{\partial y}{\partial\rho} & \frac{\partial z}{\partial\rho}\\ \frac{\partial y}{\partial\alpha} & \frac{\partial z}{\partial\alpha}\end{bmatrix}d\rho d\alpha=ab\rho d\rho d\alpha \tag{12}$$

By operating the change of variables in equation (10), it results that:

$$M_t=\frac{2G\omega\, a^3b^3}{a^2+b^2}\int_0^{2\pi}d\alpha\int_0^1\rho(1-\rho^2)d\rho=\frac{\pi\, G\omega a^3b^3}{a^2+b^2} \tag{13}$$

Equation (13) leads to the specific twist as:

$$\omega=\frac{M_t(a^2+b^2)}{\pi\, Ga^3b^3} \tag{14}$$

Equation (14), of the specific twist is usually written under the following simple form:

$$\omega=\frac{M_t}{GI_t} \tag{15}$$

where the conventional moment of inertia for twisting of elliptical section rods is obtained by identifying equations (14) and (15):

$$I_t=\frac{\pi\, a^3b^3}{a^2+b^2} \tag{16}$$

2.3. Stress state

The components τ_{xy} and τ_{xz} of the stress vector $\vec{\tau}$ expressed by the partial derivatives of the stress function Φ, according to the relations:

$$\tau_{xy} = G\omega \frac{\partial \Phi}{\partial z}; \quad \tau_{xz} = -G\omega \frac{\partial \Phi}{\partial y} \tag{17}$$

which become:

$$\tau_{xy} = -\frac{2a^2 z}{a^2 + b^2} G\omega; \quad \tau_{xz} = \frac{2b^2 y}{a^2 + b^2} G\omega \tag{18}$$

The product $G\omega$ is deduced from equation (15), thus equations (18) become:

$$\tau_{xy} = -\frac{2a^2 z}{a^2 + b^2} \frac{M_t}{I_t} = -\frac{2M_t z}{\pi\, ab^3}; \quad \tau_{xz} = \frac{2b^2 y}{a^2 + b^2} \frac{M_t}{I_t} = \frac{2M_t y}{\pi\, a^3 b} \tag{19}$$

The magnitude τ of the stress vector takes the form:

$$\tau(y,z) = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = \frac{2M_t}{\pi\, ab} \sqrt{\frac{y^2}{a^4} + \frac{z^2}{b^4}} \tag{20}$$

On the ellipse axes, the component parallel to the axis is canceled so that the stress vector is perpendicular to the axes and varies linearly along them:

$$\tau(y,0) = \frac{2M_t y}{\pi\, a^3 b}; \quad \tau(0,z) = \frac{2M_t z}{\pi\, ab^3} \tag{21}$$

On any given diameter with a slope m or $z = my$, respectively, the stress vector has the size:

$$\tau(y,z) = \frac{2M_t y}{\pi\, ab} \sqrt{\frac{1}{a^4} + \frac{m^2}{b^4}} \tag{22}$$

This vector is no longer perpendicular to the diameter but inclined towards it, so that it is tangent to the stress lines, including contour. Stresses variations along the ellipse axes and any diameter are illustrated in figure 2.

