

PAPER • OPEN ACCESS

Velocity Distribution in Rough Pipe: the Model Based on the Analytical Description of Resistance Curves in Nikuradse's Experiments

To cite this article: O Dudar 2019 *IOP Conf. Ser.: Mater. Sci. Eng.* **581** 012036

View the [article online](#) for updates and enhancements.

Velocity Distribution in Rough Pipe: the Model Based on the Analytical Description of Resistance Curves in Nikuradse's Experiments

O Dudar

Applied Mathematics and Mechanics Faculty, Perm National Research Polytechnic University, 29, Komsomolsky Ave. Perm, 614990, Russia

E-mail: olegdudar@yandex.ru

Abstract. Calculation and optimization of such complex systems as ventilating networks, water supply systems, etc. taking into account roughness of a surface and possible processes of heat and mass transfer demands use simple hydrodynamic models, which are at the same time sufficiently exact for engineering calculations. The goal of this paper is to develop such model. Prandtl's model of a two-layer flow in the smooth pipe with the linear law of velocity distribution for a near-wall laminar layer and the logarithmic law for a turbulent core is used as a basis, which spreads to a case of rough pipes. The velocity profile equation and also the equations for the coordinate Y_b of the laminar and turbulent border and its velocity U_b are derived. This paper is based on the use of the analytical description of Nikuradse's experimental curves $\lambda(\text{Re}, \delta)$ (λ is the resistance coefficient, δ is the relative roughness) which earlier was found by the author. The dependence $\lambda(\text{Re}^*, \delta)$ (Re^* is the dynamic Reynolds number) is included into all above-mentioned equations. Results of numerical calculations of Y_b and U_b as functions of Re^* for the values of δ which are the same as in Nikuradse's experiments are given. The calculation of velocity profiles for some values of Re^* and δ is also fulfilled. Comparisons with the experimental velocity profiles obtained by Nikuradse demonstrate a good agreement.

1. Introduction

Modern universal packages allow solving the most varied problems of hydrodynamics. However, their use for calculation and optimization of such complex systems as ventilating networks, water supply systems, etc. taking into account roughness of a surface and possible processes of heat and mass transfer is impossible because of significant time expenditure. The goal of this paper is to develop a simple, but at the same time exact model of a turbulent flow in a rough pipe, suitable for the solution of the above-mentioned difficult engineering problems. Prandtl's model [1-3] of a two-layer flow with the linear law of velocity distribution for a near-wall laminar layer and the logarithmic law for a turbulent core is used as a basis. The systematic verification of the assumptions which are the cornerstone of the Prandtl's mixing length theory was carried out in both real [4, 5] and numerical [6-9] experiments and proved their sufficient acceptability. However, for the analysis of a flow in rough pipes papers [1-3] used the single-layer model which is acceptable for the complete roughness (square-law) regime and gives a serious error for the regimes transitional from the smooth-pipe to the square-law regime. Most of later papers used the same approach [10-13]. At the same time a serious problem is the absence of analytical dependence $\lambda\left(\text{Re}, \frac{k}{r_p}\right)$ for transient regimes (λ is the resistance coefficient, k is the height of the uniform roughness elements, r_p is the pipe radius) entering into a velocity profile expression whereas for the limit (smooth-pipe and square-law) flow regimes such dependence is available. This paper is



based on the use of the analytical dependence $\lambda\left(\text{Re}, \frac{k}{r_p}\right)$ found earlier by the author [14] as a result of the analysis of Nikuradse's experiments [1, 2] which can be applied for the description of all flow regimes. In total, a velocity profile for the laminar and turbulent layers, the thickness of a laminar layer and the velocity on the laminar - turbulent border were found as a function of Reynolds number Re and relative roughness $\frac{k}{r_p}$.

2. Model of a liquid flow in a rough pipe

Let's assume that the roughness of a surface is uniform (sand type) and is characterized by the height of the roughness elements k .

Following Prandtl's model of a flow in a smooth pipe we presume that there is a two-layer flow, namely: for a laminar near-wall layer the Newton's friction law is applied, and for a turbulent core Prandtl's hypothesis of mixing length is used, wherein the mixing length l is proportional to the distance from a wall y ($l = \kappa y$, where κ is the von Karman's constant).

The presence of roughness requires to assume the occurrence of a virtual wall on which the no-slip condition for longitudinal velocity u is satisfied. Its location is defined by a dimensionless parameter γ according to expression $u(\gamma k) = 0$ (about virtual wall see also [15]).

Then a velocity profile can be described by the following differential equations [1, 3, 10]

$$\frac{du}{dy} = \begin{cases} \frac{\tau_w}{\mu}, & \gamma k \leq y \leq y_b \\ \frac{1}{l} \sqrt{\frac{\tau_w}{\rho}}, & y_b \leq y \leq r_p \end{cases}, \quad (1)$$

where τ_w is the viscous shear on a wall, μ and ρ are the dynamic viscosity and the density of a liquid, r_p is the pipe radius, y_b is the coordinate of a border of the laminar and turbulent layers. Making transformation to dimensionless variables, we will get

$$\frac{dU}{dY} = \begin{cases} \frac{r_p}{l^*}, & \gamma \frac{k}{r_p} \leq Y \leq Y_b \\ \frac{1}{\kappa Y}, & Y_b \leq Y \leq 1 \end{cases}, \quad (2)$$

where $U = \frac{u}{V^*}$ is the dimensionless velocity, $V^* = \sqrt{\frac{\tau_w}{\rho}}$ is the friction velocity, $Y = \frac{y}{r_p}$ is the dimensionless distance from a wall, $l^* = \frac{\nu}{V^*}$ is the dynamic length, ν is the kinematic viscosity.

3. Law of the velocity distribution

After integration (2) and use of boundary conditions we obtain the law of velocity distribution

$$U(Y) = \begin{cases} \frac{r_p}{l^*} \left(Y - \gamma \frac{k}{r_p} \right), & \gamma \frac{k}{r_p} \leq Y \leq Y_b \\ U_{\max} + \kappa^{-1} \ln Y, & Y_b \leq Y \leq 1 \end{cases} \quad (3)$$

Keeping in mind that on the border of the laminar and turbulent layers the condition $U(Y_b) = U_b$ has to be satisfied, where U_b is the velocity on this border, (3) is possible to write down in a form

$$U(Y) = \begin{cases} \frac{U_b}{Y_b - \gamma \frac{k}{r_p}} \left(Y - \gamma \frac{k}{r_p} \right), & \gamma \frac{k}{r_p} \leq Y \leq Y_b \\ U_b + \kappa^{-1} \ln \frac{Y}{Y_b}, & Y_b \leq Y \leq 1 \end{cases} \quad (4)$$

In the equation (4) Y_b , U_b , γ and κ remain unknown. From comparison (3) and (4) we can write down the system of the equations in Y_b and U_b

$$\begin{cases} \frac{U_b}{Y_b - \gamma \frac{k}{r_p}} = \frac{\text{Re}^*}{2} \\ U_b = U_{\max} + \kappa^{-1} \ln Y_b \end{cases} \quad (5)$$

In (5) it is used that

$$\frac{2r_p}{l^*} = \frac{2r_p V^*}{\nu} = \text{Re}^*, \quad (6)$$

where Re^* is the dynamic Reynolds number.

For determination of $U_{\max}$ we will apply the relation established by Prandtl between the law of resistance and the velocity distribution. Finding for a velocity profile (4) the average velocity according to the formula

$$\bar{U} = 2 \int_{\gamma \frac{k}{r_p}}^1 U(1-Y) dY,$$

after transformations we will get

$$\bar{U} = U_b \left[1 - \left(Y_b + \gamma \frac{k}{r_p} \right) + \frac{1}{3} \left(Y_b^2 + Y_b \gamma \frac{k}{r_p} + \left(\gamma \frac{k}{r_p} \right)^2 \right) \right] - \kappa^{-1} \left(\frac{3}{2} + \ln Y_b - 2Y_b + \frac{Y_b^2}{2} \right). \quad (7)$$

Though the analysis (7) shows that the linear and square summands have significant effect at small values of Reynolds number (for a smooth surface discarding of the linear and square summands leads to an error of about 9%), in this paper we will be only limited to the initial approximation for $\bar{U}$

$$\bar{U} \approx U_b - \kappa^{-1} \left(\frac{3}{2} + \ln Y_b \right). \quad (8)$$

Taking into account (5) and (8), we can write down

$$U_{\max} = \bar{U} + \frac{3}{2} \kappa^{-1}. \quad (9)$$

Then, using Darcy's formula in the form [1, 3, 10]

$$\bar{U} = \sqrt{\frac{8}{\lambda}}, \quad (10)$$

where λ is the resistance coefficient, we come to the following expression

$$U_{\max} = \sqrt{\frac{8}{\lambda}} + \frac{3}{2} \kappa^{-1}. \quad (11)$$

As a result we derive from (3) taking into account (6) and (11) the final expression for a velocity profile as a function of the resistance coefficient λ

$$U(Y) = \begin{cases} \frac{\text{Re}^*}{2} \left(Y - \gamma \frac{k}{r_p} \right), & \gamma \frac{k}{r_p} \leq Y \leq Y_b \\ \sqrt{\frac{8}{\lambda}} + \kappa^{-1} \left(\frac{3}{2} + \ln Y \right), & Y_b \leq Y \leq 1 \end{cases} \quad (12)$$