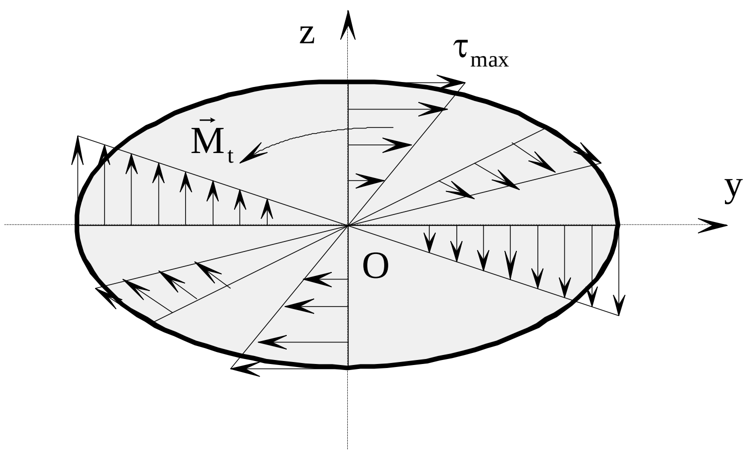
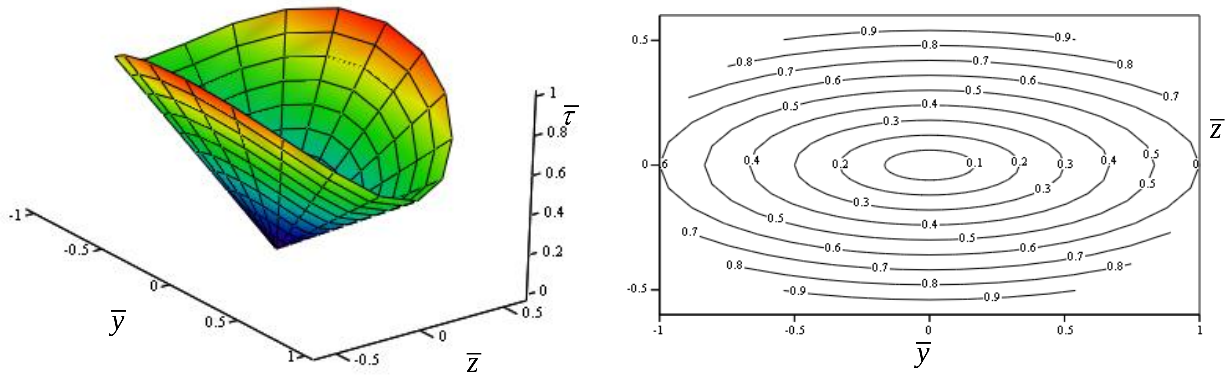


Figure 2. Variation of tangential stresses on the elliptical section.

Figure 3(a) shows the three-dimensional distribution of the stress vector value, and figure 3(b) illustrates this distribution through stress lines. The stresses are made dimensionless by division to the maximum stress $\tau_{\max}$ and the lengths by reference to the ellipse's largehalf-axis, a .



(a) Three-dimensional representation of dimensionless stress.

(b) Constant stress level lines.

Figure 3. Distribution of non-dimensional tangent twisting stresses over the elliptical section.

By analysis of the variation of tangent stress τ , it was found that, according to the general theory of twisting, tangent stress is canceled at the center of gravity of the section, reaches a maximum value τ_{max} , at the ends of the small axis and has a minimum contour value τ_{min} , at the ends of the large axis. The values of the extreme stresses are:

$$\tau_{max} = \frac{2M_t}{\pi\ ab^2}; \qquad \tau_{min} = \frac{2M_t}{\pi\ a^2b} \tag{23}$$

The maximum value above is also the maximum value on the section.

2.4. Axial displacement over the cross section

In accordance to the general methods and equation (7), the partial derivatives of the twisting function are given by:

$$\frac{\partial\psi}{\partial y} = z + \frac{\partial\Phi}{\partial z} = \frac{b^2-a^2}{b^2+a^2}\cdot z\,;$$
$$\frac{\partial\psi}{\partial z} = -y - \frac{\partial\Phi}{\partial y} = \frac{b^2-a^2}{b^2+a^2}\cdot y \tag{24}$$

By integrating the two obtained equations (24), it results that:

$$\psi(y,z) = \frac{b^2-a^2}{b^2+a^2}\cdot yz + F(z)\,;\;\; \psi(y,z) = \frac{b^2-a^2}{b^2+a^2}\cdot yz + H(y) \tag{25}$$

where $F(z)$ and $H(y)$ are functions that fulfill the role of integration constants. The two equations (25) of the function ψ must be identical. This is only possible if the functions $F(z)$ and $H(y)$ are reduced to the same constant: $F(z) =$ H(y) = C . As a result, the twisting function becomes:

$$\psi(y,z) = \frac{b^2-a^2}{b^2+a^2}\cdot yz + C \tag{26}$$

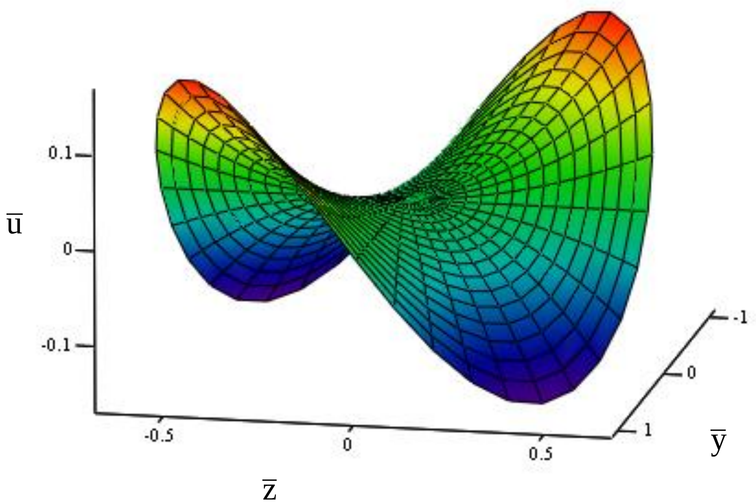
The displacement $u(y,z)$ takes the following form:

$$u(y,z) = \omega \left(\frac{b^2 - a^2}{b^2 + a^2} \right) yz + \omega C \tag{27}$$

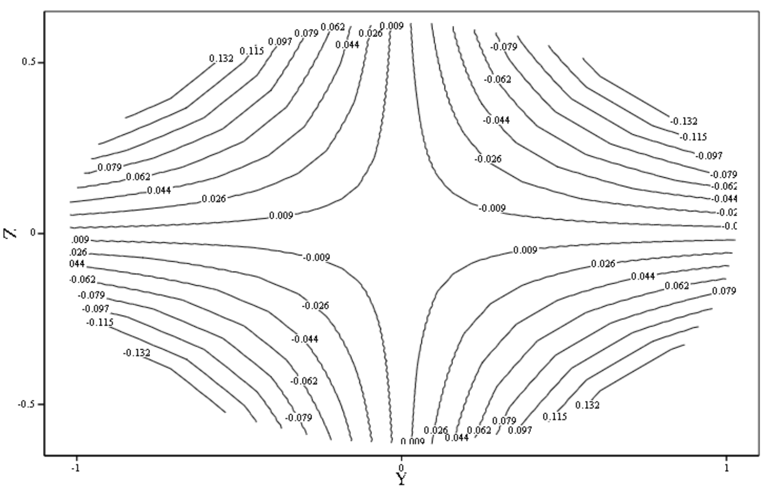
Because $u(0,0) = u_0 = 0$ the integration constant is null, $C = 0$. Final expression of axial displacement is therefore:

$$u(y,z) = \omega \left(\frac{b^2 - a^2}{b^2 + a^2} \right) yz \tag{28}$$

Equation (28) shows that the initially flat sections of the rod no longer remain flat after loading, but become hyperbolic paraboloids, as illustrated in figure 4(a) in three-dimensional view and in figure 4(b) through constant level lines. Level curves belong to a family of equilateral hyperboles. The points on the y and z axes have zero displacements and remain in their initial positions, as well as those located on the rod axis. The twisting center coincides with the center of gravity.



a. Three-dimensional representation.



b. Constant strain level curves.

Figure 4. Strain of elliptical cross-section after twisting.

3. Torsion of rods with elliptical ring cross section

A straight rod is considered with an elliptical ring cross section, bound by the following two ellipses, having a common axis and the same aspect ratio:

$$\frac{y^2}{a^2}+\frac{z^2}{b^2}-1=0\,; \quad \frac{y^2}{a^2}+\frac{z^2}{b^2}-\alpha^2=0 \tag{29}$$

where $0<\alpha<1$. The first of the two equations (29) describes the outer elliptic boundary, of *a* and *b* half-axes, while the second equation describes the inner boundary, of half-axes $\alpha\cdot a$ and $\alpha\cdot b$, smaller than *a, b*. The obtained cross section in illustrated by figure 5.

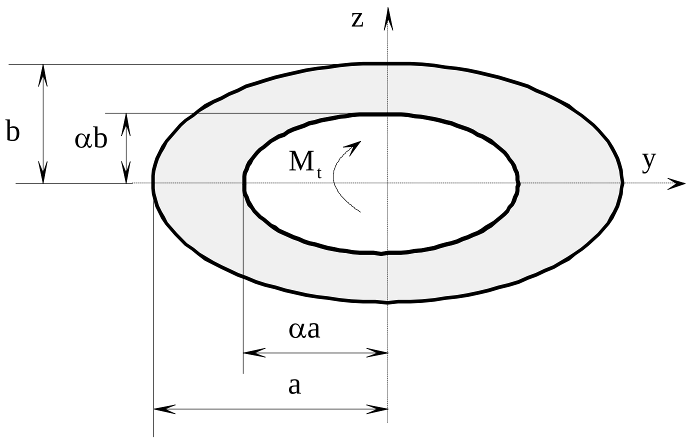


Figure 5. Elliptic ring cross section.