4. Equations for a coordinate of the laminar and turbulent border and its velocity

Substitution of (11) in (5) allows entering a parameter λ into the system of equations in Y_b and U_b :

$$\begin{cases} \frac{U_b}{Y_b - \gamma \frac{k}{r_p}} = \frac{\text{Re}^*}{2} \\ U_b = \sqrt{\frac{8}{\lambda}} + \kappa^{-1} \left(\frac{3}{2} + \ln Y_b \right) \end{cases} \quad (13)$$

Excluding U_b from a system (13), we get the transcendental equation in Y_b

$$Y_b = \gamma \frac{k}{r_p} + \frac{2}{\text{Re}^*} \left(\sqrt{\frac{8}{\lambda}} + \kappa^{-1} \left(\frac{3}{2} + \ln Y_b \right) \right). \quad (14)$$

On the contrary, excluding Y_b , we derive the transcendental equation in U_b

$$U_b = \sqrt{\frac{8}{\lambda}} + \kappa^{-1} \left(\frac{3}{2} + \ln \left(\gamma \frac{k}{r_p} + \frac{2U_b}{\text{Re}^*} \right) \right). \quad (15)$$

To use formulas (12, 14, 15), it is necessary to know dependence $\lambda \left(\text{Re}^*, \frac{k}{r_p} \right)$. For the smooth-pipe and complete roughness limit regimes of flow such dependences are known from the experiments [1-3]. For the full roughness regime the resistance coefficient λ depends only on the relative roughness $\frac{k}{r_p}$ [1, 3]

$$\frac{1}{\sqrt{\lambda}} = -2 \lg \left(\frac{k}{r_p} \right) + 1.74 = -0.869 \ln \left(\frac{k}{r_p} \right) + 1.74 \quad (16)$$

and for the smooth-pipe flow regime it depends only on the Reynolds number [2, 3]

$$\frac{1}{\sqrt{\lambda}} = 2 \lg (\text{Re} \sqrt{\lambda}) - 0.8. \quad (17)$$

Taking into account that Darcy's formula (10) can be rewritten in a form

$$\bar{U} = \frac{\bar{u}}{V^*} = \frac{\text{Re}}{\text{Re}^*} = \sqrt{\frac{8}{\lambda}} \quad (18)$$

we come to a form of record of the smooth pipe resistance law through the dynamic Reynolds number

$$\frac{1}{\sqrt{\lambda}} = 2 \lg(\text{Re}^*) + 0.103 = 0.869 \ln(\text{Re}^*) + 0.103. \quad (19)$$

Let's use the (16) and (19) formulas for the determination of the remaining not found constants.

4.1. A smooth-pipe flow regime

In this case $\frac{k}{r_p} \rightarrow 0$. Then from (16) we derive:

$$U_b - \kappa^{-1} \ln U_b - \kappa^{-1} \left(\frac{3}{2} + \ln 2 \right) + \kappa^{-1} \ln \text{Re}^* = \sqrt{\frac{8}{\lambda}}. \quad (20)$$

Substituting in the right part of (20) the experimental formula (19) we will come to the expression

$$U_b - \kappa^{-1} \ln U_b - \kappa^{-1} \left(\frac{3}{2} + \ln 2 \right) + \kappa^{-1} \ln \text{Re}^* = 2.46 \ln(\text{Re}^*) + 0.291,$$

from which we get the following numerical equalities

$$\kappa^{-1} = 2.46, \quad (21)$$

$$U_b - \kappa^{-1} \ln U_b - \kappa^{-1} \left(\frac{3}{2} + \ln 2 \right) = 0.291. \quad (22)$$

The last means that in the case of the smooth-pipe regime the dimensionless velocity on the laminar and turbulent border is constant. From (21) and (22) we obtain values of the constants

$$\kappa = 0.407, \quad U_b = 11.7. \quad (23)$$

This value of the von Karman's constant is in a good agreement with [2-4, 16]

4.2. A complete roughness regime

In this case $\text{Re}^* \rightarrow \infty$. Then it follows from the first equation of the system (13)

$$Y_b - \gamma \frac{k}{r_p} = \frac{2U_b}{\text{Re}^*} \rightarrow 0 \Rightarrow Y_b \rightarrow \gamma \frac{k}{r_p}.$$