The length of the rod, ℓ, applied torque M_t, as well as material elastic properties are considered known. The problem is to determine the rod's strain and stress states. This situation is that of a double connected domain as cross section.

3.1. Stress function

As the stress function annuls over the outer contour, the solution already obtained in the case of a filled elliptic cross section, given by equation (7) was proposed directly.

However, because in the present case the cross section is double connected, it must be verified that the stress function is also constant over the inner contour, (C_1). For this aim, the $\Phi(y,z)$ stress can be rewritten as:

$$\Phi(y,z)=\frac{a^2b^2}{a^2+b^2}\left(1-\frac{y^2}{a^2}-\frac{z^2}{b^2}\right) \tag{30}$$

The second of equations (29) and equation (30), lead to the following value of the stress function Φ over the inner contour, which represents the k_1 constant:

$$\Phi_{(C_1)}=\frac{a^2b^2}{a^2+b^2}\left(1-\alpha^2\right)=k_1 \tag{31}$$

3.2. Specific twist

In the case of the considered double connected domain, Prandtl's formula can be obtained by particularization of the general equation corresponding to a multiple connected domain, [4]:

$$M_t = 2G\omega\sum_{i=1}^{n} k_i A_i + 2G\omega\iint_A \Phi(y,z)\,dydz, \tag{32}$$

which now becomes:

$$M_t = 2G\omega\, k_1 A_1 + 2G\omega\iint_A \Phi(y,z)dydz. \tag{33}$$

The area A_1 of the surface bound by the inner ellipse is given by:

$$A_1 = \alpha^2 \pi\, ab \tag{34}$$

while the area A is:

$$A = A_0 - A_1 = (1-\alpha^2)\pi\, ab \tag{35}$$

For the calculus of the double integral in equation (33), the following substitution can be used:

$$y = a\rho\cos\theta; \quad z = b\rho\sin\theta, \quad \rho\in[\alpha, 1], \quad \theta\in[0, 2\pi] \tag{36}$$

The second term in Prandtl's formula, given in equation (33), thus becomes:

$$2G\omega\iint_A \Phi dydz = \frac{2G\omega a^3 b^3}{a^2+b^2}\int_0^{2\pi} d\theta \int_\alpha^1 \rho(1-\rho^2)d\rho = \frac{\pi\, G\omega a^3 b^3}{a^2+b^2}(1-\alpha^2)^2. \tag{37}$$

Further substitution of equations (31), (34) and (37) in (33) yield the following Prandtl's formula for elliptic ring cross sections:

$$M_t = G\omega\frac{\pi\, a^3 b^3}{a^2+b^2}(1-\alpha^4) \tag{38}$$

If the specific twist is determined by aid of the general formula:

$$\omega = \frac{M_t}{GI_t} \tag{39}$$

The conventional twisting moment of inertia has the expression, [4]:

$$I_t = \frac{\pi\, a^3 b^3}{a^2+b^2}(1-\alpha^4) = I_{te}(1-\alpha^4) \tag{40}$$

where I_{te} represents the conventional twisting moment of inertia for a filled cross section bound by the outer ellipse.

Equation (40) shows that by introducing a centred elliptical hole of $\alpha\cdot a$ and $\alpha\cdot b$ half-axes, the conventional twisting moment of inertia is reduced $(1-\alpha^4)$ times, while the area A of the cross section is reduced $(1-\alpha^2)$ times. This result shows that the decrease of the conventional twisting moment of inertia is a lot less than the corresponding area decrease, which indicates that the use of annular cross sections represents an important way of reducing material consumption.