It means that the thickness of a viscous layer tends to zero, and

$$U_b = U(Y_b) \rightarrow U\left(\frac{k}{r_p}\right) = 0.$$

Then the equation (15) takes the following form

$$\kappa^{-1} \left(\frac{3}{2} + \ln \gamma \frac{k}{r_p} \right) = -\sqrt{\frac{8}{\lambda}}. \quad (24)$$

Substituting in the right part of (24) the experimental formula (16) we will come to the expression

$$\kappa^{-1} \left(\frac{3}{2} + \ln \gamma + \ln \frac{k}{r_p} \right) = 2.46 \ln \left(\frac{k}{r_p} \right) - 4.92,$$

wherefrom

$$\ln \gamma = -4.92\kappa - 1.5 = -3.503$$

and therefore

$$\gamma = 0.0301. \quad (25)$$

In paper [17] $\gamma = \frac{1}{3}$.

5. Numerical results

To find the velocity profile $U(Y)$, the coordinate Y_b of the laminar and turbulent border and its velocity U_b using formulas (12, 14, 15) it is necessary to have the analytical dependence $\lambda\left(\text{Re}^*, \frac{k}{r_p}\right)$. In paper [14] the procedure of obtaining these dependences as a result of the analysis of Nikuradse's experimental data on a turbulent flow in rough pipes is described. The derived analytical dependence $\lambda\left(\text{Re}^*, \frac{k}{r_p}\right)$ allows to find $U(Y)$, Y_b and U_b for any combination of Re^* and $\frac{k}{r_p}$. In figure 1 the dependences $Y_b(\text{Re}^*)$ and $U_b(\text{Re}^*)$ are provided for the same values of relative roughness $\frac{k}{r_p}$ presented in Nikuradse's paper [1] for experimental curves $\lambda\left(\text{Re}^*, \frac{k}{r_p}\right)$. In figure 2 the velocity profiles obtained experimentally [1] and with the help of the analytical dependence $\lambda\left(\text{Re}^*, \frac{k}{r_p}\right)$ for two particular cases are compared. Such comparison was carried out for all experimental points presented in [1]. It showed that in 953 experimental points from 963 points the deviation did not exceed 4.5%.

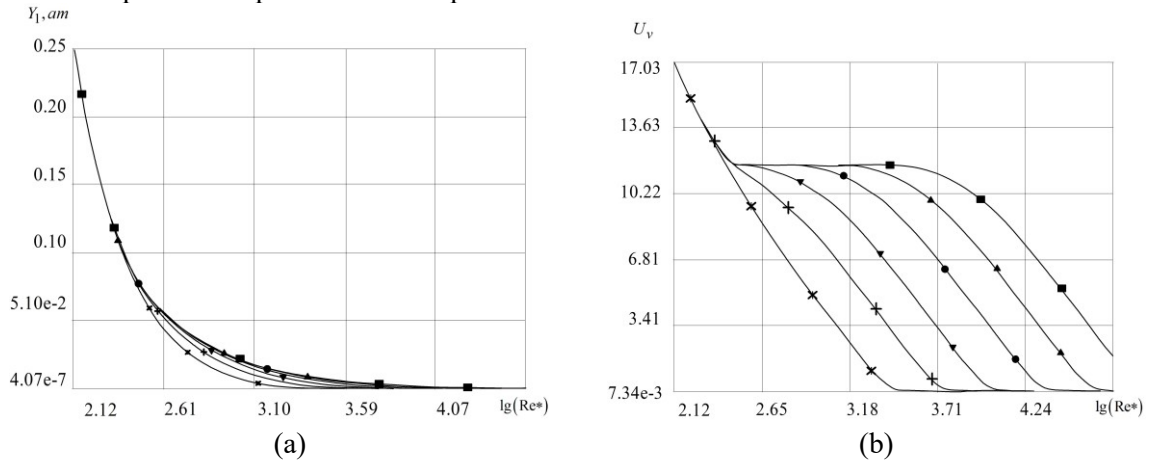


Figure 1. The coordinate Y_b (a) and the dimensionless velocity U_b (b) on the laminar and turbulent border. Legend $\blacksquare - k/r_p = 507$; $\blacktriangle - k/r_p = 252$; $\bullet - k/r_p = 126$; $\blacktriangledown - k/r_p = 60$; $+$ $- k/r_p = 30.6$; $*$ $- k/r_p = 15$.

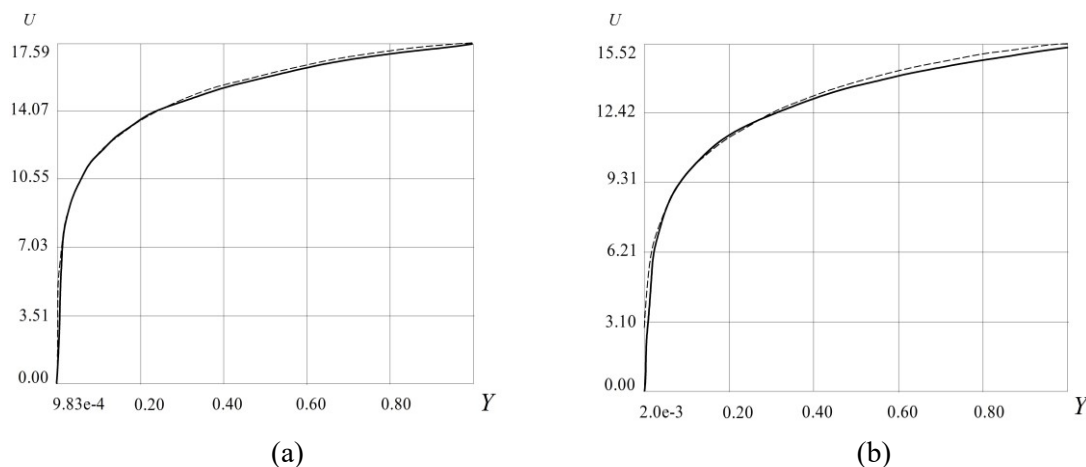


Figure 2. Velocity profile $k/r_p = 30.6$, $Re^* = 1642$ (a); $k/r_p = 15$, $Re^* = 1894$ (b)

——— – calculation; – – – – experiment.

6. Conclusions

According to the results of Nikuradse's experiments on a turbulent flow in the pipes with sand roughness

analytical dependences $\lambda \left(Re^*, \frac{k}{r_p} \right)$ are derived.

The formula of velocity distribution for any regime of a turbulent flow (smooth-pipe, square-law, transitional) expressed in terms of the resistance law for this regime is obtained.

The equations describing dependence of the coordinate of a laminar and turbulent border and its velocity on the dynamic Reynolds number and relative roughness are derived and solved.

References

- [1] Nikuradse J 1933 *Forschung auf dem Gebiete des Ingenieurwesens* **361** (B/4) 1
- [2] Nikuradse J 1932 *Forschung auf dem Gebiete des Ingenieurwesens* **356** (B/3) 1
- [3] Schlichting H 1979 *Boundary Layer Theory* (New York: McGraw-Hill) p. 817
- [4] Bailey S C C, Vallikivi M, Hultmark M and Smits A J 2014 *J. Fluid Mech.* **749** 79
- [5] Marusic I, Monty J P, Hultmark M and Smits A J 2013 *J. Fluid Mech.* **716** (R3) 1
- [6] Orlandi P, Bernardini M and Pirozzoli S 2015 *J. Fluid Mech.* **770** 424
- [7] Pirozzoli S 2014 *J. Fluid Mech.* **745** 378
- [8] Chan L, MacDonald M, Chung D, Hutchins N and Ooi A 2015 *J. Fluid Mech.* **771** 743
- [9] Chung D, Chan L, MacDonald M, Hutchins N and Ooi A 2015 *J. Fluid Mech.* **773** 418
- [10] Loitsyanskii L G 1987 *Liquids and gases mechanics* (Moscow: Nauka) p. 840
- [11] Hramtsov I V, Bulbovich R V and Pavlohradskyi V V 2013 *Volga scientific bulletin* **6** (22) 5
- [12] Bryanskaya Yu V 2013 *Magazine of Civil Engineering* **6** 31
- [13] Kondratyev A S, Nya T L and Shvydko P P 2017 *Fundamental research* **1** 74
- [14] Dudar O I 2018 Influence model of roughness on a turbulent flow in a pipe based on the analytical description of Nikuradze experiments *Nonlinear problem of the theory of hydrodynamic stability and turbulence (Electronic Materials)* ed N V Nikitin and N V Popelenskaya (Moscow: MAX Press) pp. 127 – 133
- [15] Herwig H, Gloss D and Wenterodt T 2008 *J. Fluid Mech.* **613** 35
- [16] Bernardini M, Pirozzoli S and Orlandi P 2014 *J. Fluid Mech.* **742** 171
- [17] Colebrook C F and White C M 1937 *Proc. Royal Soc. London A* **161** (960) 367