3.3. Stress state

As the stress function has the same form as in the case of rods with filled elliptical cross sections when subjected to torsion, the components of the stress vector are also yielded by the first forms of equations (19), as:

$$\tau_{xy} = -\frac{2a^2 z}{a^2+b^2}\frac{M_t}{I_t}\ ; \qquad \tau_{xz} = \frac{2b^2 y}{a^2+b^2}\frac{M_t}{I_t}$$

The size of the stress vector τ is:

$$\tau(y,z) = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = \frac{M_t}{I_t}\frac{2}{a^2+b^2}\sqrt{a^4 z^2 + b^4 y^2} \tag{41}$$

where the conventional moment of inertia I_t is given by equation (40). This leads to:

$$\tau(y,z) = \frac{2M_t}{\pi\, ab^2(1-\alpha^4)}\sqrt{\frac{z^2}{b^2} + \frac{b^2 y^2}{a^4}} \tag{42}$$

The maximum stress is reached on the outer ellipse, at the small half-axis ends and is expressed as:

$$\tau_{\max} = \frac{2M_t}{\pi\, ab^2(1-\alpha^4)} = \frac{M_t}{W_t} \tag{43}$$

while the minimum stress on the outer contour is reached at the ends of the large half-axis, as:

$$\tau_{\min} = \frac{2M_t}{\pi\, a^2 b(1-\alpha^4)} \tag{44}$$

Taking into account equation (41) for I_t, the twisting strength modulus is:

$$W_t = \frac{\pi\, ab^2}{2}(1-\alpha^4) = W_{te}(1-\alpha^4) \tag{45}$$

where W_{te} is the strength modulus of a filled elliptical cross section bound by the outer ellipse.

It can be noticed that the use of an elliptical ring cross section has a similar effect upon the conventional moment of inertia, by reducing it proportionally to $1-\alpha^4$. This result confirms that the use of elliptic ring cross sections represents an efficient solution to economize material as far as rigidity is concerned, as well as in terms of strength.

Figure 6 graphically illustrates the spatial distribution of the stress vector, made dimensionless by report to the τ_{max} stress.

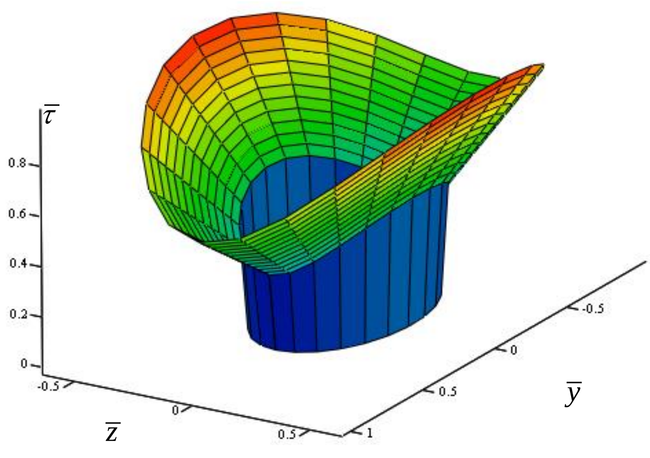


Figure 6. Stress vector τ distribution over an elliptical ring cross section with $\alpha = 0.5$ and $b = 0.6a$.

3.4. Axial displacement over the transverse cross section

As the stress function for the considered situation is the same as in the case of filled elliptical cross section, the axial displacement *u*of the cross section is also given by equation (28), in which, the definition domain is now the annular transverse cross section.A three-dimensional image of this deformed cross section is shown in figure 7.

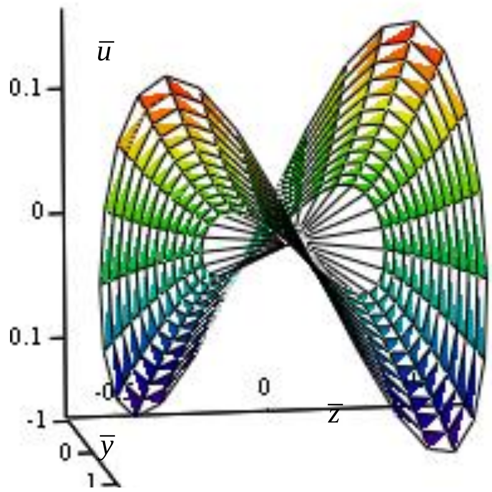


Figure 7. Strain state of an elliptical ring cross section with $\alpha = 0.5$ and $b = 0.6a$.

4. Conclusions

Based upon previously determined equations for the torsion of rods with multiple connected domains as cross sections, the present paper determines the particular solution for elliptic and elliptic ring cross sections. For both considered situations, the stress function, specific twist, stress state and axial displacements are determined analytically. By aid of the MathCAD environment, the obtained mathematical model was implemented for a specific situation and the yielded tangent stresses as well as the strained transverse cross section were represented as both 3D images and contour lines.

5. References

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